THE MOTION OF ELECTRONS BETWEEN COAXIAL CYLINDERS UNDER THE INFLUENCE OF CURRENT ALONG THE AXIS

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Abstract

Current from a large electrically heated filament to a coaxial cylindrical anode, limited by the circular magnetic field.—(1) *Motion of the electrons.* Richardson has shown that if the filament current is sufficiently large (or the plate voltage too small), its magnetic field should prevent electrons reaching the anode. The equation for the critical plate voltage is here derived in a simpler and more general form which holds for any degree of space charge. Electrons leaving the filament are deflected in the direction of the electron current in the wire and describe paths which are found, by approximate integration of the equations of motion, to be somewhat elongated cycloids. The dimensions of the paths are calculated for typical cases. No electrons reach the anode unless the plate voltage is above $V_c = 2(e/m)I^2[\log(R/r_0)]^2$ (neglecting small terms which involve the initial velocities). The *effect of space charge* is to elongate the cycloidal paths, but in practical cases the effect is small. The *effect of superposing an axial magnetic field H* is to add to V_c the critical potential $V_c' = \frac{1}{8}(e/m)H^2R^2$ due to the field H alone. This conclusion that the circular and axial magnetic fields act independently was fully verified experimentally. (2) *Plate current as a function of voltage.* Results for tubes with tungsten filaments 3/4 mm and 2.5 mm in diameter are given. The steep part of the curve points in each case to the theoretical critical voltage, but there is a foot extending to lower voltages, due to reflection and scattering of electrons. This foot is much less with a small anode (2.5 cm diam.) than with one four times the diameter. The *reflection effect* was directly proved with a special tube which enabled the current to various sections of the anode to be measured independently. If at a steep part of the curve an *alternating plate voltage is superposed*, the effective alternating current resistance is only about 35 ohms for a tube with 2.5 cm anode. (3) *Plate current as a function of filament current.* If the voltage is kept constant at a high enough value, and the filament current slowly increased, emission begins at a certain temperature and increases rapidly (temperature limited) and then decreases sharply to zero (magnetically limited). With a 2.5 mm filament in a 10 cm anode, for instance, for 800 volts potential the plate current is zero except for the range 130 to 200 amp., while for 1000 volts, it is zero except between 130 and 225 amp. *With an alternating filament current* the plate current is suppressed except during the part of each cycle when the instantaneous filament current is below the critical maximum. Thus the tube gives an intermittent unidirectional current of twice the frequency of the filament current. In a tube with filament 1 cm in diam. carrying 1600 amp. and anode 4.6 cm in diam. at 3100 volts, a plate current (space charge limited) of 45 amp. was completely controlled by the magnetic field of the filament current.

Experiments with Pring tube containing filament and metal disk.—The explanation offered by Langmuir in 1913 that the decrease of electron emission with improvement in vacuum observed by Pring and Parker was due to charge on the glass walls, is verified.

1. Introduction

THE magnetic field of heating current has frequently been mentioned as a factor which should be considered in electron tubes,[1] and the effect to be expected in cases where space charge is negligible has been calculated;[2] but no such effects have ever been observed, so far as I know.[3]

The effect is very small indeed with filaments of ordinary size (cf Tables I and II below), and is masked by the relatively enormous effect of the voltage gradient along the filament, due to the same current.

With very large filaments, on the other hand, the magnetic field of heating current may, as Richardson suggested, play an important role. The experiments here described show that in the case of symmetrical cylindrical electrodes and large heating currents the magnetic field is the most important factor, and effectively prevents electrons from reaching the anode, even with a very high anode potential. The value of heating current necessary to produce this effect is definite for each anode voltage, in agreement with theory.

2. Theory; Concentric Cylindrical Electrodes, Magnetic Field Due to Current Through Inner Cylinder

O. W. Richardson has calculated[1] the critical current necessary to prevent electrons from reaching the anode, for the case of infinitely long cylinders and negligible space-charge. In the analysis here given no restriction is placed upon space-charge or length. Symmetry is assumed, the electrodes being surfaces of revolution; and the leading-in wires to the inner cylinder are assumed straight and parallel to the axis for a sufficient length to insure uniformity of the magnetic field. The solution gives the same value of critical current as that found by Richardson, but in simpler form, and free from restrictions.

Using cylindrical coordinates r, θ, z, we shall let (see Fig. 1) X, 0, Z be the corresponding components of the electric field, and 0, $2I/r$, 0 be the components of the magnetic field.

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[1] O. W. Richardson, Proc. Roy. Soc. A, **90**, 174 (1914); Richardson and Chaudhuri, Phil. Mag. **45**, 337 (1923); H. M. Freeman, Electric J. **19**, 501 (1922); W. H. Eccles, "Continuous Wave Wireless Telegraphy," Wireless Press Ltd., p. 297 (1921)

[2] O. W. Richardson, l.c.[1]

[3] The decrease of electron emission with gas pressure in Pring and Parker's experiments (Pring and Parker, Phil. Mag. **23**, 192, 1912; J. N. Pring, Proc. Roy. Soc. **89**, **344**, 1914) was due to electrostatic effects, and not to the magnetic field of the heating current. These experiments are discussed in detail in section 15 below.

The equations of motion, for an electron starting from a point at distance r_0 from the axis, with initial velocity components u_0, v_0, w_0, are

$$\ddot{r} - r\dot{\theta}^2 = X\frac{e}{m} + 2I\frac{e}{m}\dot{z} \tag{1}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{2}$$

$$\ddot{z} = Z\frac{e}{m} - \frac{2I}{r}\frac{e}{m}\dot{r} \tag{3}$$

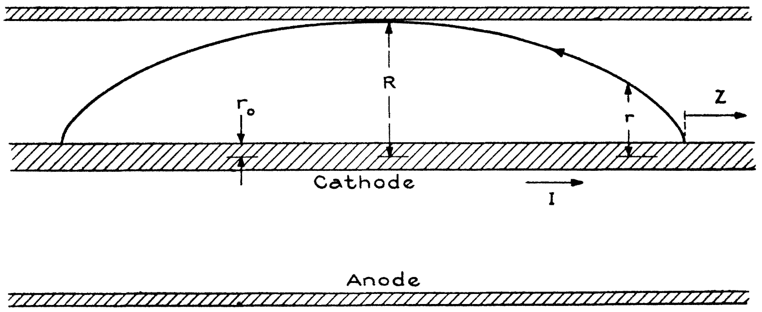

Fig. 1. Cross section sketch of cylindrical tube showing path of electron under the influence of electric field of anode and magnetic field of filament current.

Integration of Eqs. (2) and (3) with respect to time gives, respectively,

$$r^2\dot{\theta} = \text{const.} = r_0 v_0$$

whence

$$\dot{\theta} = r_0 v_0 / r^2 \tag{4}$$

and

$$\dot{z} = w_0 + \int_0^t Z\frac{e}{m}dt - 2I\frac{e}{m}\log\frac{r}{r_0} \tag{5}$$

Substituting these values of $\dot{\theta}$ and $\dot{z}$ in Eq. (1), and integrating with respect to r, gives

$$\dot{r}^2 = u_0^2 + v_0^2\left(1 - \frac{r^2}{r_0^2}\right) + 2\frac{e}{m}\int_{r_0}^{r} X dr + \left(2I\frac{e}{m}\right)^2 \int_{r_0}^{r}\frac{1}{r}\int_0^t Z dt dr$$
$$- \left(2I\frac{e}{m}\log\frac{r}{r_0}\right)^2 + 4I\frac{e}{m}w_0\log\frac{r}{r_0} \cdot \tag{6}$$

The condition that an electron shall be turned back just before it reaches the anode is $\dot{r}=0$ when $r=R$, the radius of anode; also, by definition, $\int_{r_0}^{R} X dr = V$ (potential difference between cathode and anode) and[4]

$$\int_{r_0}^{r} \frac{1}{r} \int_{0}^{t} Z dt dr = \int_{0}^{t} \left(\int_{r_0}^{r} \frac{z}{r} dr \right) dt = 0.$$

Making these substitutions in Eq. (6) and solving for V, one obtains, as the condition that an electron shall just fail to reach the anode,

$$V_c = 2\frac{e}{m}\left(I \log\frac{R}{r_0} - \frac{m w_0}{2e}\right)^2 - \frac{m}{2e}\left(u_0^2 + w_0^2 + v_0^2\left[1 - \frac{r_0^2}{R^2}\right]\right). \tag{7}$$

This equation assumes a simpler form under experimental conditions. The radius r_0 of the cathode, under practical test conditions, is of the order of one tenth that of the anode R, so that the bracket $[1 - r_0^2/R^2]$ which multiplies v_0^2 (itself a small correction term) may be taken as unity. The term in initial velocities then becomes the total initial energy per unit charge of the electron, viz., $\frac{1}{2}(m/e)(u_0^2 + v_0^2 + w_0^2)$. Representing this by its equivalent potential V_0, Eq. (7) becomes, for all practical cases,

$$V_c = 2\frac{e}{m}\left(I \log\frac{R}{r_0} - \frac{m w_0}{2e}\right)^2 - V_0 \tag{8}$$

where V_c = critical potential difference in electromagnetic units.
V_0 = initial velocity of electrons in equivalent potential units.
I = current through filament in e.m.u.
w_0 = component of initial velocity parallel to axis in cm per sec.
R, r_0 = radii of anode and cathode, respectively.

3. APPROXIMATION IN CASE OF LARGE FILAMENTS

For large values of $[I \log(R/r)]$, with correspondingly large values of V' the terms containing the initial velocities w_0 and V_0 are negligible, and Eq. (8) becomes

$$V_c = 2\frac{e}{m} I^2 \left(\log\frac{R}{r_0}\right)^2. \tag{9}$$

Changing to practical units of volts, amperes and the base of logarithms to 10, and inserting $e/m = 1.769 \times 10^7$, this becomes

$$V_c = .01876\, I^2 [\log_{10}(R/r_0)]^2. \tag{10}$$

Since the range of operating temperature for any given type of cathode is very narrow, the normal direct current I necessary to maintain this

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[4] $Zdr = 0$ since dr is at right angles to Z. Likewise, since r is always finite, $(Z/r)dr = 0$

temperature can be expressed in terms of the diameter of the filament. Thus for tungsten filaments[5] at 2500°K, $I_{amp}=1530\ d_{cm}{}^{3/2}$. Eq. (10) can therefore be written

$$V_c \text{ (volts)} = 352{,}000\ [\log_{10}(R/r_0)]^2\ r_0{}^3 \text{ (cm)}. \qquad (11)$$

4. EXAMPLES

Eq. (11) shows that the critical voltage increases as the cube of the cathode diameter, if the ratio of anode to cathode diameter is kept the same. For example, if the anode diameter is 10 times that of the cathode, a tube with a 1 cm diam. cathode will pass no current at voltages below 44,000 volts; a tube with a 1 mm cathode will begin to pass current at 44 volts; and a tube with 1/10 mm (4 mils) diameter cathode will have a critical potential of only .044 volts (approximate, since initial velocities are important in this case). These values are collected in Table I.

TABLE I

Critical voltages for tungsten filaments at 2500°K with direct current heating. Ratio of anode to cathode diameter = 10.

Diameter of anode	Diameter of cathode	Critical voltage
0.1 cm	.01 cm	.044 (approx.)
1.0	0.1	44.
10.0	1.0	44000

The critical voltage increases less rapidly than the cube of the cathode diameter when, as is more nearly practical, the anode diameter is kept constant, instead of the ratio of anode to cathode diameter. A few examples, shown in Table II, will give an idea of the critical voltages for different sizes of filament under these conditions. The values of critical voltage are calculated from Eq. (11), and refer to electrons starting from the cathode with zero initial velocity. The correction to be applied for initial velocities of 1 volt ($u_0=v_0=w_0=\pm 6\times 10^7$cm/sec.) is given after each value of critical voltage. These corrections are calculated from Eq. (8), which, in practical units, takes the form

$$V\text{(volts)} = .01876[I\text{(amp.)}\log_{10}(R/r_0) - 1.23\times 10^{-7}w_0]^2 - V_0\text{(volts)} \qquad (8)$$

TABLE II

Critical voltages for tungsten filaments at 2500°K with direct current heating. Anode diameter = *5 cm*

Diam. of filament	Critical voltage	
0.0025 cm	0.0075	− 1.4
0.025	3.6	± 4.
0.100	127.	± 23.
0.250	1140.	± 70
1.00	21600.	±300.
2.50	62300.	±500.

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[5] Langmuir, Phys. Rev. **7**, 302-30 (1916)

It will be noted that initial velocities of 1 volt affect the critical potential by 1 per cent even at 20,000 volts. This effect is due to the component of initial velocity parallel to the axis of the tube. Eq. (8) shows that the proportional effect of the w_0 term upon the critical potential is (approx.) $2\sqrt{mw_0^2/2e}/\sqrt{V}=2\sqrt{W_0/V}$, where W_0 is the component of initial velocity parallel to the axis in equivalent volts, while the proportional effect of the term representing the total velocity V_0 (in equivalent volts) is V_0/V. Thus when the total initial energy (in volts) is 1 per cent of the critical potential, the effect of the component parallel to the axis will be 20 per cent. This offers a possible method for further study of the tangential components of emission velocities, provided ambiguities due to reflection of electrons from anode and cathode can be avoided.

5. External Cathode

Eq. (7) applies also equally well to electrons traveling inwards from an external cathode to an internal anode, or to ions formed at a distance r_0 from the axis of the tube and traveling either inwards or outwards. When the electrons or ions travel inwards, r_0 in the above equations is to be taken as radius of the outer cylinder from which the electrons start, and R that of the inner cylinder. The numerical value of the ratio R/r_0 is, in this case, the inverse of what it would be if the electrons or ions started from the inner cylinder. The effect of this inversion on Eq. (7) is to change the sign of $\log(R/r_0)$ and to increase the multiplier of v_0^2. Neither of these changes affects appreciably the value of the critical potential; the change in sign of $\log(R/r_0)$ not at all, since negative values of w_0 occur as frequently as positive, and the increased value of the term in v_0^2 only slightly, since it still remains a small correction term. Hence the critical voltage will be the same for internal and external anode, except for the slightly larger correction (still small) for tangential initial velocities noted above.

It is also evident from Eq. (8) and the examples already given, that even the lightest ions will be entirely unaffected by the magnetic field of any attainable filament currents.

6. Combined Circular and Axial Magnetic Fields

It is easy to calculate the effect of an externally impressed uniform magnetic field H, parallel to the filament, superimposed upon the internally impressed field $2I/r$ due to the heating current. The electrodes are assumed as before to be concentric cylinders of revolution. Using cylindrical coordinates as before, the equations of motion are

$$\ddot{r}-r\dot{\theta}^2=X\frac{e}{m}-H\frac{e}{m}r\dot{\theta}+2I\frac{e}{m}\ \frac{\dot{z}}{r} \tag{12}$$

$$\frac{1}{r}\ \frac{d}{dt}(r^2\dot{\theta})=H\frac{e}{m}\dot{r} \tag{13}$$

$$\ddot{z}=Z\frac{e}{m}-2I\frac{e}{m}\ \frac{\dot{r}}{r} \tag{14}$$

where X, Z, are the components of electric field in the r and z directions respectively.

Integrating Eqs. (13) and (14) with respect to time, substituting the resulting values of the z in Eq. (12), and integrating this with respect to r, gives

$$r^2=2\frac{e}{m}V-\left(r_0v_0-r_0^2\frac{He}{2m}\right)^2\left(\frac{1}{r^2}-\frac{1}{r_0^2}\right)-\left(\frac{He}{2\,m}\right)^2(r^2-r_0^2)$$
$$-\left(2I\frac{e}{m}\log\frac{r}{r_0}\right)^2+4I\frac{e}{m}\log\frac{r}{r_0}w_0+u_0^2. \tag{15}$$

Placing, as before, $\dot{r}=0$ when $r=R$ (radius of anode), as the condition that electrons shall just be able to reach the anode, and collecting terms, the critical voltage V_c is found to be

$$V_c=H^2\frac{e}{8m}R^2\left(1-\frac{r_0^2}{R^2}\right)+H\frac{r_0v_0}{2}\left(1-\frac{r_0^2}{R^2}\right)+2\frac{e}{m}\left(I\log\frac{R}{r_0}-\frac{mw_0}{2e}\right)^2$$
$$-\frac{m}{2e}\left(u_0^2+w_0^2+v_0^2\left[1-\frac{r_0^2}{R^2}\right]\right). \tag{16}$$

If V_c is of the order of 20,000 volts, all the terms containing initial velocities are negligible,[6] and Eq. (16) reduces to the very simple form

$$V_c=\frac{e}{m}\left[\frac{H^2R^2}{8}+2I^2\left(\log\frac{R}{r_0}\right)^2\right]. \tag{17}$$

Changing to practical units of volts, amperes, and inserting $e/m=1.77\times 10^7$,

$$V_c=.0221H^2R^2+.0188I^2[\log_{10}(R/r_0)]^2. \tag{18}$$

V_c is the critical voltage, i.e. the voltage that is just sufficient to drag electrons from the cathode of radius r_0 to the anode of radius R, in the presence of the magnetic field due to I amperes flowing through the cathode combined with H gauss impressed parallel to the cathode.

It is to be noted that this critical voltage for the combined action of the two magnetic fields is the sum of the critical voltages of the respective

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[6] See examples above and Phys. Rev. **18**, 34 (1921)

fields when they act separately [cf Eq. (10) above and Eq. (8), Phys. Rev. **18**, 34 (1921)].

7. PATHS OF THE ELECTRONS WHEN SPACE CHARGE IS NEGLIGIBLE

The differential equation of the path may be obtained at once from Eqs. (5) and (6) above. Neglecting initial velocities, and considering the central portion of the tube, where $Z=0$, these equations give

$$\dot{z}^2 = [2I(e/m)\log(r/r_0)]^2$$

and

$$\dot{r}^2 = 2(e/m)V_r - [2I(e/m)\log(r/r_0)]^2$$

whence

$$\frac{dr}{dz} = \sqrt{\frac{V_r}{2I^2(e/m)\,[\log(r/r_0)]^2} - 1}. \tag{19}$$

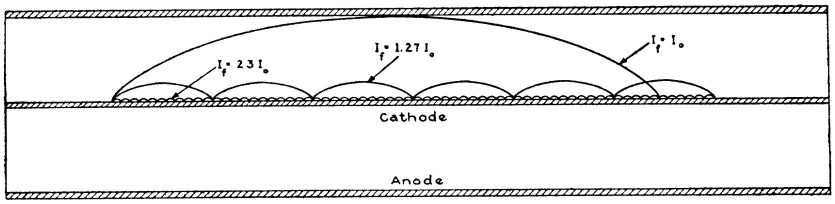

Fig. 2. Cross section of tube with 40 to 1 ratio of anode to cathode diameter, showing paths of electrons for three different values of current through the cathode, viz., critical current I_0, just sufficient to prevent electrons reaching anode; 1.27 I_0; and 2.3 I_0. In the last case the maximum radial excursion of the electrons is only 1/40 the distance from cathode to anode.

The path will therefore depend upon the degree of space-charge, which determines the potential distribution V_r.

8. *If space-charge is negligible*, the potential V_r is given by the electrostatic equation for concentric cylinders

$$V_r = [V_R/\log(R/r)]\log(r/r_0).$$

Combining this with Eq. (9) gives

$$V_r = 2I^2(e/m)\log(R/r_0)\log(r/r_0). \tag{20}$$

In this R and V_R represent the radius and potential of the imaginary cylindrical surface at which the electron turns back. It may be any point between cathode and anode, depending upon the value of I.

Substituting V_r from (20) in Eq. (19) one obtains the very simple equation for the path of an electron, of zero initial velocity, moving in charge-free space between concentric cylinders of radii r_0 and R

$$dz = dr\sqrt{\log(r/r_0)/\log(R/r_0)}. \tag{21}$$

Eq. (21) is not directly integrable, but approximate solutions are easily obtained for given values of (R/r_0). Figs. 2 and 3 show typical paths, corresponding to $R/r_0=2$, 10, and 40; and Table III gives the distance traveled along the axis per jump, for a series of values of (R/r_0).[7]

TABLE III

Calculated distances along axis traveled by electrons, before returning to cathode, assuming negligible space-charge.

r_0 = radius of cathode; R = radius of imaginary cylinder to which path is tangent;
z_0, z_1 = successive values of z (measured along axis)
at which electron returns to cathode.

Ratio of radii R/r_0	Distance traveled along cathode. z_1-z_0	$\frac{z_1-z_0}{R-r_0}$
1.01	$.0314\times r_0$	3.14
1.1	.322	3.22
1.2	.657	3.29
1.3	1.005	3.34
1.4	1.360	3.40
1.5	1.720	3.44
2.0	3.716	3.72
3.0	8.154	4.08
5.0	18.16	4.54
10.0	46.60	5.11
20.0	112.4	5.92
40.0	251.2	6.42

It will be noted from Figs. 2 and 3 that the paths are of the general form of cycloids. For small ratios of (R/r_0) they are exactly cycloidal, as may easily be verified by replacing the logarithms in Eq. (21) by the first terms in their series expansions. This condition is very nearly satisfied for the smallest paths shown in Fig. 2. For larger values of R/r_0 the distance

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[7] These values were obtained as follows.

For ratios of R/r_0 less than 1.5, Eq. (21) was put in the form

$$dz=Re^x(x-a)dx/\sqrt{ax-x^2};$$

where $x=\log(r/R)$; $a=\log(r_0/R)$.

Series expansion of e^x and term by term integration gave

$$z=Ra\sin^{-1}(1-2x/a)\left\{1-\tfrac{1}{2}(1-a)-(3a/8)\,(1-\tfrac{1}{2}a)-(5a^2/16)(\tfrac{1}{2}-\tfrac{1}{6}a)+\,.\;\;.\;\right\}$$
$$+R\sqrt{ax-x^2}\left\{(1-a)+(1-\tfrac{1}{2}a)\,(\tfrac{1}{2}x+3a/4)+(\tfrac{1}{2}-\tfrac{1}{6}a)\,(\tfrac{1}{3}x^2+5ax/12+5a^2/8)+\right\}$$

This gives results accurate to .1 per cent for $R/r_0\leq 1.5$.

For ratios of R/r_0 greater than 1.5, the integration was performed in two stages. The first part of the path, for $r_0<r<.9R$, was obtained from step by step integration of Eq. (21), using Simpson's rule. The remainder of the path cannot be obtained in this way, since the function becomes infinite at $r=R$. It was obtained from the approximate equation

$$z=2R\sqrt{a}\left\{x-\tfrac{1}{3}x^3(1+1/2a)+\tfrac{1}{5}x^5(\tfrac{1}{2}+1/2a-1/8a^2)-\tfrac{1}{7}x^7(\tfrac{1}{6}+1/4a-1/8a^2+1/16a^3)+..\right\}$$

This equation is the integral of the series expansion of Eq. (21), which takes the form $dz=-2Re^{-x^2}\sqrt{a-x^2}$, upon substitution of $\log(R/r)=x^2$; $\log(R/r_0)=a$. It gives accurate results for $.7R<r<R$, for all values of R/r_0 greater than 1.5.

between intercepts becomes progressively greater than that of the cycloid, being approximately twice as great at $R/r_0=40$. (cf Table III and Fig. 2).

The most noteworthy feature in Fig. 2 is the rapid shrinking of the path with increase in heating current above the critical value. An increase in heating current of 27 percent above the critical value decreases the maximum excursion to 1/4, and for a heating current only slightly more than double the critical value the excursion is only 1/40 of the critical excursion. The shrinking is even greater when space charge is present, the maximum radial excursions corresponding to $I/I_0=1.27$ and 2.3 respectively being approximately 1/14 and 1/60 of the distance from cathode to anode (see next section).

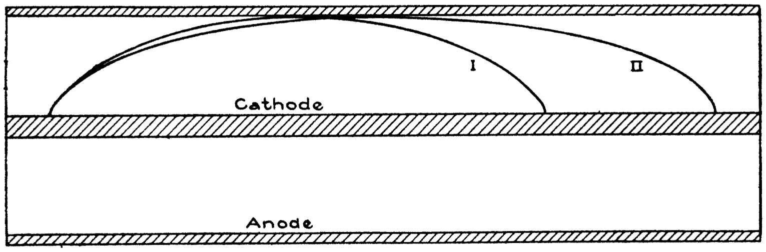

Fig. 3. Cross section of tube with 10 to 1 ratio of anode to cathode diameter, showing path of electron when just prevented from reaching anode by magnetic field of heating current. Curve I represents the path when space-charge is negligible (small emission from cathode), curve II when maximum space charge is present. The actual path will always be intermediate between these two, generally near curve I.

9. PATHS OF ELECTRONS WHEN SPACE-CHARGE IS PRESENT

The effect of the space charge is to weaken the electric field near the cathode, and hence to lengthen the path in the Z direction (parallel to axis). With maximum space-charge the field at the cathode is zero, and the path has its longest possible value. This condition seldom exists, however, since in all practical cases the degree of space-charge is far below the maximum that the space can hold under cut-off conditions, with full line voltage between cathode and anode. An exact solution of the path under maximum space-charge is therefore of little interest. An approximate solution, which is easily obtained, will serve satisfactorily as an upper limit to the length of path. The actual path will in most cases be nearer to the lower limit, which is the zero space charge case already discussed.

The approximate solution assumes the space-charge distribution given by Langmuir's equation[8] (for zero magnetic field)

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[8] Langmuir, Phys. Rev. **2**, 450 (1913)

$$i=\frac{2\sqrt{2}}{9}\sqrt{\frac{e}{m}}\,\frac{V^{3/2}}{\beta^2 r}\,.$$

This distribution differs but slightly from the actual maximum space-charge in the magnetic field. To the same degree of approximation the factor β^2 may be taken as unity. This gives for the potential at any point r

$$V_r=V_R(r/R)^{2/3}. \tag{22}$$

Substituting this value of V_r in Eq. (19) one obtains, for the path of an electron of zero initial velocity moving between concentric cylinders of radii r_0 and R in the presence of maximum space-charge

$$dz=\frac{\log(r/r_0)dr}{\sqrt{(r/R)^{2/3}[\log(R/r_0)]^2-[\log(r/r_0)]^2}}\,. \tag{23}$$

This equation may be integrated in the same manner as Eq. (21), viz., step by step integration from $r=r_0$ to $r=0.9R$, and series expansion with direct integration from $.9R$ to R.[9] Fig. 3 shows the path calculated in this way, for a ratio R/r_0 of anode to cathode diameter equal to 10. It is almost identical in shape with the path for zero space charge, except that it is stretched 30 per cent in the z (axial) direction.

The actual path in all cases will lie between these two limiting paths, given by curves I and II in Fig. 3. Under practical conditions, the actual space charge will be only a small fraction, of the order of 1 per cent, of the maximum, so that the path will in most cases be very close to that represented by curve I.

The small relative space-charge is due to the combination of high voltage and small electrode spacing, requiring very large currents for full space-charge, and to the fact that most of the turned-back electrons are lost. The electrons hop up the filament, from negation to positive end, in the same direction as the conduction electrons in the filament. Hence upon their return to the filament they strike it with a velocity corresponding to the voltage-drop in the section of filament jumped, generally a considerable fraction of a volt, and most of them enter. This is in contrast to the action in the case of a magnetic field parallel to the filament, where the returning electrons strike the filament with only

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[9] In order to obtain a rapidly convergent series the independent variable is changed to $x=\log(R/r)$. Eq. (23) then becomes $dz=-R(a-x)e^{-x}dx/\sqrt{a^2e^{-2x/3}-(a-x)^2}$, where $a=\log(R/r_0)$. Expansion of the right hand side of this equation in powers of x and integration gives

$$z=\frac{R}{c}\left\{2ax^{1/2}-(2/3)x^{3/2}(1+a-ka/2)+(2/5)x^{5/2}[1+a/2-k/2(1+a)+3k^2a/8]+\cdots\right\}$$

where $c=\sqrt{2a-(2/3)a^2}$; $k=1-(2/9)a^2(1-2x/9)/(2a-2/3a^2)$. This equation may be used from $r=R$ down to $r=.7R$

their emission velocities, or miss it entirely if it is small, so that full space charge eventually builds up even when the emission is quite small.

The degree of true reflection (without appreciable loss of energy) may be tested by its effect on the sharpness of cut-off. The electrons which just miss the anode will travel a considerable fraction of the length of the tube before returning to the cathode (cf Fig. 3). If then they are reflected without loss of energy they will emerge with initial energies of the order of magnitude of the whole potential drop in the cathode. Initial velocities of this order will have an enormous effect on the cut-off voltage (cf Eq. (8) and examples in Table II). The existence of such an effect was tested by the use of an equipotential cathode, which consisted of a nickel cylinder 1 cm in diameter, coated on the outside with barium oxide, with an insulated molybdenum core through which the heating current could be passed. The anode was a water-cooled copper tube of 1 inch inside diameter. No difference in volt-ampere characteristic could be detected when the potential drop in the cathode was changed from zero to that of the molybdenum core, indicating that reflection was not an important factor in this tube. The difference between anode and cathode diameter was so small in this tube that the paths were short and the voltage drop in the filament per path was only of the order of the Maxwell initial velocities. A more sensitive test with a larger diameter tube, divided into sections, is described below, and gives definite evidence of reflection.

EXPERIMENTAL TESTS

10. *Tests with small filaments.* Filaments less than 20 mils ($\frac{1}{2}$ mm) diameter showed no measurable effect of magnetic field. The small effect that undoubtedly was present (cf Tables I and II) was obscured by the much greater effect of the voltage gradient due to the same current and inseparable from the magnetic field. With a 30 mil ($\frac{3}{4}$ mm) diameter filament and a large anode (10 cm diam.), the magnetic effect is well marked. Fig. 4 shows volt-ampere characteristics of such a tube, for two different values of filament current. Both characteristics differ from the theoretical space-charge characteristic shown in curve I, by an amount much greater than the voltage drop in the filament (13 and 21 volts respectively). The theoretical voltage at which measurable current should begin to flow to the anode, with 36 amperes filament current, as given by Eq. (8) with Maxwell's distribution of initial velocities, is shown at E_0. This agrees with the experimental value as closely as is to be expected, in view of the filament drop of 21 volts and a small "tail," always observed, due to electrons deflected by collision with gas molecules. The

lack of steepness of the curve above this voltage is due mainly to the shortness of the tube (see tests in section 11).

Fig. 5 shows a similar characteristic with a larger filament of 100 mils (2.5 mm) diam. in the same anode (10 cm diam. by 30 cm long). The characteristic for "zero heating current" was taken with the filament switch open, hence no magnetic field. With slight initial overheating of the filament there was ample time to read plate-current before the filament cooled down to normal temperature. This is the ordinary 3/2 power space-charge characteristic. The theoretical critical voltage, taking into account Maxwell initial velocities, is shown at E_0.

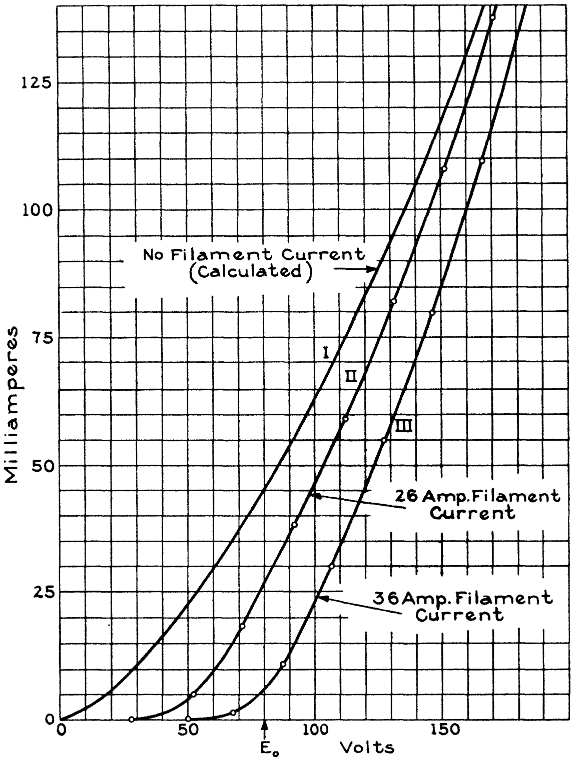

Fig. 4. Volt ampere characteristic of tube containing a 30 mil (.75 mm) filament in the axis of an anode 10 cm in diameter and 30 cm long. The calculated critical voltage for curve III is shown at E_0.

The effect of magnetic field is more striking when the filament current is varied, keeping anode voltage constant. Fig. 6 shows such a characteristic, and Fig. 7 a series of characteristics for different anode voltages, taken with the same tube as Fig. 5. As the filament temperature is slowly raised, no emission occurs until the heating current reaches 130 amp. This is due to temperature limitation. From this point on the emisson rises rapidly according to Richardson's equation, entirely unaffected by magnetic field, to a definite maximum, e.g. 2 amp. at 1600 volts. With further increase of heating current the emission falls sharply and rapidly to zero. In this falling region the emission is controlled entirely by magnetic field, and is independent of temperature. Space-charge has no effect upon any part of the characteristic at 1600 volts, but becomes a factor at higher voltages, limiting the maximum plate current to approximately 10 amperes at 2200 volts.

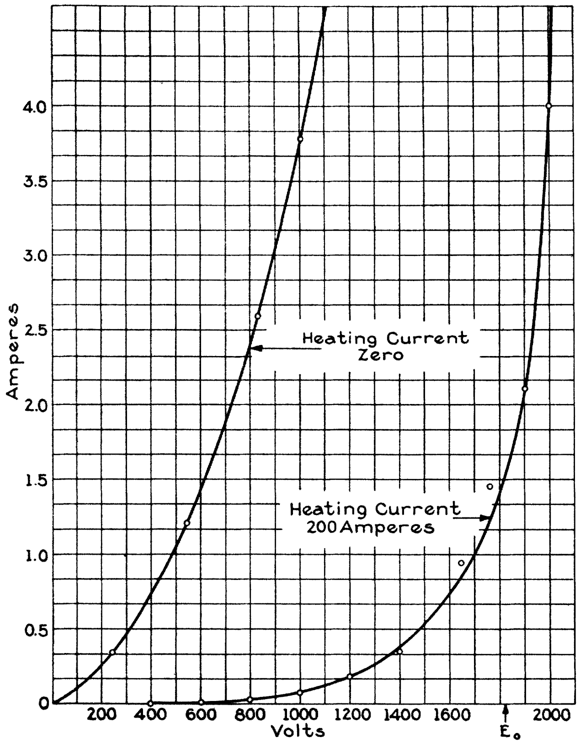

Fig. 5. Volt ampere characteristic of tube containing a 2.5 mm filament in axis of 10 cm diameter cylindrical anode. The calculated critical voltage for 200 amp. heating current is shown at E_0.

There is thus a definite range of filament current, narrower the lower the voltage, in which the tube will pass electron current. Below 700 volts no emission can be obtained at any steady filament current.

If the filament current is varied rapidly, as with alternating current, so that the filament remains at constant high temperature, the emission is governed either by the instantaneous value of the heating current, or by space charge, according to which gives the lower value of emission. Fig. 8 shows a series of oscillograms taken in this way, with the same tube as Fig. 5, 6, and 7. The anode voltage was maintained constant. The lower curve B in each film is filament current, the middle curve A emission current from filament to anode. It will be noted that the emission is zero whenever the instantaneous value of filament current is above a critical value. Full space-charge current flows during that part of each cycle in which the heating current is below the critical value, and no current during the part when the field is above the critical value.

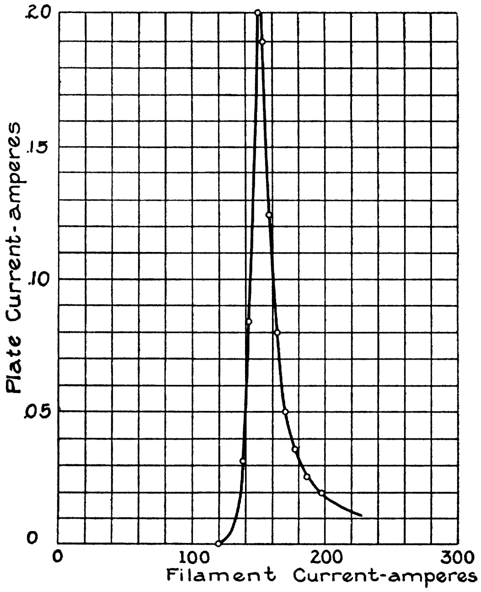

Fig. 6. Variation of plate current with filament current in a tube with 2.5 mm filament and 10 cm diameter anode at 800 volts.

The middle film was taken with a constant potential of 250 volts on the anode. During the greater part of each cycle the filament current is above the critical value for 250 volts, and the plate current is zero, as it should be. When the filament current falls to the value 73 amperes, plate current begins and increases rapidly to the maximum value allowed by space-charge. As soon as the filament current again reaches the same value of 73 amperes in the opposite direction, the plate current begins to fall, and drops rapidly to zero. The plate current is thus interrupted twice each cycle, since both directions of filament current are alike in their magnetostrictive effect.

The second (lower) film was taken with 505 volts on the plate. The space current is larger and flows for a longer part of each cycle, beginning and ending when the filament current reaches the value 101 amperes. The critical value given by Eq. (2) is 102 amperes. The filament current in this test (184 r.m.s. amperes) was larger than that used for the first film, and the lower curve shows distortion due to saturation in the filament transformer.

For the upper film a plate voltage of 1050 was used, and a larger filament current (201 amperes). The filament current curve shows bad saturation distortion, so that the distance from the point where it crosses the axis to the point where it reaches its critical value is very different above and below the axis. In spite of this, the plate current begins and ends accurately at the critical filament current of 157 amperes, which agrees fairly well with the theoretical value of 149 amperes.

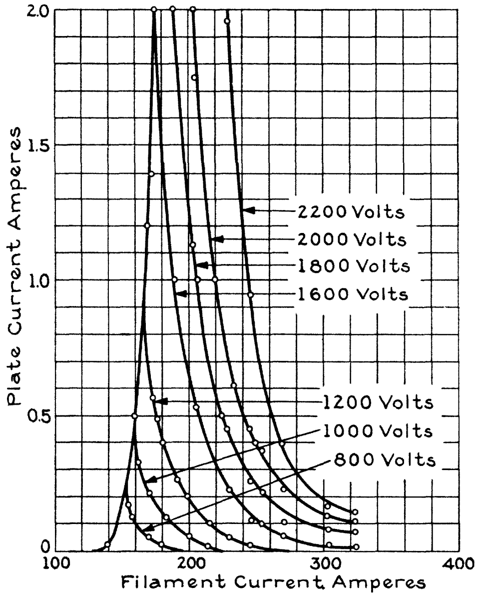

Fig. 7. Plate current-filament current characteristics of tube with 2.5 mm cathode and 10 cm diameter anode, at different (constant) anode potentials.

11. *Reflection of electrons at the cathode.* The "transition" current in Fig. 5, that is, the current at voltages below E_0, is larger than can be accounted for by gas collisions or lack of symmetry. It was suspected that this current was due to electrons reflected at the filament, which in this case, on account of the large tube diameter and consequent long paths (cf Fig. 2, which is drawn to scale for this tube) might have initial

velocities of the order of 10 volts. This hypothesis was tested by the following experiment. A tube was constructed with a 5 cm diameter anode divided into 3 sections, viz., a central section 15 cm, and two end sections each 5 cm long (Fig. 9). The cathode was a 100 mil (2.5 mm) tungsten wire, the same as in the previous test. All three anodes were maintained at the same voltage, and the current to each measured. Fig. 10 shows these currents as a function of voltage. Curves I, II, and III refer to the currents to sections 1, 2, 3, Fig. 9. It is seen that the current to section 1 (curve I) begins at approximately the correct voltage, but rises very slowly due to the fact that this section receives electrons from

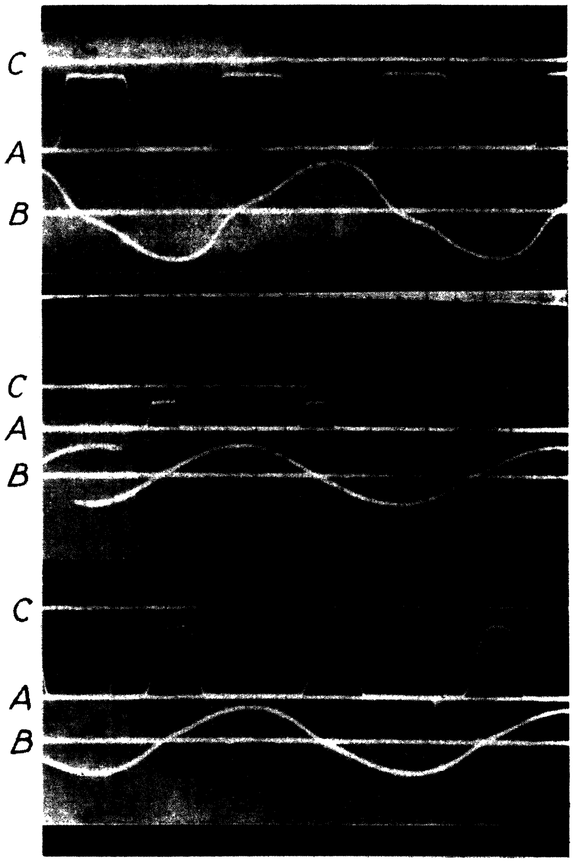

Fig. 8. Three oscillograms showing control of plate current by filament current, at constant plate voltage, in a tube with 2.5 mm filament and 10 cm diameter anode. In each film the lower curve B is filament current, and the middle curve A plate current. The upper curve (single line) is a calibration for the plate current curve.

only a tiny portion of the filament, at its very end. With increasing voltage, it receives current from an increasing portion, up to 1/5 of the filament at very high voltage (paths normal to cathode).

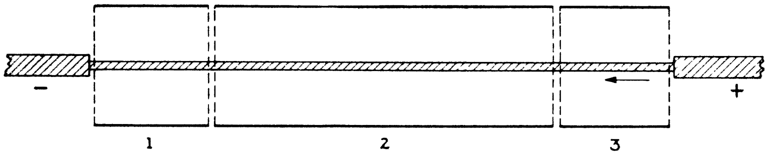

Fig. 9. Cross-section of special tube with anode divided in 3 sections. for testing reflection of electrons from the cathode.

The current to the middle section (curve II) begins at a voltage 200 volts lower than the theoretical critical value, and is nearly half an ampere at the critical voltage. From this point on it rises very rapidly.

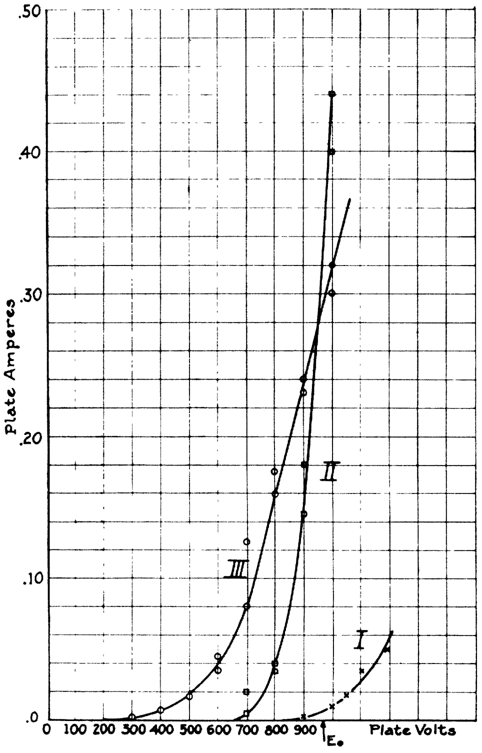

Fig. 10. Currents to three sections of anode shown in Fig. 9, as function of voltage.

This section receives primary electrons from approximately 3/5 of the filament, and in addition some electrons which have completed one path and have been reflected. Those electrons which retain at reflection the full energy of their first flight, about 4 volts,[10] and emerge nearly parallel to the axis, should be able to reach the anode at 900 volts, according to Eq. (8). This leaves 150 milli-amperes to be accounted for by gas collisions—somewhat high, but possible, since the vacuum was not measured.

The current to section 3 begins at about 200 volts. This is too low to be accounted for by either gas collisions or reflection from the filament. It is accounted for, however, by reflection from the filament lead, which was a 3/8 inch diameter molybdenum rod, concentric with the filament. If the electrons impinging upon this rod were reflected, as they must be to a slight extent, they could reach the anode at 320 volts, according to Eq. (8), even if their initial energy at reflection were zero, and at 250 volts if their initial energy were 4 volts.

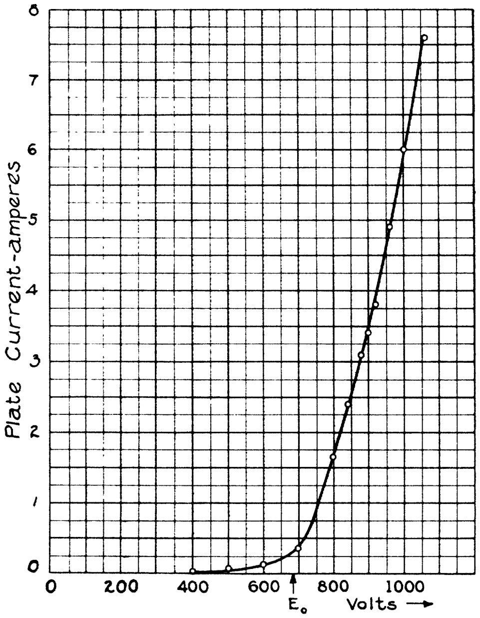

Fig. 11. Volt-ampere characteristic of tube with 2.5 mm filament and 2.5 cm diameter anode.

The shape of the characteristics in Fig. 5 and 10 is thus satisfactorily explained by reflection. It is quite possible, however, that there are other factors present, such as high frequency (Barkhausen) oscillations,[11] which have not been investigated.

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[10] This is the voltage drop in the length of filament (15 cm) traversed in the first flight. The returning electron strikes the filament at a point 15 cm nearer the positive end than the point at which it left.

[11] Barkhausen and Kurz, Phys. Zeits. **21**, 1 (1920)

12. *Experiments with anodes of small diameter.* The effects of reflection may be made negligible by using tubes of small diameter, so that the paths are short. On the other hand, it is difficult to maintain the same relative degree of symmetry in small tubes. Fig. 11 shows the characteristic of a tube with a 2.5 mm diameter filament and a 2.5 cm diameter copper anode, water cooled.

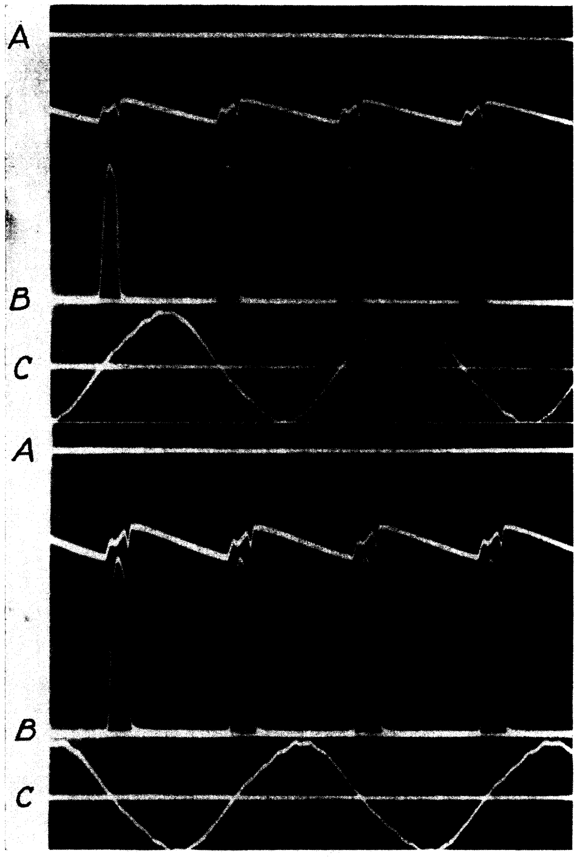

Fig. 12. Oscillogram showing plate current in a tube with 1 cm diameter tungsten cathode and 4.6 cm diameter anode, as function of instantaneous value of filament current. The lower curve C in each film is filament current, the middle curve B plate current, and the upper curve A plate voltage (deflection down).

The critical voltage is much lower than in Figs. 5 and 6, as predicted by Eq. (8) for smaller anode diameter, but is well defined, and the transition currents are not greater than are to be expected from the slight asymmetry and residual gas. The low alternating-current resistance of only 35 ohms (the inverse slope of the curve at 1000 volts) is especially to be noted. This is about a hundred times smaller than the resistance of similar pliotrons at the same voltage, and is the main point of difference between magnetically and electrostatically controlled pure electron currents.

Fig. 13 shows the characteristic of a tube with a 2.5 cm diameter anode and a 1/2 cm diameter oxide-coated nickel cathode. In order to increase its mechanical strength at high temperature this cathode was supplied with a molybdenum core. The point to be noted is that the characteristic is considerably better than in Fig. 5, in spite of lack of uniformity in the spacing (only 4 mm) between anode and cathode.

13. *Experiments with larger filaments.* A series of tests with larger filaments is being carried out in this laboratory by Mr. J. H. Payne. Fig. 12 shows two typical oscillograms taken by Mr. Payne with one of these tubes. The filament was a tungsten rod 1 cm in diameter and 56 cm long, the anode a copper tube of 4.6 cm internal diameter. The filament was heated with 60 cycle alternating current, and the anode potential maintained as nearly constant as possible.

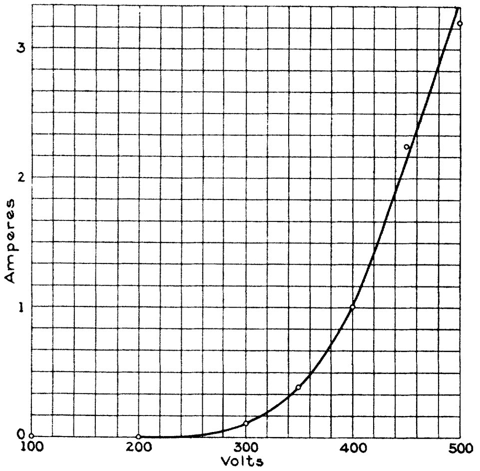

Fig. 13. Volt-ampere characteristic of tube with 2.5 cm diameter anode and 1/2 cm oxide-coated nickel filament.

In film 1 (above) the average potential was 1700 volts, which is recorded by the upper curve (base line A, deflection downwards). This potential was furnished by a 3000 volt d.c. generator, with a 50μf condenser across its terminals. The linear portions of the curve show the charging rate of this condenser, while the wavy portions are due to the rapid rate of discharge (36 amperes). The lower curve C gives the filament current (1600 amperes, 60 cycles) and the middle curve B the plate current. It is seen that the plate current is completely suppressed except during the short fraction of each cycle when the filament current is below a definite critical value. During these short periods, twice each cycle, it rises to 36 amperes (space-charge limit), and then falls abruptly to zero as the filament current again reaches the critical value.

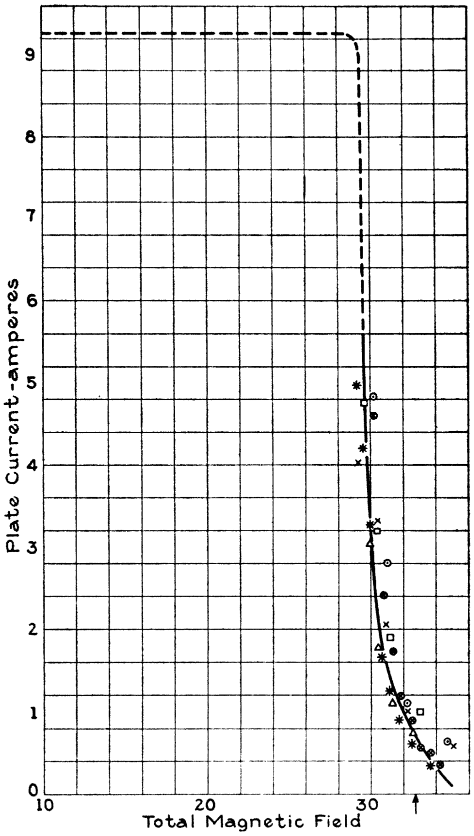

Fig. 14. Plate current-magnetic field plot of seven tests with combined radial and axial fields. Each test (denoted by a particular symbol in the plot) was made with a different proportion of axial field.

In film 2, below, the average anode potential was 3100 volts. Corresponding to this higher potential, the critical value of filament current was larger, so that the electron current flowed for a slightly longer fraction of each cycle, and attained the higher (space-charge limited) value of 45 amperes.

The values of critical filament current and voltage shown by these oscillograms agree completely with those predicted by Eq. (10).

14. *Tests of combined external and internal control.* It is predicted by Eq. (18), section 6, that under the combined influence of an externally impressed axial magnetic field, and an internally impressed circular field, the critical voltage should be the sum of the critical voltages that would be required for each of these fields separately. This prediction was tested over the whole range of combined fields from 100 per cent axial to 100 per cent circular field, and found to hold quite accurately. In Fig. 14 are plotted the results of 7 such tests for different field combinations from entirely axial to entirely circular field, for a tube with 2.5 cm diameter anode and 2.5 mm diameter tungsten filament. The anode potential was maintained constant at 1000 volts, and the total magnetic field varied by varying either filament current or solenoid current, or both. The effective magnetic field is taken, according to Eq. (16), as

$$\sqrt{[.0221\ H^2R^2 + .0188(I - 1.22\times 10^{-7}\ w_0)^2]},$$

correction being made for the initial velocities w_0 in accordance with Maxwell's distribution law. The plot of plate current against effective magnetic field should give identical curves for all these tests. The experimental points obtained in the different tests are indicated by different symbols. It is seen that they fall on a common curve within experimental error. The fact that the cutoff occurs at a value slightly lower than the theoretical (indicated by the arrow) is due to imperfect centering of the filament in the small anode.

15. *Pring and Parker's experiments.* In 1912 Pring and Parker[3] reported a series of experiments with large carbon filaments, in which the current from the filament to a small metal disk decreased rapidly as the gas pressure was reduced, reaching values of the order of 10^{-12} amperes for moderate temperatures, and 10^{-9} amperes for the highest temperatures. The authors interpreted this as evidence that thermionic emission from carbon is due to chemical action between the carbon and gas, and would be zero in a perfect vacuum. O. W. Richardson,[1] defending the electron emission theory, suggested that the electrons might have been deflected by the magnetic field of the filament current, and showed that the filament current was of the right order of magnitude to prevent electrons reaching a symmetrical cylindrical anode of radius equal to the distance between disk and filament. Langmuir[12] suggested that the effect was due to electrostatic charges on the glass walls, and cited experiments in well exhausted lamps which showed exactly similar effects.

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[12] Langmuir, Phys. Rev. **2**, 484 (1913)

While the nature of thermionic emission is no longer in question, it seemed desirable to determine experimentally the relative importance of space charge and magnetic field in tubes of the type used by Pring and Parker (and by many previous investigators). This is easily accomplished by observing the current from filament to disk, first with the filament current flowing, and then with the filament circuit open but the filament still hot. A tube was constructed, as nearly as possible of the same dimensions as Pring and Parker's (see Fig. 15, copied from Dr. Pring's paper) but with a tungsten filament instead of carbon. This filament, 2.5 mm in diameter and 10 cm long, was mounted axially in a 17 cm bulb. The anode was a 1.2 cm disk of molybdenum, 7.25 cm from the filament. The distance from the disk to the glass wall was 1.5 cm.

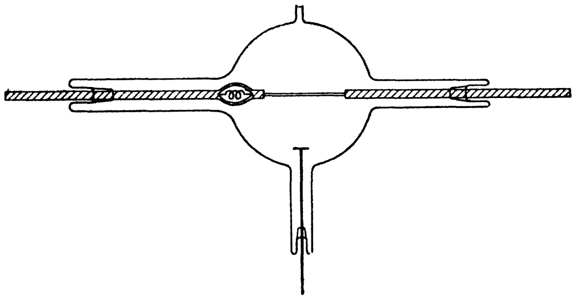

Fig. 15. Cross section of Pring's tube.

A current of 200 amp. heated the filament to 2500°K. This current was allowed to flow for 15 sec. (to give approximately constant temperature) and the plate current was read. The filament switch was then opened and the plate current again read, before the filament cooled appreciably. In order to make sure that there was no temperature limitation, the tests were repeated at 250, 300 and eventually 400 amp. At the latter temperature the time of heating was gradually increased until the filament melted at 4.5 seconds. At the lowest temperature the filament was capable of emitting 10 amp. and at the highest temperatures over 1000 amp.; *yet the current to the plate was less than* 10^{-9} *amperes, both with filament current flowing and not flowing.* The plate voltage was 250, and the vacuum good, of the order of 10^{-5} bars. It is evident that the smallness of the plate current was not due to lack of thermionic emission. It is also evident that under these conditions, which correspond closely to those used by Pring and Parker, it was not due to filament current, since the plate current was essentially zero even where no filament current was flowing.

The obvious explanation is charging up of the glass walls, as suggested by Langmuir. This was proved by the following experiment. A tungsten wire was sealed through the glass wall of the tube just described and half imbedded in the inner surface so as to make contact with a thin coating of tungsten deposited by evaporation from the filament.[13] The walls could thus be charged to any desired potential, and the current from filament to plate measured for each potential. The results, given in Table IV, show (1) that the current to the disk decreased uniformly with increasing negative potential of the glass, and became zero in the absence of magnetic field at −21 volts. Electrons with initial velocities of 21 volts, though not to be expected from Maxwell's distribution law, are nearly always present in measurable quantity in 3 electrode tubes under similar conditions,[11] so that the absence of current in the previous experiment, and in the similar low gas pressure experiments of Pring and Parker, is fully accounted for by the potential of the glass. The limitation of the current is due to space-charge. The double function of gas ions (poor vacuum) in discharging the glass and annihilating space-charge, and so allowing more current to the disk, has been fully described by Langmuir.[12] (2) The results also show that when the glass is positive enough to allow current to flow to the disk, this current is greatly decreased by the magnetic field of the filament current. Further tests with the glass coating insulated, but with the disk much nearer (3.6 cm) to the filament, so that the shielding effect of the glass was less, gave 1 milliamp. to the disk at 250 volts with no filament current, and .075 milliamp. with filament current of 88 amp. Hence the magnetic effect suggested by Richardson is verified, though it must have been masked, under the conditions of

TABLE IV

Current in Pring and Parker tube, as function of potential of glass.
Gas pressure 7 bars He; anode potential 250 volts.

Potential of glass	Current from filament to disk (1) with no filament current	(2) with 67 amp. filament current
0 volts	550 μ amp.	8.0 μ amp.
−3	475	5.0
−6	350	2.0
−9	225	.75
−12	50	.15
−15	4	.00
−18	0.2	.00
−21	0.00	.00

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[13] To obtain contact it was necessary to reduce the oxide on the wire, which was accomplished by filling the bulb with low pressure hydrogen, and dissociating this by the hot filament. Atomic hydrogen reduces WO_3 at room temperature.

Pring and Parker's experiments, by the much greater effect of space-charge.

I gratefully acknowledge the able assistance of Mr. F. R. Elder in the experimental part of this work.

Research Laboratory,
General Electric Co.,
Schenectady, N. Y.
January 14, 1925

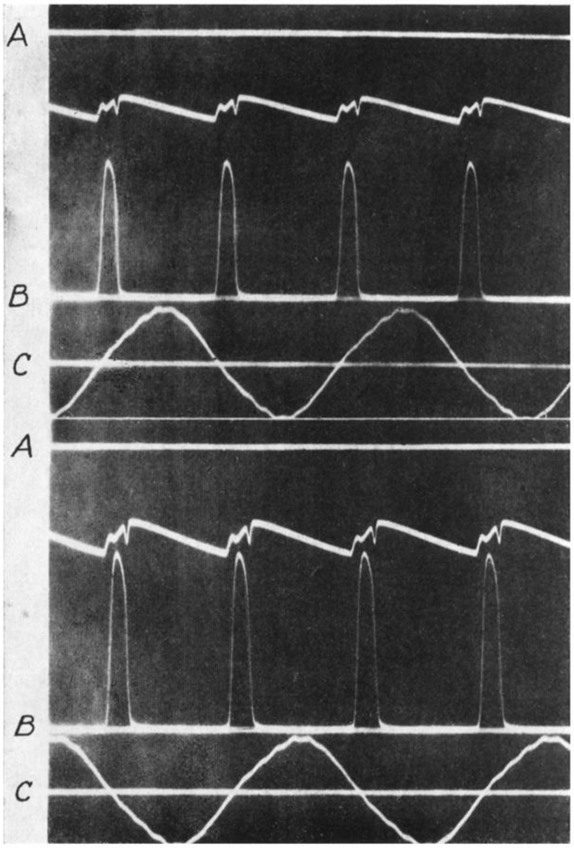

Fig. 12. Oscillogram showing plate current in a tube with 1 cm diameter tungsten cathode and 4.6 cm diameter anode, as function of instantaneous value of filament current. The lower curve C in each film is filament current, the middle curve B plate current, and the upper curve A plate voltage (deflection down).

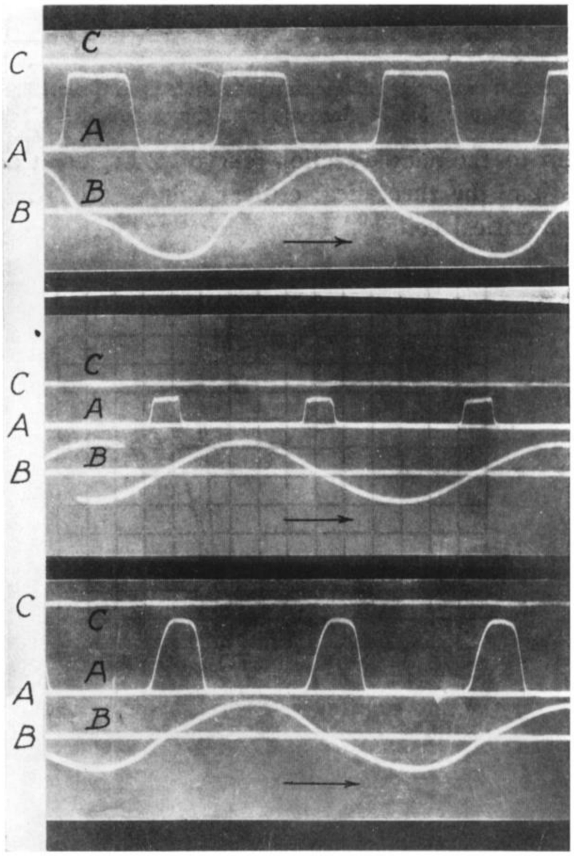

Fig. 8. Three oscillograms showing control of plate current by filament current, at constant plate voltage, in a tube with 2.5 mm filament and 10 cm diameter anode. In each film the lower curve B is filament current, and the middle curve A plate current. The upper curve (single line) is a calibration for the plate current curve.