ACOUSTIC WAVE FILTERS; AN EXTENSION OF THE THEORY

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ABSTRACT

Extension of theory of acoustic wave filters to high frequencies.—In the simple form of acoustic wave filter theory already published, the wave-length was assumed long in comparison with the length of a section of the conduit. Experimentally additional bands appeared at higher frequencies and these are now discussed. Advantage is taken of the recurrence at higher frequencies of the vanishing of the acoustic impedance of a section of the conduit, and approximate values of the components of this impedance are established for the neighboring region of frequencies. *Formulas for low-pass and single-band-pass filters*, using these values, are found to give the experimental cutoffs of the additional bands with satisfactory accuracy. By appropriate design an additional band can be made practically to disappear. Tables showing agreement of experiment with theory are given. The theory is thus extended to higher frequencies but not to those for which additional resonance frequencies of the branches occur.

IN a discussion[1] of an approximate theory of acoustic wave filters for low-frequency-pass, high-frequency-pass and single-band-pass acoustic wave filters, it became evident that correct expressions for the acoustic impedances in the transmitting conduit and in its branches could not readily be obtained, and attention was confined to the securing of approximate formulas. The theory based on these approximations can be relied upon only if the wave-length is larger than several times the length of a section in the transmitting conduit.

Eqs. (5) and (6) of the preceding paper state that there is transmission through such a filter only between the frequencies represented by the following values of the ratios of the acoustic impedance in a section of the transmission line and in a branch: $Z_1/Z_2=0$ and $Z_1/Z_2=-4$. Transmission occurs at all frequencies at which Z_1 and Z_2 are of opposite signs and the absolute value of Z_1 is not greater than four times that of Z_2. It is not difficult to see that there will be transmission at frequencies other than the ranges specified by the values of Z_1 and Z_2 used in the simple theory.[1] For in every case considered Z_1 and Z_2 were composed of inertances and capacitances in series and in parallel or in a combination of these. Zobel[2] has shown that if there are two reactances which increase

[1] Stewart, Phys. Rev. **20**, 528 (1922)
[2] Zobel, Bell System Technical Journal **2**, 1 (Jan. 1923)

with frequency, their series and parallel combinations will also increase with frequency. His discussion concerns electrical reactances but is readily seen to be applicable to the acoustic case. In the acoustic filters the impedances were[1] combinations of the reactances, $iM\omega$ and $-i/C\omega$, ω being 2π times the frequency, M the inertance and C the capacitance. The change of these reactances with frequency is positive and hence, according to the above theorem, the changes of Z_1 and of Z_2 with frequency are positive. Since there is a band of transmission from the frequency where $Z_1=0$ to the frequency where $Z_1=-4Z_2$, one may expect that in the acoustic filters there will be a transmission band with one frequency limit wherever $Z_1=0$, and this value will occur at the resonance frequency of a section of the transmitting conduit. Since in all the acoustic filters thus far constructed a simple tubular conduit is used for the transmission line, the resonance frequencies are practically harmonic; also the frequencies for which $Z_1=0$ can be predicted, and, as will be shown, this leads to the location of additional transmission bands not indicated by the earlier approximate theory.[1] This procedure is possible, however, only in case the expression for Z_2 is known. Now it happens that the filters enumerated in this article have branches such that the formulas for Z_2 are approximately correct over the entire range of frequencies considered, and consequently the above method of locating the bands is applicable.

Low-frequency-pass Filters

The formulas for low-frequency-pass filters found most acceptable in the previous report were $f_1=0$, and $f_2=(1/\pi)\ (M_1C_2+4M_2C_2)^{-1/2}$, wherein f_1 and f_2 are the limiting frequencies of a band of zero attenuation, M and C are respectively the inertance and capacitance, and the subscripts of the last two refer respectively to the conduit and branch as indicated by Z_1 and Z_2. In obtaining these equations it was assumed that C_1 was zero. The justification was chiefly that the length of a section of the conduit was very short compared to a wave-length and the fluid could be assumed as moving as a whole. There seems to be no simple escape from the policy of assuming approximate values of Z_1, and the present discussion really concerns a method of approximating the values of the reactance components of Z_1 in higher frequency regions near where $Z_1=0$, (i.e. near the resonance frequencies of each section of the conduit) so that predicted values of the recurring bands may be obtained. If Z_1 is a tube of length l_1, open at both ends, we have the resonance frequencies $na/2l_1$ where n is an integer. We know that in the actual case there will be such resonance frequencies if the sizes of the openings are sufficient. Such a tube in the resonance frequency region may be regarded as approximately equivalent to an inertance M_1 and a capacitance C_1 in series, or $Z_1 = i(M_1\omega - 1/C_1\omega)$. In the simpler theory[1] Z_1 was merely $iM_1\omega$. The value of Z_2 is, however, carried over into this extension of the theory without change because it is sufficiently exact. Its value is $i(M_2\omega - 1/C_2\omega)$. The procedure involves the adoption of these values of Z_1 and Z_2, substituting them in the standard equations for the attenuation limits, $Z_1/Z_2 = 0$ and $Z_1/Z_2 = -4$. There is then obtained the following:

$$2\pi f_1 = (M_1C_1)^{-1/2};\ 2\pi f_2 = [(1/C_1 + 4/C_2)/(M_1 + 4M_2)]^{1/2}. \qquad (1)$$

These equations occur in the previous article[1] but were not applicable to the filters at low frequencies. Here they will be accepted under the condition that they apply only in the frequency region of resonance $[Z_1 = 0]$. The equivalent M_1 and C_1 are not separately determined by the vanishing of Z_1, but by an additional consideration to be mentioned. Knowing the equivalent inertance and capacitance, M_1 and C_1, the attenuation limits can be obtained from Eq. (1). The essence of the method is the knowledge of the values of the *equivalent* M_1 and C_1 in each frequency region where $Z_1 = 0$. The method assumes no change in the form of Z_2 or in the values of M_2 and C_2.

Single-band Filters

Adopting the above form of Z_1 and the value of

$$Z_2 = iM_2\omega(M_2'C_2\omega^2 - 1)/(M_2C_2\omega^2 + M_2'C_2\omega^2 - 1)$$

as previously[1] used for a single band filter we get the following values of attenuation limits,

$$2\pi f_1 = (M_1C_1)^{-1/2},\ 2\pi f_2 = [(M_2C_2(1 + M_2'/M_2)\ )]^{-1/2}$$
$$2\pi f_3 \text{ or } 2\pi f_4 = (M_2C_2)^{-1/2}[A \pm (A^2 - B)^{1/2}]^{1/2}$$

wherein

$$A = \tfrac{1}{2}C_1^{-1}(M_1C_1 + M_2C_2 + M_2'C_2 + 4M_2C_1)(M_1 + M_1M_2'/M_2 + 4M_2')^{-1}$$
$$B = M_2C_2C_1^{-1}(M_1 + M_1M_2'/M_2 + 4M_2')^{-1}$$

In accord with what has been stated, the new approximate expressions later to be obtained for M_1 and C_1 are to be accepted only in the neighborhood of the frequency f_1 corresponding to $Z_1 = 0$. Hence if the computed values of f_3 or f_4 prove to be far removed from this frequency region, they should receive no consideration. In the cases of filters here recorded, f_4 falls in a region of much lower frequency where the old approximation for Z_1 holds and hence f_4 cannot be considered, while f_2 is the same form as in the earlier theory and is one of the cutoffs of the first band, formerly described as a single band. Hence in the additional bands f_1 and f_3, only, will be considered In the case of a more complicated filter a variable single band pass filter,[3] Z_2 has the more complicated expression,

$$Z_2 = i[M_2''\omega(M_2C_2\omega^2 + M_2'C_2\omega^2 - 1) + M_2\omega(M_2'C_2\omega^2 - 1)](M_2C_2\omega^2 + M_2'C_2\omega^2 - 1)^{-1}.$$

Here M_2'' is the inertance of the orifice introduced in series with the side branch. When this new value of Z_2 is used, the equations for f_1 and f_3 are as follows:

$$2\pi f_1 = (M_1C_1)^{-1/2};\ 2\pi f_3 = [A + (A^2 - B)^{1/2}]^{1/2} \quad (3)$$

where $B^{-1} = C_1C_2[M_1M_2 + M_1M_2' + 4M_2''\ (M_2 + M_2') + 4M_2M_2'];$ (3)

$$A = \tfrac{1}{2}B[M_2C_2 + M_2'C_2 + C_1\ (4M_2'' + 4M_2 + M_1)\ ].$$

Application of Equations

If the values of M_1 and C_1 were known, the equations (1) and f_1 and f_3 of equations (2) and (3) could be at once applied, the former to low-pass and the latter to single-band-pass filters. In the earlier work[1] the value of M_1 for a section of a conduit was $\rho l_1/S_1$, wherein l_1 was the length and S_1 the area of the conduit and ρ was the density of the gas. Also C_1 was $V_1/\rho a^2$, where V_1 was the volume considered and a was the velocity of sound. If these two expressions are applied to a section of a conduit, the product is $M_1C_1 = l_1^2/a^2$, but this cannot be correct in the region $Z_1 = 0$, for here, as shown by the formula for Z_1, $M_1C_1 = 1/\omega^2$ and, from elementary acoustics, at resonance $1/\omega = l_1/n\pi a$. Consequently it will here be assumed that the *equivalent* M_1 and C_1 are such that $(M_1C_1)^{1/2} = l_1/n\pi a$. Obviously this product will occur if, instead of the earlier expressions for M_1 and C_1 the following somewhat empirical values be used: $M_1 = \rho_1 l_1/S_1\alpha$ and $C_1 = V_1/\rho a^2\beta$, wherein $\alpha\beta = n^2\pi^2$. By trial it was shown that a fairly satisfactory confirmation of the equations could be obtained if the following values of α and β were adopted. For low-pass filters, $\alpha = 1$, $\beta = n^2\pi^2$; for single-band-pass filters, $\alpha = n$ and $\beta = n\pi$. The reason no attention in this connection is given to high pass filters is that throughout the attenuation region the assumption of relatively long wave-length is sufficiently accurate, and transmission seems to be excellent at higher frequencies. There are, however, effects in this transmission region which may be discussed at a later time.

The values of the other symbols in our equations are as follows: $M_2 = \rho/c$ with low-pass; $M_2 = \rho l_2/S_2$ with single-band-pass filters, wherein

[3] Stewart, Phys. Rev. **22**, 502 (1923)

c is the conductivity of the orifice into the volume in the side branch; $M_2' = \rho/c'$, wherein c' is the conductivity of the orifice into the side branch and $C_2 = V_2/\rho a^2$. These, together with $M_1 = \rho l_1/S_1 a$ and $C_1 = V_1/(\rho a^2 \beta)$ are substituted in Eqs. (1), (2) and (3) as indicated, and the additional bands are found, one for each value of n. Attention should be called to the fact that it is possible to cause one and sometimes more of the additional transmission bands to disappear by using in series filters

TABLE I

No.	No. sects.	l_1 (cm)	r_1 (cm)	V_2 (cm^3)	f_m (obs.)	f_1'* (calc.)	f_{11} (obs.)	f_{11} (calc.)
H−2	3	2.66	.243	21.6	450	524	5300	4820
H−3	5	5.0	.243	43.1	230	291	3200	3115
H13−5	5	4.0	1.42	44.4	920	1360	3400	3125
$H13_1$−10C†	10	4.0	1.42	44.4	1000	1118	3675	3520
$H13_2$−10C	10	4.0	1.42	44.4	660	952	3100	3040
H13− 5C	5	8.0	1.42	44.4	825	875		2060
H13− 3C	3	12.0	1.42	44.4	700	750	1500	1433
H13− 2C	2	16.0	1.42	44.4	670	664	1110	910
H−15−16	4	1.67	.75	5.94	2200	2710	6400	6660
H−15− 8	4	1.67	.75	5.94	1700	2120	5600	4580
H−16−16	4	1.67	.75	2.62	3200	4085	7200	7030
H−16− 8	4	1.67	.75	2.62	2500	3200	5800	5700
H−17−16	4	1.67	.75	4.36	2700	3175	6400	6870
H−17− 8	4	1.67	.75	4.36	2200	2480	5500	4720
H−18	2	1.00	.401	1.59	2800	3265		9600
R−1	3	1.58	1.19	7.1	2700	2840	5050	5370
R−2	4	1.58	1.19	7.1	2750	2840	5200	5370
R−3	4	1.32	1.19	5.87	3350	3170	5750	5910
R−4	4	1.45	1.19	6.48	3025	3010	6400	5350
R−5	6	2.58	1.19	11.82	2000	2095	3800	3940
R−6	6	1.58	1.19	7.1	2075	2100	3900	3930
R−6	6	1.58	1.19	7.1	2700	2700	5300	5040

No.	f_{12} (obs.)	f_{12} (calc.)	f_{21} (obs.)	f_{21} (calc.)	f_{22} (obs.)	f_{22} (calc.)	f_{31} (obs.)	f_{31} (calc.)
H−2	6500	6460				12920		
H−3	3400	3440	6200	6250	6800	6880		
$H13_1$−10C	3900	4300	7600	6970		8600		
$H13_2$−10C	4500	4300	6300	5710	7700	8600		
H13− 5C		2150	4300	3720	4700	4300	5900	5570
H13− 3C	1650	1528		2820		2866		4080
H13− 2C	1300	1074	1900	2100	2000	2150	2900	3070

No.	f_{32} (obs.)	f_{32} (calc.)	f_{41} (obs.)	f_{41} (calc.)	f_{42} (obs.)	f_{42} (calc.)	f_{51} (obs.)	f_{51} (calc.)	f_{52} (obs.)	f_{52} (calc.)
H13−5C	6800	6450		8600						
H13−3C		4300		5400		5733	6800	6750	7400	7160
H13−2C	3200	3225	4000	4040	4070	4300	5300	5000	5800	5370

* All calculated values were obtained by use of Eq. (1) except f_1' (calc.) for which the original equations were used.

† The additional designation, C indicates the presence of a core in the series section, thus changing the area.

possessing the same transmission region at low frequencies and yet differently located transmission bands at high frequencies. This is possible since the additional bands depend only on the conduit dimensions, if branches are used for which the Z_2's can be expressed with fair accuracy throughout the entire range. It is also possible to determine the dimensions that will cause the limits of an additional band to coalesce and thus remove the band.

TABLE II

No.	No. Sects.	l_1 (cm)	r_1 (cm)	V_2 (cm^3)	c	F_1 (obs.)	F_1 (calc.)	F_2 (obs.)	F_2 (calc.)
1	3	5.0	.243	42.7	.69	190	224	260	346
2	3	3.66	.243	21.19	.69	275	318	375	575
3	3	5.0	.243	28.02	.69	270	283	370	430
4	3	2.66	.243	14.81	.69	430	436	520	707
5	3	5.00	.243	28.02	.69	190	234	270	410
6	3	2.66	.243	14.81	.69	350	349	500	680
9	3	2.66	.243	22.7	.420	285	275	455	499
9_1	3	2.66	.243	22.7	.420	285	275	350	387
9_2	3	2.66	.243	22.7	.420	285	275	390	439
10	4	1.77	1.12	6.11	.556	775	850	1150	1180
20_1	5	2	1.42	20.4	1.67	670	673	1075	1390
20_2	5	2	1.42	20.68	2.3	680	684	1200	1555
20_3	5	2	1.42	20.0	2.25	860	885	1550	1638
20_4	5	2	1.42	20.66	3.18	890	910	1560	1770

No.	f_{11} (obs.)	$f_{11}(2)$ (calc.)	f_{12} (obs.)	$f_{12}(2)$ (calc.)	f_{21} (obs.)	$f_{21}(2)$ (calc.)	f_{22} (obs.)	$f_{22}(2)$ (calc.)
1	2300	2125	3740	3440	5000	4210	7700	6880
2	3400	3720	6200	6450				
3	2500	2725	3600	3440	5450	4620	7000	6880
4	3300	4500	6600	6450				
5	2675	2730	3475	3440	5900	5410	7600	6880
6	3500	4440	6150	6450				
9	4300	3470	6300	6450				
9_1	2700	*1935	5500	*6450				
9_2	4100	*2460	6200	*6450				
10	3900	3400		9730				
20_1	3000	2995	6800	8600				
20_2	2975	3775	6800	8600				
20_3	3650	3500	6200	8600				
20_4	3600	3972	6800	8600				

* Computed from Eq. (3).

EXPERIMENTAL RESULTS

Following a method already described,[1] transmission measurements were made of a number of filters. The data concerning them including their dimensions and cut-off frequencies are presented in the accompanying tables. In Table I are the data for the low-pass filters having the experimental cut-off's $f_m, f_{11}, f_{12}, f_{21}, f_{22}$, etc. The corresponding calculated frequencies were computed according to equation (1) excepting f_1' which was made according to the original equations. The subscripts 11 and 12 indicate the limits of the first band, 21 and 22 of the second, etc. In Table II are recorded the data for the single-band-pass filters. Here F_1 and F_2 are the cut-offs of each of the "single bands" experimentally determined and F_1(calc.) and F_2 (calc.) are the corresponding cut-offs as determined by the original equations. Some of these values, as likewise some of the f_1' values in Table I, have already been published.[1] The columns f_{11} and f_{12} are the experimental values, and f_{11} (2) and f_{12} (2) are the values computed according to equation (2). In the cases of 9_1 and

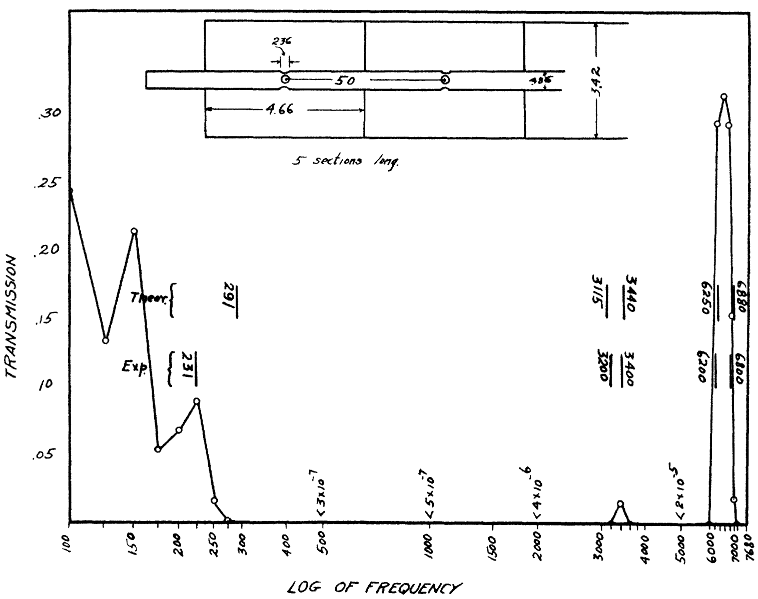

Fig. 1. A large attenuation range and two additional bands in a low-pass filter.

9_2 filters Eq. (3) is used instead of Eq. (2). A blank in a table indicates that the measurement or computation was above the range of frequencies used, or that the filter was no longer available for measurements, or that the transmission was too small to permit the location of the band. Throughout both tables there is a fairly satisfactory agreement of computation with experiment.

Fig. 1 shows a low frequency pass filter the dimensions of which are such that the additional bands do not appear until the frequency has reached ten times the cut-off frequency. In Fig. 2 the first additional band is nearer. In Fig. 3 is the case of a single band filter. These three

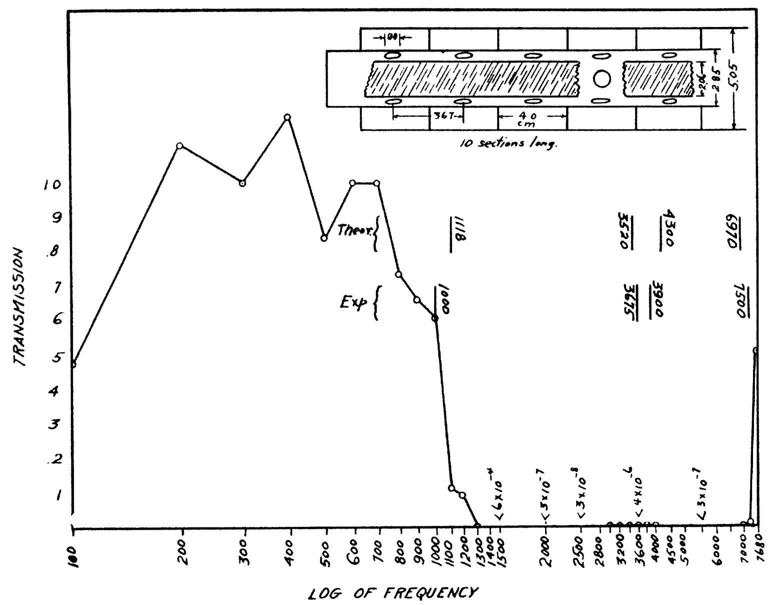

Fig. 2. An almost vanishing additional band in a low-pass filter.

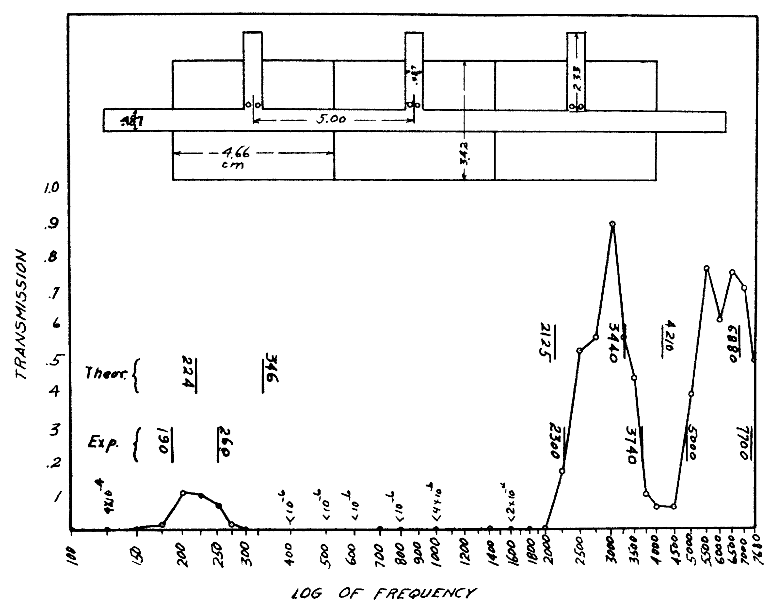

Fig. 3. A "single band" filter with two additional bands.

cases are selected not because of the excellence or inferiority of the filters as such but because they illustrate the agreement with theory. The experimental and theoretical values of the cut-offs are indicated on the figures and the agreement is found to be very satisfactory, the nature of the problem being considered.

Referring to the theorem that both Z_1 and Z_2 should have positive slopes when plotted with frequencies as abscissas, one can reach a conclusion concerning the location of the additional transmission bands relative to the resonance frequencies. Such a plot for low frequency-pass filters shows that in this region Z_2 is positive. Hence since the band extends between the frequencies where $Z_1/Z_2=0$ and $Z_1=-4Z_2$, and since Z_1 passes through zero from negative to positive with increasing frequency, the transmission band must be on the lower frequency side of the resonance value. In computations, this is not always true, presumably because of the approximate and semi-empirical character of the equations. For fifty-three bands computed for low-pass filters, only five did not have the resonance frequency as the upper limit and in each of these cases the band was very narrow. A similar graphical study of the single-band-pass filter shows that the resonance frequency is the upper limit of the additional transmission bands. No exception to this rule occurrs in the computed values of the transmission limits.

Conclusions

There has been given an approximate theory for the appearance of additional bands in low-pass and "single-band-pass" filters, the latter type of filter taking its name from the former theory.[1] This additional theory assists in design and at the same time affords an understanding of the phenomena.

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