THE INTENSITY OF THE SCATTERING OF X-RAYS BY RECOILING ELECTRONS

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Abstract

Theoretical calculation of the intensity of scattering of x-rays by recoiling electrons.—(1) *On the Compton hypothesis of moving electrons* with a velocity in the direction of the incident ray equal to $c\alpha/(1+\alpha)$, where $\alpha = h/mc\lambda_0$, a calculation according to the usual electrodynamics when correlated with Compton's previous result, gives the same total scattering as that obtained by Thomson. The fact that experiments with very short wave-lengths do not agree with this result shows that Compton's hypothesis is not satisfactory without further modification. (2) *If we assume the virtual moving oscillators* proposed by Bohr, Kramers and Slater, in view of the similarity of these to moving electrons, the result is the same, and the lack of agreement with experiment indicates that the exact form of the correspondence principle set up by these authors fails to answer the intensity problem in the scattering of x-radiation.

1. Introduction

In his quantum theory of the scattering of x-rays[1] A. H. Compton emphasizes the analogy between the change of wave-length of the ray scattered by the recoiling electron and the classical Doppler effect of radiation from a moving source. As Compton points out, the change in frequency of the scattered radiation is the same as if the ray were scattered by electrons moving in the direction of propagation with a velocity $c\beta$, where $\beta = \alpha/(1+\alpha)$ and $\alpha = h\nu_o/mc^2$, ν_o being the frequency of the primary beam, h Planck's constant, c the velocity of light, and m the mass of the scattering electron. He calls $c\beta$ the effective velocity of the scattering electrons.

Assuming that all the scattering electrons are moving in the direction of the incident beam with an effective velocity $c\beta$, and applying probability considerations to the emission of quanta from them, Compton deduced expressions for the intensity I_θ of the scattering at any angle θ with the incident ray, and for the total scattering absorption coefficient σ, which have recently been put to experimental test by several investigators.[2]

In the present paper a calculation is made of the scattering according to usual electrodynamics on the basis of the same hypothesis as that assumed by Compton and it may therefore be regarded as a supplement

[1] A. H. Compton, Bull. Nat. Res. Council, No. 20, p. 19 (1922); and Phys. Rev. **21**, 207 and 483 (1923).

[2] N. Ahmad and E. C. Stoner, Proc. Roy. Soc. A **106**, 8 (1924); E. A. Owen, N. Fleming and W. E. Fage, Proc. Phys. Soc. London **36**, 355 (1924).

to a part of Compton's work. In view of the similarity between the moving scattering electron just mentioned and the virtual moving oscillator suggested by Bohr, Kramers, and Slater,[3] the scattering by the latter will also be discussed.

2. The Scattering of X-Rays by Moving Electrons

Let us consider two reciprocal Euclidean systems S and S', such that all the points of S' have the same constant velocity $c\beta$ relative to S. Let a set of right-handed axes XYZ be fixed in S so that the X axis has the direction of the velocity of S'. Let a similar set of axes $X'Y'Z'$, parallel to XYZ respectively, be fixed in S'. With respect to S' the scattering electrons are at rest.

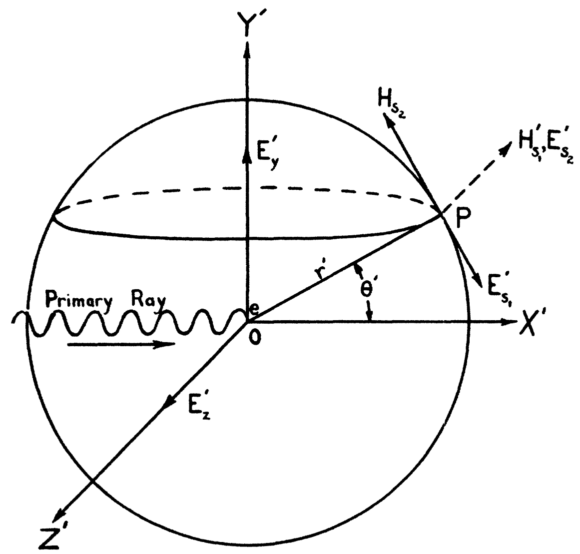

Fig. 1. Electric and magnetic vectors in S'.

Imagine, as in Fig. 1, that an electron at the origin O is accelerated by an incident beam which has its electric vectors E_y' and E_z' along Y' and Z' axes respectively. It is desired to determine the values of the scattered electric and magnetic fields E_s' and H_s' due to this accelerated charge at a point P distant r' from O at a time r'/c, and also the corresponding electric and magnetic fields E_s and H_s in S at the same instant. Since the velocity of light is the same in the two systems, the time at P will be r/c in S when it is r'/c in S'. Hence the result of the transformation about to be carried through will give E_s and H_s at the time r/c.

[3] N. Bohr, H. A. Kramers, and J. C. Slater, Phil. Mag. **47**, 785 (1924).

Let θ' be the angle in S' which the line OP makes with the X' axis. Without loss of generality this line may be supposed to lie in the $X'Y'$ plane. Let E_{S1}' and E_{S2}' be the electric vectors of the scattered radiation at P due to the actions of E_y' and E_z' respectively. Then

$$E_{s1}' = E_y' e^2 \cos\theta' / mc^2 r' \ , \tag{1}$$

$$E_{s2}' = E_z' e^2 / mc^2 r' \ . \tag{2}$$

The Lorentz-Einstein transformations give

$$x' = r' \cos\theta' = kr(\cos\theta - \beta) \ ,$$

$$y' = r' \sin\theta' = r \sin\theta \ ;$$

whence

$$\left.\begin{aligned} r' &= kr(1-\beta\cos\theta) \ , \\ \cos\theta' &= (\cos\theta - \beta)/(1-\beta\cos\theta) \ , \\ \sin\theta' &= \sin\theta / k(1-\beta\cos\theta) \ , \end{aligned}\right. \tag{3}$$

where

$$k = 1/\sqrt{1-\beta^2} \ .$$

By the well-known transformation equations for E_y' and E_z' and taking account of the relations $E_y = H_z$ and $E_z = -H_y$, we obtain

$$\left.\begin{aligned} E_y' &= kE_y(1-\beta) \\ E_z' &= kE_z(1-\beta) \ . \end{aligned}\right. \tag{4}$$

Substituting the values for E_y', E_z', r' and $\cos\theta'$ given by Eqs. (3) and (4) in Eqs. (1) and (2) and reducing,

$$E_{s1}' = \frac{e^2 E_y}{mc^2 r} \frac{(1-\beta)(\cos\theta - \beta)}{(1-\beta\cos\theta)^2} \ , \tag{5}$$

$$E_{s2}' = \frac{e^2 E_z}{mc^2 r} \frac{(1-\beta)}{(1-\beta\cos\theta)} \ . \tag{6}$$

Referring to Fig. 1 we see the directions of the vectors E_{S1}', H_{S1}', E_{S2}', and H_{s2}', where H_{s1}' and H_{s2}' are the magnetic fields of the scattered radiation at P due to the actions of E_y' and E_z' respectively; and, of course, $E_{s1}' = H_{s1}'$ and $E_{s2}' = H_{s2}'$ as regards their magnitudes.

Since E_s' and H_s' are supposed to be the total electric and magnetic vectors of the scattered beam at point P, it can be readily seen that

$$\left.\begin{aligned} E_{sx}' &= E_{s1}' \sin\theta' \ , \\ E_{sy}' &= -E_{s1}' \cos\theta' \ , \\ E_{sz}' &= -E_{s2}' \ , \\ H_{sx}' &= -H_{s2}' \sin\theta' = -E_{s2}' \sin\theta' \ , \\ H_{sy}' &= -H_{s2}' \cos\theta' = \ \ E_{s2}' \cos\theta' \ , \\ H_{sz}' &= -H_{s1}' \qquad\quad = -E_{s1}' \ , \end{aligned}\right\} \tag{7}$$

where E_{sx}' and H_{sx}' represent the x-components of E_s' and H_s' respectively and so forth.

Making use of Eqs. (7) and transforming from S' into S, we have

$$\left.\begin{aligned} E_{sx} &= E_{s1}' \sin\theta' , \\ E_{sy} &= -kE_{s1}'(\cos\theta'+\beta) , \\ E_{sz} &= -kE_{s2}'(1+\beta\cos\theta') ; \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} H_{sx} &= -E_{s2}'\sin\theta' , \\ H_{sy} &= kE_{s2}'(\cos\theta'+\beta) , \\ H_{sz} &= -kE_{s1}'(1+\beta\cos\theta') ; \end{aligned}\right\} \tag{9}$$

From Eqs. (8)

$$E_s^2 = E_{sx}^2+E_{sy}^2+E_{sz}^2 = k^2(1+\beta\cos\theta')^2(E_{s1}'^2+E_{s2}'^2) . \tag{10}$$

Similarly, we find from Eqs. (9) that

$$H_s^2 = E_s^2 \text{ or } H_s = E_s , \tag{11}$$

as they should be.

On substituting the values of E_{s1}' and E_{s2}' from Eqs. (5) and (6) we have

$$E_s^2 = H_s^2 = \frac{e^4}{m^2c^4r^2}\,\frac{(1-\beta^2)\,(1-\beta)^2}{(1-\beta\cos\theta)^6}\left\{(\cos\theta-\beta)^2E_y^2+(1-\beta\cos\theta)^2E_z^2\right\} . \tag{12}$$

Now we proceed to calculate the energy radiated from the scattering electron. Describe about the origin a sphere of radius r. Consider an element $d\sigma$ of the surface of this sphere such that the radius vector drawn to it from the origin makes an angle θ with the X-axis. The energy in the primary beam which passes O in a time dt and is scattered in the direction of this radius vector by a moving electron starting from O, will reach the surface of the sphere at a time r/c and will take a time

$$(1-\beta\cos\theta)dt/(1-\beta)$$

to pass through this surface, where $c\beta$ is equal to the velocity of the scattering electron in the X-direction. Hence, as the flow of energy per unit cross section per unit time is given by Poynting's vector as

$$\mathcal{S}_s = (c/4\pi)\left[E_sH_s\right] ,$$

where the small square bracket indicates a vector product, the energy passing through $d\sigma$ during the time dt is given by

$$Rdt = \mathcal{S}_s d\sigma(1-\beta\cos\theta)dt/(1-\beta) .$$

Therefore the energy flux is equal to

$$\bar{\mathcal{S}}_s = \frac{Rdt}{d\sigma dt} = \frac{c}{4\pi}\,E_s^2\,\frac{(1-\beta\cos\theta)}{(1-\beta)} . \tag{13}$$

Combining this relation with Eq. (12) and taking account of the fact that on the average

$$E_y^2 = E_z^2 = \tfrac{1}{2}E^2 \ ,$$

we find

$$\bar{\mathcal{S}}_s = \frac{c}{4\pi}\,\frac{e^4}{m^2c^4r^2}\,\frac{1}{2}\,E^2\frac{(1-\beta^2)\,(1-\beta)}{(1-\beta\cos\theta)^5}\left\{(\cos\theta-\beta)^2+(1-\beta\cos\theta)^2\right\} \ . \quad (14)$$

The energy flux in the incident radiation is given by

$$\mathcal{S} = (c/4\pi[EH] = (c/4\pi)E^2 \ .$$

If intensity is defined as the amount of energy per second per unit area, then the ratio of the intensity I_θ of the radiation scattered at an angle θ to a distance r, to the intensity I of the primary ray, must be equal to

$$I_\theta/I = \bar{\mathcal{S}}_s/\mathcal{S} = \frac{e^4}{2m^2c^4r^2}\,\frac{(1-\beta^2)\,(1-\beta)}{(1-\beta\cos\theta)^5}\left\{(\cos\theta-\beta)^2+(1-\beta\cos\theta)^2\right\} \ . \quad (15)$$

Let N be the number of free electrons which are effective in scattering, then, substituting for β its value $\alpha/(1+\alpha)$ and reducing, we have

$$I_\theta = I\frac{Ne^4(1+2\alpha)}{2m^2c^4r^2}\,\frac{\left\{1+\cos^2\theta+2\alpha(1+\alpha)\,(1-\cos\theta)^2\right\}}{\left\{1+\alpha(1-\cos\theta)\right\}^5} \ . \quad (16)$$

In the forward direction, where $\theta = 0$, the intensity of the scattered beam is thus

$$I_0 = I\frac{Ne^4(1+2\alpha)}{m^2c^4r^2} \ . \quad (17)$$

From Compton's calculation the intensity of the ray scattered at $\theta = 0$ is[4]

$$I_0 = \frac{3}{8\pi}\,\frac{nh\nu_0}{r^2}(1+2\alpha) \ , \quad (18)$$

where n is the number of quanta scattered per second. In spite of the fact that for a ray scattered directly forward the velocity of recoil is zero, we should, on the hypothesis of the moving scattering electron, identify Eq. (17) with Eq. (18). Thus

$$n = \frac{8\pi}{3}\,\frac{INe^4}{h\nu_0 m^2c^4} \ . \quad (19)$$

Following Compton's argument we find that the scattering absorption coefficient is

$$\sigma = \frac{nh\nu_0}{I} = \frac{8\pi}{3}\,\frac{Ne^4}{m^2c^4} = \sigma_0 \ , \quad (20)$$

[4] Compton, loc. cit.[1] p. 493, Eq. (24).

where N is the number of scattering electrons per unit volume, and σ_0 is the scattering coefficient calculated by J. J. Thomson.[5]

The total energy truly scattered can be obtained by integrating I_θ given by Eq. (16) over the surface of the sphere surrounding the scattering material, i.e.,

$$\epsilon_s=\int_0^\pi I_\theta \cdot 2\pi r^2 \sin\theta d\theta=\frac{8\pi}{3}\frac{INe^4}{m^2c^4}\frac{1+\alpha}{1+2\alpha}.$$

Hence the true scattering coefficient is

$$\sigma_s=\frac{8\pi}{3}\frac{Ne^4}{m^2c^4}\frac{1+\alpha}{1+2\alpha}=\sigma_0\frac{1+\alpha}{1+2\alpha}. \tag{21}$$

The difference between σ and σ_s is termed by Compton the coefficient of true absorption due to scattering, which has its value

$$\sigma_a=\sigma-\sigma_s=\sigma_0\frac{\alpha}{1+2\alpha}. \tag{22}$$

It is seen that according to the present calculation Compton's hypothesis leads to a value for the scattering absorption coefficient exactly equal to that calculated by Thomson. This is not successful in accounting for the experimental results for hard x-rays and γ-rays and, therefore, indicates that Compton's postulate is not satisfactory without some modification. However, from the above results it seems quite possible that by making a different assumption about the motion of the scattering electron a different value for σ could be obtained. A slight alteration of Compton's hypothesis might thus give results in accord with experiment.

3. The Scattering of the Radiation by the Virtual Moving Oscillator

Bohr, Kramers and Slater[6] have recently put forward an alternate interpretation of the Compton effect. According to these authors' view the scattering of the radiation is considered as a continuous phenomenon associated with the emission of coherent secondary wavelets by each of the illuminated electrons. The reaction from each electron on the incident radiation field is similar to that to be expected on the usual electrodynamics from an electron executing forced vibrations under the action of the illuminating field and moving with a velocity equal to that of the imaginary moving source postulated by Compton. At the same

[5] J. J. Thomson, Conduction of Electricity through Gases, 2d ed., p. 325.

[6] Bohr, Kramers and Slater, loc. cit.[3] p. 799.

time, however, these authors assume that the illuminated electron possesses a certain probability of taking up in unit time a finite amount of momentum in any given direction so that a statistical conservation of momentum is secured in a way quite similar to that in which is obtained the statistical conservation of energy in the phenomena of absorption of light, discussed by them in the same paper.

As is well known, in the case of the spectrum problem, Bohr's correspondence principle has led to comparing the reaction of an atom on a radiation field with the reaction it would have on a field which on the classical electrodynamics should be produced by a set of virtual harmonic oscillators having frequencies corresponding to the various possible transitions between stationary states. Similarly in the case of the scattering, the picture of Bohr, Kramers and Slater naturally leads to the conclusion that the intensity of the scattered radiation should be distributed in the same way as that of the scattering from the above-mentioned virtual moving oscillator. In view of the analogy between such an oscillator and the scattering electron discussed in section 2, the calculation of the scattering given there is obviously applicable here. Thus Eqs. (16) and (21) give respectively the intensity of the scattering by the virtual moving oscillator at an angle θ with the primary beam, and the total scattering coefficient.

On the hypothesis of the moving oscillator it seems necessary to take account of the work done by the radiation pressure of the incident beam on the scattering electron as well as the scattered energy. This work, of course, represents a type of true absorption resulting from the scattering process. It takes the place of what on Compton's theory is the kinetic energy of the recoiling electron. A possible way of doing this is to adopt the method attempted in section 2. As already pointed out, such a calculation leads to results not in accord with experiments.

Now let us turn our attention to the true scattering coefficient given by Eq. (21). This certainly fails to account for the very recent experiments on the scattering of γ-rays by Ahmad and Stoner, and by Owen, Fleming and Fage.[2] It has a value as low as $\sigma_0/2$ only in the limit when $\alpha=\infty$, whilst the total atomic absorption coefficient for hard γ-rays has been found to be only of the order of $\sigma_0/4$. This appears to make Eq. (21) incompatible with the observed facts.

The failure to answer the intensity problem certainly is a difficulty in the way of Bohr, Kramers, and Slater's interpretation of the Compton effect. Incidentally this seems to point favorably to the conclusion that the scattered radiation occurs as definitely directed quanta rather than as spherical waves, because, basing his discussion on the former idea, Jauncey has deduced satisfactory expressions for the scattering.[7]

4. Final Remarks

On Compton's hypothesis the scattering of polarized x-rays is just a special case of the problem considered in section 2. Proceeding as in that section and assuming that the electric vector of the incident ray is in the $X'Y'$-plane, we find that the electric vector of the scattered radiation at P in S is given by

$$E_s=\frac{e^2}{mc^2}\,\frac{kE(1-\beta)\,(1-\beta^2)}{r(1-\beta\cos\theta)^3}\,(\cos\theta-\beta)\ ,$$

and the intensity of the ray scattered at an angle θ to a distance r is

$$I_\theta=I\frac{e^4}{m^2c^4r^2}\,\frac{(1+2\alpha)\left\{(1+\alpha)\cos\theta-\alpha\right\}^2}{\left\{1+\alpha(1-\cos\theta)\right\}^5}\ . \tag{23}$$

Putting $\theta=\pi/2$ in Eq. (23) we have the intensity $I_{\frac{1}{2}\pi}$ in the direction of the electric vector, while putting $\theta=0$ we have the intensity I_0 scattered in the direction perpendicular to the electric vector. The ratio $I_{\frac{1}{2}\pi}/I_0$ is not zero as it should be according to the classical theory.

From Eq. (23) it is seen that I_θ is not equal to zero except $\theta=\theta_p$, where

$$\cos\theta_p=\beta=\alpha/(1+\alpha)\ .$$

This agrees with the result obtained by Jauncey from his corpuscular quantum standpoint.[8]

In conclusion, the writer wishes to express his sincere thanks to Prof. A. H. Compton for suggesting the problem, and for constant advice and helpful criticism.

Ryerson Physical Laboratory,
University of Chicago,
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[7] G. E. M. Jauncey, Phys. Rev. **22**, 233 (1923).
[8] G. E. M. Jauncey, Phys. Rev. **23**, 313 (1924).