ON THE EXPERIMENTAL DETERMINATION OF THE VISCOSITY OF VIBRATING SOLIDS

By S. L. Quimby

Abstract

Effect of viscosity on longitudinal vibrations in bars.—*The theoretical development* is based on the assumption, due to Stokes, that the stress in the medium due to viscosity is proportional to the first power of the time rate of shearing strain. The equation of propagation of a plane longitudinal sound wave along a slender bar is made to include the viscous stress which arises from the shearing strain associated with this type of disturbance. *Comparison with experiment.* Specimen bars are excited to longitudinal vibration by means of a high frequency, sinusoidal alternating electric field impressed on a piece of piezo-electric quartz cemented to one end of the bar. The amplitude of vibration is observed by measuring the torque on a Rayleigh disk suspended in air immediately off the other end of the bar. Resonance curves are obtained showing the relation between the square of the particle velocity at the end of the bar and the frequency of excitation. The experimental curves for *hard drawn copper, aluminum, and glass* are in admirable agreement with those deduced from the theory. Curves for soft *annealed copper and silver,* however, exhibit discrepancies which indicate the presence of viscous forces varying according to higher powers of the strain velocity.

Coefficient of viscosity as determined from longitudinal vibrations.—Where the agreement is good, comparison of the observed with the theoretical resonance curves yields the value of the coefficient of viscosity of the substance multiplied by $(1+\sigma)$, where σ is Poisson's ratio. The values obtained for this quantity for Al, Cu, and plate glass, are 545, 2880 and 2440 c.g.s. units, respectively. These are in the neighborhood of 10^3, in marked disagreement with the values of 10^8 obtained by other investigators using quite different methods. It is possible that irreversible changes involving dissipation of energy take place in slow bending which are absent in rapid vibrations.

Velocity of sound in solids.—For *aluminum, hard drawn copper and plate glass* the values obtained from the resonance frequencies are 5070, 3650 and 5710 m/sec., accurate to about 1 per cent.

Measurement of small changes of elasticity of bars.—Changes of less than .01 per cent can be detected by this method.

Differential frequency meter for measuring small changes of frequency in a high frequency generating set is described, sensitive to a change of less than 1 cycle per sec. in a frequency of 50,000.

THE following research is an experimental study of the theory of longitudinal vibrations in a viscous medium as developed by Stokes. The method involves a comparison between the observed resonance curves of a vibrating bar and the theoretical resonance curves which follow from the fundamental equation obtained by Stokes. The validity of the theory having been established, the coefficient of viscosity of the medium may be calculated.

A. Theory

Viscosity of matter is a term denoting the phenomenon of transfer of coordinated molecular motion within the medium into random motion or heat. The mechanism of this process lies evidently in those modifications in the normal intermolecular cohesive forces, which accompany molecular collision. Such extra intermolecular forces, averaged over a time interval long compared with their own periods, will constitute an equivalent stress in the medium, and it is the object of any theory of viscosity to evaluate this viscous stress in terms of the coordinated motion (strain) which produces it. Now it is a matter of common experience that viscous force, in fluid media at least, depends on the time rate of strain and not on the strain itself. Hence the obvious first approximation is to assume the viscous stress at any point in the medium to be proportional to the strain velocity at that point. Stokes,[1] however, goes one step further than this. The most general type of small strain in an isotropic medium may, by the method of the ordinary theory of elasticity, be exhibited as a uniform dilatation accompanied by two simple shears. Stokes argues that, inasmuch as the viscous forces arise from the mutual actions of neighboring molecules, in the case of a dilatation uniform in all directions these forces will on the average balance one another and can therefore contribute nothing to the normal elastic stress at the point in question. Thus in Stokes' theory viscosity is inseparably associated with shearing motion, and the coefficient of viscosity μ of a homogeneous medium is defined by the equation $S=\mu\, ds/dt$, in which ds/dt is the time rate of shearing strain and S the contribution of the viscous forces to the corresponding shearing stress. It is now desired to find the way in which these viscous stresses modify the equations of propagation of a longitudinal sound wave in the medium.

If the coordinate axes coincide with the principal axes of strain, the components of elastic stress accompanying displacements u, v, w, in an isotropic medium are given by expressions of the type,[2]

$$P=k\delta+2n\left(\frac{\partial\mu}{\partial x}-\frac{1}{3}\delta\right)\text{ , etc . ,}$$

where

$$\delta=\frac{\partial\mu}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\,.$$

[1] Stokes, Camb. Phil. Soc. Trans. **7,** 287 (1845); Math. and Phys. Papers I, p. 75.

[2] Rayleigh, Theory of Sound, II, p. 313.

It will be observed that, in these equations, the bulk modulus k appears only as a coefficient of a dilatation and the modulus of rigidity n only as a coefficient of shears. Hence, on Stokes' assumptions, if the coefficient n be replaced by the operator $(n+\mu\, \partial/\partial t)$ the contributions of the viscous forces to the total stress will be fully taken into account.[3] Accordingly, the principle stresses in an isotropic viscous medium will be given by the expressions

$$P=k\delta+2n\left(\frac{\partial u}{\partial x}-\frac{1}{3}\delta\right)+2\mu\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}-\frac{1}{3}\delta\right), \tag{1}$$

$$Q=k\delta+2n\left(\frac{\partial v}{\partial y}-\frac{1}{3}\delta\right)+2\mu\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}-\frac{1}{3}\delta\right), \tag{2}$$

$$R=k\delta+2n\left(\frac{\partial w}{\partial z}-\frac{1}{3}\delta\right)+2\mu\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}-\frac{1}{3}\delta\right). \tag{3}$$

The equations of motion in the medium may now be written in the form,

$$\begin{aligned}\rho\frac{\partial^2 u}{\partial t^2}&=\frac{\partial P}{\partial x},\\ \rho\frac{\partial^2 v}{\partial t^2}&=\frac{\partial Q}{\partial y},\\ \rho\frac{\partial^2 w}{\partial t^2}&=\frac{\partial R}{\partial z}.\end{aligned} \tag{4}$$

The boundary conditions appropriate to the propagation of a longitudinal wave along a slender bar whose axis lies parallel to X are that the resultant elastic and viscous stresses parallel to Y and Z be separately equal to zero. These conditions applied to Eqs. (2) and (3) yield the relations

$$\begin{aligned}&Q=R=0,\\ &\frac{\partial v}{\partial y}=\frac{\partial w}{\partial z}=-\sigma\frac{\partial u}{\partial x},\\ &\frac{\partial^2 v}{\partial y\partial t}=\frac{\partial^2 w}{\partial z\partial t}=-\sigma\frac{\partial^2 u}{\partial x\partial t},\end{aligned} \tag{5}$$

where σ is Poisson's ratio. Thus the resultant longitudinal stress in the bar is given by

$$P=k(1-2\sigma)\frac{\partial u}{\partial x}+\frac{4}{3}(1+\sigma)n\frac{\partial u}{\partial x}+\frac{4}{3}(1+\sigma)\mu\frac{\partial^2 u}{\partial x\partial t}. \tag{6}$$

[3] Ibbetson, Mathematical Theory of Elasticity, p. 494

A physical interpretation of this analysis shows that the type of strain associated with longitudinal vibrations in a slender bar is an elongation of amount $\partial u/\partial x$ per unit length accompanied by a lateral contraction of amount $\sigma\ \partial u/\partial x$ per unit length. Such a strain involves two simple shears each of amount $(2/3)(1+\sigma)\partial u/\partial x$ in mutually perpendicular planes at angles of 45° with the direction of elongation. Phenomena resulting from the viscous forces called into play by these shearing motions form the subject of the present investigation.

If the values of P, Q, and R given by Eqs. (5) and (6) be now substituted in Eqs. (4) these become

$$\rho\frac{d^2u}{dt^2}=G\frac{d^2u}{dx^2}+\frac{4}{3}(1+\sigma)\mu\frac{d^3u}{dx^2dt^2}\ , \tag{7}$$

$$\rho\frac{d^2v}{dt^2}=0,$$

$$\rho\frac{d^2w}{dt^2}=0\ .$$

where G is Young's modulus.

Due to the finite cross section of the bar the boundary conditions assumed in the derivation of Eq. (7) are not exact. Lateral stress and consequent lateral motion will exist. In physical terms, the inertia associated with this lateral motion will produce a decrease in the effective value of σ, the ratio of lateral contraction to longitudinal extension in the bar. In an argument based on energy considerations, Rayleigh has shown that the effect of lateral motion in a circular bar of radius r and length l is to increase its natural period T in the ratio $T/T'=1/(1+\Delta/2)$, where $\Delta=m^2\pi^2\sigma^2r^2/2l^2$, and m is the number of half waves in the bar. Thus, since $G=2(1+\sigma)n$,

$$\frac{T^2}{T'^2}=\frac{G'}{G}=\frac{1+\sigma'}{1+\sigma}=(1-\Delta)\ ,$$

and the effect of neglecting the modification in the viscous stress due to the inertia of the lateral motion will be to make the μ calculated from Eq. (7) too low by an amount $\mu\Delta$. In the present experiments this correction is less than the experimental error.

B. Apparatus

1. The Vibrating System

Fig. 1 is a diagram of the vibrating system used throughout these experiments. A piece of piezo-electric quartz 2 in. long, 1 in. wide, and

3/16 in. thick is cemented with a very thin layer of hard shellac, softened by heating, to one end of a bar of the material under investigation having the same breadth and thickness. The slab of quartz is so cut from the crystal that the optic axis lies parallel to Y and an electric axis parallel to Z. Thin sheets of tinfoil are pasted on the X-Y faces of the quartz. If a sinusoidal potential difference be established between these sheets of tinfoil a pressure strictly proportional to the potential difference will be developed in the quartz,[4] and this pressure will be communicated across the boundary to the bar. In this manner it is possible to set up in the bar longitudinal vibrations of great purity.

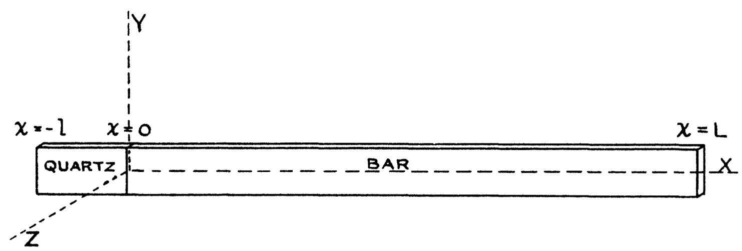

Fig. 1. The vibrating system.

The following mathematical analysis of the vibrating system is based upon certain assumptions, namely:

1. The effect of the shellac cement is negligible. The experimental results justify this assumption.

2. The effect of dissipation through radiation into the surrounding air is negligible. This follows both experimentally and from a theoretical development in which radiation is included.

3. The effect of viscosity in the quartz is negligible. Unpublished results on the dissipation in vibrating quartz plates obtained in this laboratory by C. G. Stone indicate that the viscous coefficient of quartz is extremely small. The validity of this assumption is further experimentally verified in what follows. The equation of propagation of a plane wave of sound in the quartz is accordingly

$$\frac{d^2u}{dt^2} = V_2^2 \frac{d^2u}{dx^2} \tag{8}$$

in which u is the particle displacement and V_2 is defined by the equation $V_2 = \sqrt{G_2/\rho_2}$, where G_2 is Young's modulus for quartz and ρ_2 its density.

The present experimental method demands an expression relating the particle velocity at the end of the bar with the frequency of excitation of

[4] W. G. Cady, Proc. Inst. Radio Eng. **10**, 83 (1922).

the system. If the potential difference E impressed on the quartz is of the form

$$E = E_0 e^{int} ,$$

it follows that the piezo-electric pressure Π developed in the quartz may be represented by the equation

$$\Pi = \Pi_0 e^{int} .$$

The amplitude of vibration at all points in the system will vary harmonically with the time and

$$u = A_2 e^{i(nt+k_2x)} + A_2' e^{i(nt-k_2x)}$$

will be a solution of Eq. (8) giving the motion in the quartz provided that

$$k_2 = n/V_2 .$$

Similarly

$$u = A_1 e^{int+(a+ik_1)x} + A_1' e^{int-(a+ik_1)x} \tag{9}$$

will be a solution of Eq. (7) giving the motion in the bar provided that

$$a \equiv \frac{2}{3} \frac{n^2}{\rho_1} \frac{\mu'}{V_1^3},$$
$$k_1 \equiv n/V_1 ,$$
$$\mu' \equiv (1+\sigma)\mu ,$$

and 16 $n^2\mu'^2/9\rho_1^2V_1^4$ is small compared with unity. The constants A_1, A_1', and A_2, A_2' are evaluated with the aid of the equations expressing the continuity of pressure and velocity at the boundaries. Thus at $x = -l$,

$$G_2[du/dx]_2 + \Pi = 0 ;$$

and at $x = 0$,

$$G_2[du/dx]_2 + \Pi = G_1[du/dx]_1 + \frac{4}{3}\mu'[d^2u/dxdt]_1,$$

and

$$[du/dt]_2 = [du/dt]_1 ;$$

and at $x = L$,

$$\frac{4}{3}\mu'[d^2u/dxdt]_1 + G_1[du/dx]_1 = 0 ,$$

where the subscripts 1 and 2 refer to conditions in the bar and quartz respectively. The values of A_1, and A_1' obtained from these equations are substituted in Eq. (9), which then constitutes a complete solution for the motion in the bar. Thus it readily follows that

$$|V_0|^2 = \Pi_0^2\{\cos y - 1\}^2/\left[\{V_1^2\rho_1^2\cos^2 y + V_2^2\rho_2^2\sin^2 y\}\{\cos h^2 z - \sin^2(w-\beta)\} + \tfrac{2}{3}n(V_2/V_1)\rho_2\mu'\sin y \cos y \sin h2z\right] \tag{10}$$

where $|V_0|$ = the amplitude of the particle velocity at the end of the bar,

$$w = nL/V_1\ ,\ y = nl/V_2\ ,$$

$$z = \frac{2}{3}\,\frac{n^2L}{\rho_1 V_1^3}\mu'\ ,\ \beta = \tan^{-1}\frac{V_1\rho_1\cos y}{V_2\rho_2\sin y}\ .$$

Since z is of the order of 10^{-2} the hyperbolic functions may be replaced by their series expansions and the third and higher powers of z discarded. When this is done Eq. (10) becomes

$$|V_0|^2 = \frac{\Pi_0^2\{\cos y - 1\}^2}{\{V_1^2\rho_1^2\cos^2 y + V_2^2\rho_2^2\sin^2 y\}\{[4n^4L^2/9\rho_1^2V_1^6 + K]\mu'^2 + \cos^2(w-\beta)\}}\ , \tag{11}$$

where
$$K = \frac{8n^3V_2\rho_2L\sin y\cos y}{9\rho_1V_1^4\{V_1^2\rho_1^2\cos^2 y + V_2^2\rho_2^2\sin^2 y\}}\ .$$

In this expression all the terms on the right hand side except μ' and Π_0 are known.

The observations recorded below were all taken for frequencies between 37 and 60 kilocycles, while the resonance curves themselves are only about 100 cycles wide. It follows that as the frequency is varied in the neighborhood of a resonance point the term in μ'^2 remains sensibly constant. Resonance will therefore occur when cos $(w-\beta)$ is a minimum, i.e., when

$$(w-\beta) = (m\pi - \tfrac{1}{2}\pi)\ ,$$

where m is the number of half waves in the bar. This gives the following relation between V_1, L and the resonance frequencies of the system, viz.,

$$2f_0L/V_1 = m + \delta. \tag{12}$$

where f_0 = a resonance frequency,

and $\delta = \beta/\pi - \tfrac{1}{2}$.

2. The Power Set

Fig. 2 is a diagram of the oscillating circuit supplying power to the vibrating system. The vacuum tube is a 250 watt General Electric Co. type P pliotron. E is a standard Kelvin electrostatic voltmeter, and C_2 a vernier condenser. The set is capable of producing a very pure sinusoidal voltage across the quartz ranging in value from 1 to 5 kilovolts and in frequency from 35 to 65 kilocycles. The voltage is regulated by varying the current in the field coils of the generator.

3. The Vibration Indicator

Variations in the particle velocity at the end of the bar are measured by observing the torque on a Rayleigh disk suspended by a fine quartz fiber in the air immediately off the end of the bar.

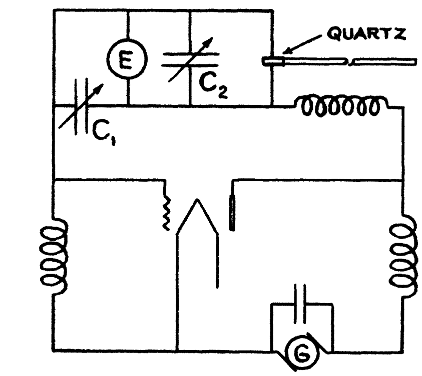

Fig. 2. The power circuit.

Fig. 3 is a schematic diagram of this apparatus viewed from above. The Rayleigh disk, 0.2 cm in diameter and 0.005 cm thick, is stamped with a die from a sheet of aluminum and mounted on the tip of a fine glass

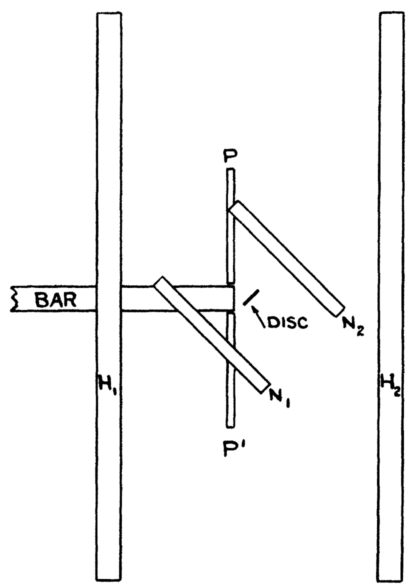

Fig. 3. Schematic diagram of the vibration-meter.

staff. Near the top of the staff are placed two mirrors, back to back. One of these carries a small magnet. The whole is hung so that the disk comes about 0.2 cm from the center of the end of the bar, which fills an aperture in the plane surface PP'. The whole apparatus is in a space in which the earth's magnetic field has been carefully neutralized by coils not shown in the figure. The plane of the disk is adjusted to make an angle of 45° with the surface PP'. In this position a spot of light reflected from one of the mirrors falls upon a fiducial mark on a scale.

When the bar is in vibration there is a torque on the disk tending to set it parallel to PP', due to the vibration of the air in which it is situated. The moving system is returned to its original position by balancing this torque with an equal and opposite magnetic torque produced by sending a suitable current through a pair of Helmholtz coils, H_1 and H_2, Fig. 3. When a balance is obtained, as indicated by the position of the spot of light, the vibration torque will be proportional to the current in the coils, which is then read from a milliammeter. The magnitude of the vibration torque is proportional to the square of the velocity of the air about the disk.[5] It is assumed that the motion of the air at the disk follows that of the end of the bar. It therefore follows that the currents observed on the milliammeter should vary with the frequency of excitation of the vibrating system in accordance with the relation expressed by the right hand member of Eq. (11). The theoretical working formula of the instrument is, therefore,

$$T=c\frac{E'^2\{\cos y-1\}^2}{\{V_1^2\rho_1^2\cos^2 y+V_2^2\rho_2^2\sin^2 y\}\ \{[4n^4L^2/9\rho_1^2V_1^6+K]\mu'^2+\cos^2(w-\beta)\}}, \quad (13)$$

where T = torque in milliamperes,

E' = root mean squared voltage applied to quartz,

c = a constant.

The constant c includes the piezo-electric constant of the quartz and the ratio between the mean squared velocity of the end of the bar and the corresponding torque in milliamperes on the Rayleigh disk.

If the prolongation of the glass staff does not pass through the center of the disk there will be another torque on the disk due to the radiation pressure of the sound. To eliminate this it is necessary to take another series of observations with the disk rotated through 180°. This procedure reverses the radiation torque and leaves the velocity torque unchanged. The reversal of the moving system is accomplished by means of a temporary magnetic field in the two small coils N_1 and N_2, which are mounted to rotate by hand about a vertical axis. The present disk gives identical results in both positions.

[5] Rayleigh, op. cit.[2] II, p. 44.

4. The Differential Frequency-meter

Changes in the frequency of excitation of the vibrating system in the neighborhood of a resonance point are measured by adjusting the note from a calibrated audio-frequency oscillator to consonance with the beat frequency obtained by coupling the power set to a small oscillator whose frequency is maintained constant.

Fig. 4 is a diagram of the circuits employed. Circuit I is an oscillator designed to supply a very constant frequency differing by about 2500 cycles from that of the power set. Both these frequencies are impressed on the grid of the detector tube II, which rectifies the 2500 cycle beat note. This beat note from circuit II is, in turn, mixed with an audio-frequency note from circuit IV and impressed on the grid of tube III, to which is connected a loud speaking telephone.

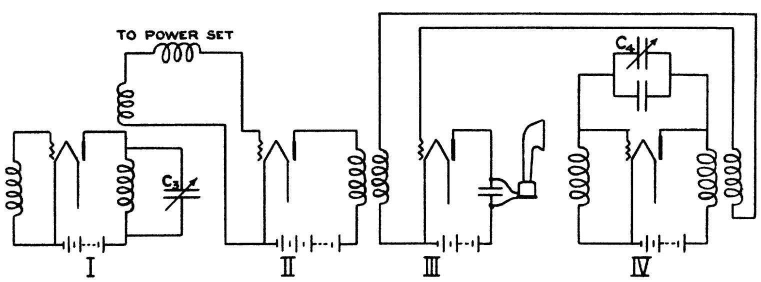

Fig. 4. The differential frequency-meter circuit.

The audio-oscillator IV has been accurately calibrated by comparing its note with that from a siren. C_4 is a vernier condenser which will vary the frequency about 300 cycles in 2500. A scale on C_4 indicates directly the *difference* between the (audio) frequency corresponding to any setting on C_4 and the frequency when C_4 is 0.

It follows that any small change in the frequency of the power set will be given at once by the difference in the scale readings on C_4 corresponding to values of C_4 which produce consonance between the audio note and the beat note. As consonance may readily be determined to better than one beat per second, the apparatus measures a change in frequency of the power set of one part in 50,000.

5. Method of Manipulation

The operation of making a complete run on a specimen bar is as follows:

1. The vibrating system, having been prepared as described above, is suspended in position by fine threads and grounded to prevent stray charges from affecting the disk.

2. The earth's magnetic field is carefully neutralized so that the Rayleigh disk swings under the negligible torque of the fine quartz fiber. The disk is then conveniently held in position by a current in the rotating coils.

3. A resonance frequency is selected by varying C_1 (Fig. 2). C_2 is then adjusted until the system is vibrating on the lower portion of the resonance curve.

4. C_4 (Fig. 4) is set at 0 and C_3 adjusted for consonance between the audio and beat notes.

5. The vibrating system is carried back and forth through resonance by varying C_2, torques being observed on the milliammeter and corresponding frequency differences on the condenser C_4.

6. The absolute resonance frequency is measured with a precision of 1/4 per cent on a wave meter especially calibrated by the U. S. Bureau of Standards.

C. Results

The method just described offers two points of attack in the verification of Eq. (11), namely, (a) comparison of the observed harmonic resonance frequencies with those calculated using Eq. (12), and (b) comparison of the experimental resonance curves with those given by Eq. (13).

1. Harmonic Resonance Frequencies

Eq. (12) predicts the harmonic resonance frequencies of the vibrating system with great precision. The method of using this equation was to calculate V_1, the velocity of sound in the medium, using the observed values of f_0 and L and the computed value of δ. For calculating δ sufficiently good values of V_1 were obtained by setting $\delta=0$ in Eq. (12), and of V_2 by vibrating the quartz by itself. V_1 should, of course, remain constant irrespective of the value of f_0 and L.

The results are shown in Table I.

Table I

Material	L	f_0	$m+\delta$	V
Glass	97.2 cm	39,020	13.265	5.718×10^5 cm/sec.
		41,720	14.215	5.706
		44,480	15.142	5.711
Aluminum	90.0	37,400	13.308	5.059
		40,230	14.258	5.079
		45,550	16.154	5.077
Copper	91.67	43,920	22.083	3.647
	58.14	44,250	14.076	3.655
	24.60	45,000	6.072	3.646
	24.60*	41,900*	5.717*	3.606*

* Made with a piece of quartz 1 inch long.

Used simply as a means for determining the velocity of sound in solid media the precision of the method is seen to be limited only by that of the device used for measuring the absolute resonance frequency.

2. The Resonance Curves

Resonance curves typical of the large number observed are shown in Fig. 5. The points are experimental and the solid lines are graphs of Eq. (13).

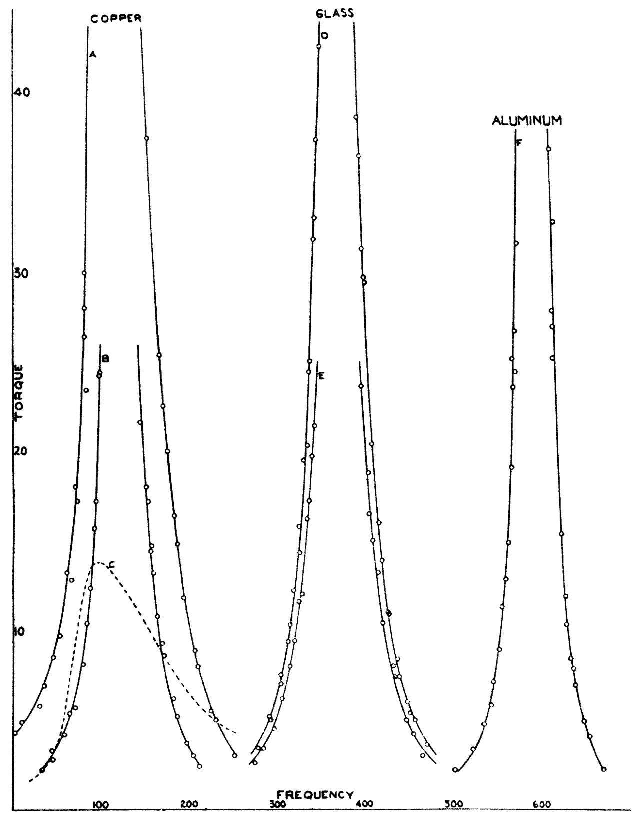

Fig. 5.

Resonance curves of vibrating bars.

Curve	Material	L	f_0	E′
A	Copper	24.6 cm	45,000	1400 volts
B	Copper	91.7	43,920	2200
C	Annealed copper	24.6	45,000	2000
D	Glass	97.2	39,020	2000
E	Glass	97.2	44,480	1500
F	Aluminum	90.0	45,550	1400

The scale of ordinates for curve F is double that for the others.

It is not possible to obtain points on the higher parts of the curves since the curves in this region are extremely steep. A very small fluctuation in the frequency of the power set results in a large increase in the amplitude of vibration and a correspondingly large increase in the piezoelectric reaction of the vibrating quartz on the power set. This, in turn, produces a further change in frequency. The result is that, in passing through resonance, when the curve reaches a certain steepness the electro-mechanical system becomes unstable and the frequency jumps abruptly to some point on the other branch of the curve.

The process of expressing the experimental data in the form of Eq. (13) involves the determination of the two arbitrary constants c and μ'. These are evidently mutually independent. The observed values of c proved to be constant within a few per cent as long as the bar was left undisturbed in its position relative to the disk. The values of μ' were reproducible under all circumstances.

The average values of $\mu' = (1+\sigma)\mu$ for the materials investigated are as follows:

Aluminum	$\mu' = 545 \pm 4$ c. g. s. units
Copper	$\mu' = 2880 \pm 46$
Glass	$\mu' = 2440 \pm 53$

The glass strip was cut from a plate of automobile windshield glass and had a specific gravity of 2.52. The aluminum was sawed from a piece of commercial sheet aluminum having a specific gravity of 2.63. The copper came in the form of hard drawn copper bar having a specific gravity of 8.85.

In order to verify the assumption that any dissipation in the quartz is negligible, experiments were made on eight copper bars varying in length from 91.67 cm to 24.60 cm. The resulting values of μ' showed no systematic variation.

Curve C (Fig. 5) was obtained from the same bar as Curve A after the copper had been annealed by heating to a cherry red and plunging into water. The indicated shift of the resonance frequency is not significant as the curves were superimposed arbitrarily. Curves taken on a long annealed copper bar and on a bar of fine silver show the same characteristics. It is impossible to represent the observations on these substances by means of Eq. (13). In addition to the change in the shape of the curves, the harmonic frequencies of these bars did not follow Eq. (12) but fell off consistently from the calculated values as the frequency increased.

D. Discussion

The results obtained with aluminum, hard drawn copper, and glass indicate that the equation of Stokes accurately expresses the propagation of sound in a solid medium for which the coefficient of viscosity is comparatively small. When substances of greater viscosity are examined, marked discrepancies appear. Now it will be recalled that the assumptions upon which Stokes' development is based are first, that Hooke's law is obeyed in the medium, and second, that the viscous forces vary as the first power of the relative particle velocity. For the amplitudes of vibration obtained in the experiments now under review (about 10^{-5} cm) the first assumption probably remains valid. It seems likely, however, that, as the viscosity of the medium increases dissipative forces appear which vary according to higher powers of the velocity of strain.

The tangential and normal coefficients of viscosity of various solids have been measured by other investigators using methods radically different from that here described. In these researches the tangential coefficient of viscosity is obtained by observing the logarithmic decrement of a torsion pendulum, the elastic member of which is a fine wire of the material under examination.[6] The normal coefficient of viscosity is calculated from the logarithmic decrement of a pendulum consisting of a flat strip of material clamped at the upper end and loaded at the lower end.[7] The strains in the medium are of the same order of magnitude as those produced in this investigation. The periods of vibration, however, are of the order of 1 second. The values for the coefficients obtained by these methods are uniformly of the order of 10^8 c.g.s. units, to which must be compared the values of the order of 10^3 c.g.s. units given above. The magnitude of the difference between these figures indicates that the two widely different methods of investigation employed have, in fact, dealt with two entirely different phenomena. Indeed, it can readily be shown that if the coefficient of viscosity for sound vibrations of a metallic bar were of the order of 10^8 the bar would be wholly incapable of sustained vibration. The motion would be aperiodic. In accord with this view two suggestions appear worthy of notice.

The first deals with the period of vibration. It is possible that a slow bending or twisting of a substance in which the elastic limit is not exceeded nevertheless produces minute irreversible changes in its internal

[6] C. E. Guye and others, Arch. des Sciences **26,** 136 (1908); **29,** 289 (1910); **30,** 133 (1911); Journ. de Phys. 1912, p. 620.
Iokibe and Sakai, Phil. Mag. **42,** 397 (1921).

[7] Honda and Konno, Phil. Mag. **42,** 115 (1921).

structure accompanied by loss of mechanical energy. If the time required to produce such changes was of the order of a second, a possible explanation might be found for the apparent difference in response of the medium to rapid and to slow vibration.* The second suggestion deals with the method of support of the specimen. In the pendulum experiments the specimen was clamped rigidly at one end. This introduces the possibility of losses of energy *in the medium* near the point of attachment which are not contemplated in the theoretical treatment of the pendulum. Boudouard[8] has shown that a large part of the effect of the viscosity of the medium is, in fact, localized in a region very near the point of support. Boudouard experimented with thin strips of iron about 25 cm long clamped in a vise at one end and excited to transverse vibration at 30 cycles per second by means of an electromagnet. The strains produced in this way were kept well below the elastic limit of the iron. In spite of this the strips broke in two after a period of excitation which varied from five minutes to fourteen hours according to the carbon content and heat treatment of the specimen. The break invariably occurred near the point of support and the broken material showed striations.

In the opinion of the writer a satisfactory solution of the problem demands the development of a method for measuring the coefficient of viscosity of solids which will not necessitate clamping the specimen and which will operate continuously over the range of frequencies between 1 and 100 cycles per second. The industrial importance of this problem has been dealt with in a paper by Henry Le Chatelier.[9]

In conclusion, attention is directed to the fact that the apparatus developed in this research is capable of measuring a change in the elastic coefficient of a freely suspended solid of one hundredth of one per cent. Irrespective of the shape of the resonance curves, a change in the resonance frequency of the vibrating system of 2 cycles in 50,000, or 0.004 per cent, can be measured. Since f^2_0 varies as the elasticity it follows that a change in the elastic coefficient of 0.008 per cent will produce a measurable change in f_0. The method may be used safely up to a temperature of 80°C, above which there is danger of softening the shellac cement between quartz and bar. As an example of this application of the apparatus, it may be noted that a change in resonance frequency of 23 cycles in 44,000 was observed when an iron bar was placed in a longi-

* When a specimen of iron is magnetized it is known that marked changes in the internal structure of the iron persist for several seconds after the application of the magnetizing field.

[8] Boudouard, Comptes Rendus **150**, 696 (1910); **152**, 45 (1912).

[9] H. Le Chatelier, Revue de Metallurgie 1909, p. 888.

tudinal magnetic field. Further researches on the relation between magnetization and elasticity are in progress in this laboratory.

The writer wishes to express his appreciation of the kindness of Professor J. H. Morecroft, who loaned from his laboratory the high power oscillator and the excellent piece of quartz used in these experiments, and to acknowledge his indebtedness to Professor A. P. Wills, whose counsel and encouragement have been of much help in the course of this research.

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December 5, 1924.