A NOTE ON SCHOTTKY'S METHOD OF DETERMINING THE DISTRIBUTION OF VELOCITIES AMONG THERMIONIC ELECTRONS

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Abstract

Limiting conditions for Schottky's formula for the thermionic current from a filament to a coaxial cylinder.—The formula must fail when, due to space charge, the potential at any distance x $(r<x<R)$ from the axis is less than $Vr^2(R^2-x^2)/x^2(R^2-r^2)$, V being the potential of the filament with respect to the cylinder, and r and R the radii of filament and cylinder respectively. This is more restrictive than the condition for failure which has been previously assumed.

IT IS proposed in this note to examine the conditions under which the formula developed by Schottky and employed by Germer in the preceding article may be expected to represent the results of experimental observations. This formula [Eq. (1) of the preceding article] is an expression for the thermionic current i flowing from a straight filament to a coaxial cylinder against a retarding potential V, subject to specific assumptions regarding the distribution of velocities among the electrons as they leave the filament. For $V=0$ the formula gives $i=i_0$, where i_0 is the total emission from the filament. If V is given this value, the observed current is ordinarily much less than i_0. This is due, as is well known, to an effect of space charge. The presence of negative charge between the filament and cylinder causes the potentials of all points in the interspace to be less than the common potential of the bounding surfaces. If the emission is uniform in density and character over the surface of the filament the equipotential surfaces are coaxial cylinders, and the distribution of potential along a radius is characterized by a single minimum. The current reaching the cylinder in this case is made up only of those electrons that manage to cross the surface of minimum potential. If the potential of the cylinder is fixed at zero and the potential of the filament is increased from zero the surface of minimum potential moves outward, and reaches the cylinder for some particular value of V, say V'. The potential gradient at the cylinder is then zero, and no minimum of potential occurs in the interspace. It is not unnatural perhaps to expect that Schottky's formula will fail for all values of V less than V', and be applicable to observations in the range above V'. This, in fact, appears to have been Schottky's own view of the matter.[1] It may be shown, however, that the formula must fail at $V=V'$, and also for a certain range of V greater than V'.

Following Schottky we write u and v for the radial and tangential velocity components of an electron leaving the filament in a plane normal to the axis of the system, and u_1 and v_1 for the corresponding components of the velocity with which the electron reaches the cylinder. If the filament potential is V, we have, from energy considerations

$$\tfrac{1}{2}m\left[(u^2-u_1^2)+(v^2-v_1^2)\right]=Ve$$

and from the conservation of angular momentum in a radial field

$$vr=v_1R$$

where r and R represent the radii of filament and cylinder respectively. From these

$$u_1^2=u^2+v^2(1-r^2/R^2)-2Ve/m$$

so that

$$u^2+v^2(1-r^2/R^2)-2Ve/m\geq 0 \tag{1}$$

is a condition that must be satisfied if the electron is to reach the cylinder. It is not, however, a sufficient condition, for to reach the cylinder the electron must cross all intermediate potential surfaces, and at each such surface a condition similar to (1) must be satisfied. If V_x is the potential at a radial distance x, the electron can cross the cylinder of radius x only if

$$u^2+v^2(1-r^2/x^2)-(2e/m)(V-V_x)>0\ . \tag{2}$$

Or differently expressed

$$V_x>V-(m/2e)\left[u^2+v^2(1-r^2/x^2)\right]\ ; \tag{3}$$

that is, V_x must be greater than the expression on the right if the electron is to cross the cylinder of radius x.

Assuming that the necessary condition (1) is satisfied we obtain by combining (1) and (3)

$$V_x>\left[V-\frac{mu^2}{2e}\right]\frac{r^2(R^2-x^2)}{x^2(R^2-r^2)}$$

as a condition that must be satisfied for all values of x between r and R if the electron is to reach the cylinder.

For the electrons emitted from the filament the least value of u will be zero, which means that all electrons whose velocity components satisfy condition (1) will reach the cylinder if and only if the potential

[1] Schottky, Ann. der Phys. **44,** 1011 (1914); Verh. d. Deutsch. Phys. Ges. **16,** 490 (1914).

at a distance x from the axis is greater than $Vr^2(R^2-x^2)/x^2(R^2-r^2)$. The equation

$$V_x' = V\frac{r^2(R^2-x^2)}{x^2(R^2-r^2)}$$

defines a distribution of critical potentials with the value zero at $x=R$ and V at $x=r$. It thus coincides at these limits with any actual distribution of potential. The condition that Schottky's formula shall be applicable is then that the curve of actual potentials shall lie above this curve of critical potentials except at the ends.

It is important in the first place to find whether this condition is ever satisfied. If the distribution of actual potentials in the absence of space charge should fail to satisfy this condition, Schottky's formula would fail in all circumstances. Without space charge the distribution of actual potentials is given by

$$V_x = V \log (R/x)/\log (R/r)$$

and the ratio of actual potential to critical potential is

$$\frac{V_x}{V_x'} = \frac{x^2(R^2-r^2) \log (R/x)}{r^2(R^2-x^2) \log (R/r)}.$$

It may be shown by means of the expansion

$$\log n = 2\frac{(n-1)}{(n+1)}\left[1 + \frac{1}{3}\left(\frac{n-1}{n+1}\right)^2 + \frac{1}{5}\left(\frac{n-1}{n+1}\right)^4 + \cdot\cdot\cdot\right]$$

that this ratio is greater than unity for all values of x between r and R. The condition is therefore satisfied in this limiting case. It does not follow, however, that the condition can be satisfied with space current flowing, for unless in the absence of space charge

$$\text{(i)} \qquad \frac{dV_x/dx}{dV'_x/dx} > 1 \qquad \text{at } x=R$$

and

$$\text{(ii)} \qquad \frac{dV'_x/dx}{dV_x/dx} > 1 \qquad \text{at } x=r$$

the presence of any space charge whatever will cause the curve of actual potentials to lie below the critical curve through a part of its range.

For the logarithmic distribution $dV_x/dx = -(V/x)\log(R/r)$, and for the critical distribution $dV_x'/dx = -2R^2r^2V/x^3(R^2-r^2)$, these functions satisfy (i) and (ii) for all values of R/r greater than unity. It is therefore possible for the curve of actual potentials to lie above the critical curve with space current flowing, provided this current is not too great.

If V is fixed and i_0 increased from zero by raising the temperature of the filament, the curve of actual potentials moves downward except for its end points. At some critical value of i_0 the curve of actual potentials touches the critical curve and Schottky's formula fails. Since the slope of the critical curve is negative for all values of x including $x=R$, it is clear that this occurs before a minimum has developed in the curve of actual potentials and before the gradient at the cylinder has become zero. The actual condition for failure is thus seen to be more restrictive than ordinarily assumed.

If we compare the gradient of the logarithmic curve at the cylinder with that of the critical curve, we find that

$$\left(\frac{dV_x/dx}{dV_x'/dx}\right)_{x=R}=\frac{R^2-r^2}{2r^2\log(R/r)}$$

and that for $R/r \geqq 30$, which is a requirement if the approximate form of Schottky's formula is to be used, this ratio is 135 or greater. The gradient of the critical curve is thus very small in this region compared to that of the logarithmic. This means that so far as a failure in this region is concerned the condition for failure is not very different from that which has usually been assumed.

It is not certain, however, that the failure does occur in this region. At the filament the ratio of slopes of the critical and logarithmic curves is

$$\left(\frac{dV_x'/dx}{dV_x/dx}\right)_{x=r}=\frac{2R^2}{R^2-r^2}\log\frac{R}{r}$$

and for $R/r \geqq 30$, this has values 6.8 or greater. As these are much smaller than the corresponding ratios at the cylinder there is the definite possibility that the failure may first occur at the filament or even somewhere in the interspace. The question that is involved can be answered, it would seem, only by calculating the form of the potential curve in the general case with space current flowing. It is not proposed in this note to enter upon any such formidable calculations.

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