QUANTUM THEORY OF THE UNMODIFIED SPECTRUM LINE IN THE COMPTON EFFECT

By G. E. M. Jauncey

Abstract

The theory is based on the following assumptions: (a) The energy of the primary quantum plus the kinetic energy of the electron in its Bohr orbit (supposed circular) is equal to the energy of the scattered quantum plus the kinetic energy of the recoil electron after removal from the Bohr orbit and before it has escaped from the electrostatic field of the atom; (b) the vector sum of the momenta of the primary quantum and the electron in its orbit is equal to the sum of the momenta of the scattered quantum and the recoil electron just after removal from its orbit; (c) in order for the scattering to take place according to (a) and (b) the difference between the energies of the primary and scattered quanta must at least equal the binding energy of the electron in its Bohr orbit. From these assumptions it is found that there are certain positions of the electron in its orbit from which the electron cannot be ejected by the primary quantum. In these positions the mass of the whole atom is added to that of the electron, and the change of wave-length is negligible. For MoKα rays scattered by the K electrons of C, the theory indicates that there are no unmodified rays scattered at angles less than 23°, and that the change of wave-length of the modified rays scattered at 90°, lies between .01 and .06 A. These conclusions are in general agreement with experimental results of A. H. Compton and of Clark, Duane and Stifler.

I. Introduction

The experiments of Compton[1], Ross[2], Compton and Woo[3], and others on the change of wave-length produced when x-rays are scattered, show that part of the rays are scattered with and part without change of wave-length. Compton calls the former the modified, and the latter the unmodified rays. The modified x-rays show a change of wave-length in accord with the formula given in Compton's quantum theory of scattering,[4]

$$\lambda_\phi - \lambda_0 = h(1-\cos\phi)/mc, \tag{1}$$

where ϕ is the angle of scattering. The unmodified scattered rays are however not in agreement with this theory. Compton in a later paper[5] suggests that the existence of the unmodified rays may be explained on one of two hypotheses. The first hypothesis is that the binding energy

[1] A. H. Compton, Phys. Rev. **22**, 408 (1923)
[2] P. A. Ross, Phys. Rev. **22**, 201 and 524 (1923)
[3] Compton and Woo, Proc. Nat. Acad. Sci. **10**, 271 (1924)
[4] A. H. Compton, Phys. Rev. **21**, 483 (1923)
[5] A. H. Compton, Phil. Mag. **46**, 897 (1923)

of the electron is such that the energy imparted to the electron during the process of scattering is insufficient to eject it from the atom; while the second hypothesis is that the primary quantum spreads itself over several electrons and that the quantum is therefore scattered by a group of electrons, the group having a mass of several times that of a single electron. Compton[5] favors the second hypothesis. It seems peculiar however that electrons should sometimes act singly and at other times in groups. Further if there is any truth in the second hypothesis then there should be scattering by groups of two, three, and more electrons. These groups would give one-half, one-third, and so on, respectively, of the change of wave-length predicted by Eq. (1). There is however no experimental evidence that there are modified lines between the unmodified line and the modified line corresponding to Eq. (1). It appears therefore that the second hypothesis is untenable.

Another fact which is unexplained by Compton's simple theory is that the modified spectrum line always has a width greater than that of the unmodified spectrum line and also than that of the primary spectrum line. Some of the excess width of the modified spectrum line is due to the variation of ϕ in Eq. (1). However this explains only a small part of the excess width. In a recent paper[6] the writer has been able to explain qualitatively this width by taking account of the motion of the electrons in their Bohr orbits in the atoms of the scattering substance. The change of wave-length for a given scattering angle then varies with the position of the electron in its Bohr orbit at the time of being hit by the primary x-ray quantum. This suggests that there may be certain positions of the electron in its orbit from which the impinging x-ray corpuscle or quantum may be unable to eject it. In this case the mass of the whole atom would be added to that of the electron. This would cause m in Eq. (1) to be large and therefore the change of wave-length to be negligible. By slightly altering the assumptions given in the paper on the width of the modified spectrum line[6] it is possible to explain the existence of the unmodified rays and also the width of the modified spectrum line. The theory will now be developed.

II. Theory

Assumption I.—The energy of the primary x-ray corpuscle plus the kinetic energy of the electron in its Bohr orbit immediately before impact is equal to the energy of the scattered corpuscle plus the kinetic energy of the recoil electron immediately after impact and before the

[6] G. E. M. Jauncey, Phil. Mag. in print, and Phys. Rev. **24,** 204 (1924)

recoil electron has moved an appreciable distance from the position in the Bohr orbit at which it was struck by the primary x-ray corpuscle.

Let us suppose for simplicity that the Bohr orbit of the electron in the atom is circular. The writer has shown[6] that the momentum p of the electron in a circular orbit is given by

$$p=(mc\cdot\sqrt{2\alpha_s-\alpha_s^2})/(1-\alpha_s) \qquad (2)$$

where $\alpha_s=h/mc\lambda_s$ and λ_s is a critical absorption wave-length of the scattering substance. Also it may be shown that the kinetic energy K of the electron in a circular Bohr orbit is given by

$$K=mc^2\alpha_s/(1-\alpha_s)\ . \qquad (3)$$

Applying assumption I we therefore have

$$hc/\lambda_0+K=hc/\lambda_\phi+mc^2\{1/\sqrt{1-\beta^2}-1\}\ , \qquad (4)$$

where λ_0 is the wave-length of the primary x-rays, λ_ϕ that of the scattered x-rays, and βc is the velocity of the recoil electron immediately after being removed from its Bohr orbit. It should be noted that βc is not the velocity of the electron after it escapes from the electrostatic field of the atom.

Assumption II.—The vector sum of the momentum of the primary x-ray corpuscle and that of the electron in its Bohr orbit before impact is equal to the vector sum of the momentum of the scattered x-ray corpuscle and that of the recoil electron immediately after impact and before it escapes from the electrostatic field of the atom.

Referring to Fig. 1, the direction of the primary x-rays is along OX; OA represents the momentum of the primary x-ray corpuscle; OB represents the momentum of the scattered x-ray corpuscle; AC that of the electron before impact, and BC that of the electron after impact. OB is in the plane XOY, while BC and AC are not necessarily in this plane; AD is the projection of AC on the plane XOZ; ψ and θ are the angles CAD and XAD respectively, and ϕ is the angle of scattering.

We now find BC^2 which represents $m^2\beta^2c^2/(1-\beta^2)$, the square of the momentum of the recoil electron, and we have, after dividing by m^2c^2,

$$\beta^2/(1-\beta^2)=\alpha_0^2+\alpha_\phi^2+b^2+2\alpha_0\, b\cos\psi\cos\theta-2\alpha_0\alpha_\phi\cos\phi$$
$$-2\alpha_\phi\, b\cos\psi\cos\phi\cos\theta-2\alpha_\phi\, b\sin\psi\sin\phi\ , \qquad (5)$$

where $\alpha_0=h/mc\lambda_0$, $\alpha_\phi=h/mc\lambda_\phi$ and $b=p/mc$.

From Eq. (4) we also have, after dividing throughout by m^2c^4,

$$\beta^2/(1-\beta^2)=\alpha_0^2+\alpha_\phi^2+B^2+2\alpha_0+2B-2\alpha_\phi+2\alpha_0 B-2\alpha_0\alpha_\phi-2\alpha_\phi B \qquad (6)$$

where $B=K/mc^2$. By equating the right hand sides of Eqs. (5) and (6) we may solve for α_ϕ. However since $\alpha_\phi/\alpha_0=\lambda_0/\lambda_\phi$ we shall write the solution in the form,

$$l=\frac{u-v(\cos\psi\cos\theta\cos\phi+\sin\psi\sin\phi)}{1-v\cos\psi\cos\theta} \tag{7}$$

where $l=\lambda_\phi/\lambda_0$, $u=1+\alpha_0(1-\alpha_s)\operatorname{vers}\phi$, $v=\sqrt{2\alpha_s-\alpha_s^2}$.

For circular orbits the length of AC (Fig. 1) is constant and therefore the locus of C is a sphere. In Eq. (7) u and v are constants for a given value of the scattering angle ϕ. We may take v as representing the length of AC; then the value of l is a function of the angles ψ and θ which give

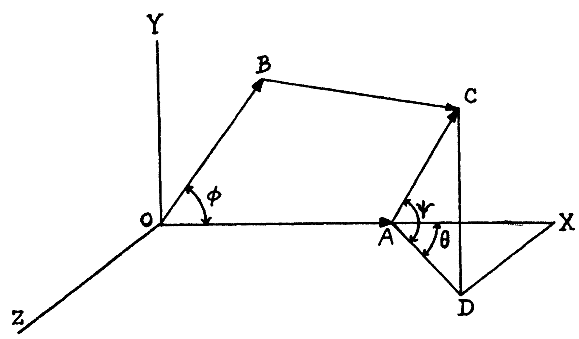

Fig. 1.

the direction of AC. Let us move the origin of coordinates to A; then, if x, y and z are the coordinates of C, we have $v\cos\psi\cos\theta=x$ and $v\sin\psi=y$ and hence, substituting in Eq. (7),

$$l=\frac{u-x\cos\phi-y\sin\phi}{1-x}. \tag{8}$$

For a given value of ϕ, we have from Eq. (8) that the surface $l=$constant is a plane. Further, since there is no term in z in Eq. (8), the plane $l=$constant is parallel to the z axis. Hence the locus of C for $l=$constant on the sphere $x^2+y^2+z^2=v^2$ is the circle where the plane $l=$constant cuts the sphere. The maximum and minimum values of l, and therefore of $\lambda_\phi(\phi=\text{constant})$, are for those values of l which cause two of the planes given by Eq. (8) to touch the sphere. Since these planes are parallel to the z axis the problem reduces to one of two dimensions. The condition that the straight line, Eq. (8), touches the circle $x^2+y^2=v^2$ is that

$$l^2(1-v^2)-2l(u-v^2\cos\phi)+u^2-v^2=0 \tag{9}$$

Solving Eq. (9) for l we have

$$l=\frac{u-v^2\cos\phi\pm v\sqrt{1+u^2-2u\cos\phi-v^2\sin^2\phi}}{1-v^2}. \tag{10}$$

Let l_1 be the maximum value of l as given by Eq. (10) and l_2 the minimum value. Then the maximum change of wave-length on scattering for a given value of ϕ is $\lambda_\phi-\lambda_0=(l_1-1)\lambda_0$, while the minimum change of wave-length is $\lambda_\phi-\lambda_0=(l_2-1)\lambda_0$. On our first and second assumptions the change of wave-length of the scattered x-rays may have any value between these extreme values. So far we have not considered the binding energy of the electron except in so far as we have used it to obtain the momentum of the electron before scattering in its Bohr orbit. Now, however, we take account of the binding energy in the following assumption.

Assumption III.—In order that the scattering process may take place according to Assumptions I and II the kinetic energy of the electron immediately after being hit must be great enough to overcome the binding energy of the electron in its Bohr orbit.

This assumption restricts the possible values of $\lambda_\phi-\lambda_0$. The least possible value of λ_ϕ is such that the energy of the primary x-ray corpuscle equals that of the scattered x-ray corpuscle plus the binding energy of the electron, or

$$hc/\lambda_0=hc/\lambda_\phi+hc/\lambda_s\,. \tag{11}$$

This gives the least possible value of $\lambda_\phi-\lambda_0$ as equal to $\lambda_0^2/(\lambda_s-\lambda_0)$. Let the least possible value of l be designated by l_3; then

$$l_3=\lambda_s/(\lambda_s-\lambda_0)\,. \tag{12}$$

For those directions of AC (Fig. 1) which make $l<l_3$, the electron is not ejected from its Bohr orbit and the mass of the whole atom is added to that of the electron so that scattering takes place without change of wave-length. This type of scattering occurs for all values of the scattering angle ϕ for which $l_2<l_3$.

The number of x-ray corpuscles scattered at a given angle ϕ for which l is between the values l and $l+dl$ is proportional to the area on the sphere $x^2+y^2+z^2=v^2$ between the planes (see Eq. 8) $l=l$ and $l=l+dl$. This area is proportional to the difference between the perpendiculars from A (Fig. 1) on the planes $l=l$ and $l=l+dl$. The perpendicular from A on the plane $l=l$ is

$$P=(l-u)/\sqrt{1-2l\cos\phi+l^2}\,. \tag{13}$$

The area on the sphere between $l=l_1$ and $l=l_3$ is

$$A_{13}=k\int_{l_3}^{l_1}dP=k\Big[P\Big]_{l=l_3}^{l=l_1}\,, \tag{14}$$

where k is a constant of proportionality. Similarly the area on the sphere between $l=l_3$ and $l=l_2$ is

$$A_{32}=k\left[P\right]_{l=l_2}^{l=l_3} \tag{15}$$

The ratio of A_{13} to A_{32} gives the ratio of the number of modified to that of the unmodified x-ray corpuscles scattered at a given angle ϕ. However each of the modified x-ray corpuscles has less energy than an unmodified corpuscle. As an approximation we may take the average value of l for the modified corpuscles as being the arithmetic mean of l_1 and l_3 so that on the average a modified corpuscle has an energy of $2/(l_1+l_3)$ times that of an unmodified corpuscle. The ratio of the energy of the modified rays to that of the unmodified rays scattered at a given angle ϕ is then

$$E_{mod}/E_{unmod}=2A_{13}/(l_1+l_3)A_{32}\ , \tag{16}$$

l of course having the value unity for the unmodified rays.

The intensity of a certain wave-length may be defined as being measured by $dE/d\lambda_\phi$ or by dE/dl. The energy dE is proportional to dP/l, since dP is proportional to the area on the sphere between $l=l$ and $l=l+dl$ from which the modified x-ray corpuscles come, and l is inversely proportional to the energy of each corpuscle. Hence the intensity of the wave-length λ_ϕ is given by $I=\text{constant}\times(dP/dl)/l$.
Hence from Eq. (13) we have

$$I=\text{constant}\times\frac{1-(l+u)\cos\phi+lu}{l(1-2l\cos\phi+l^2)^{3/2}}\ . \tag{17}$$

III. Application to Experiment

For molybdenum Kα x-rays scattered by the K electrons of carbon we have $\lambda_0=0.71$ A and $\lambda_s=45$ A, making $\alpha_0=0.034$ and $\alpha_s=0.00054$. In this case we have α_0 fairly small and α_s very small so that Eq. (10) may be written in the approximate form

$$l=1+\alpha_0\ \text{vers}\ \phi\pm 2\sqrt{2\alpha_s}\sin\tfrac{1}{2}\varphi\ . \tag{18}$$

Hence for scattering at 90° by the K electrons of carbon $l=1.034\pm0.046$. Now the least possible value of l other than unity according to assumption III is $\lambda_s/(\lambda_s-\lambda_0)$ or 1.016. Hence $l_1=1.080$, $l_2=0.988$, and $l_3=1.016$. There is in this case therefore both a modified and an unmodified ray. Substituting values in Eq. (16) we have the ratio of the energies equal to 2.2. A curve is given in Fig. 2 showing the variation of the ratio E_{mod}/E_{unmod} with the scattering angle ϕ. It is of interest to note that the ratio becomes zero at $\phi=23°$, hence for $\phi<23°$ there are no modified rays. Above 23° the ratio has a finite value and becomes larger as ϕ increases to 180°. This is qualitatively in accord with Compton's curves.[1]

Giving l in Eq. (17) values between 1.016 and 1.080 and putting $\phi=90°$ we may plot the curve between I and λ_ϕ since l is proportional to λ_ϕ. This plot is represented by the full curve of Fig. 3. It is seen that the intensity is greatest for the smallest possible wave-length of the modified x-rays. The full curve of course could only be realized experi-

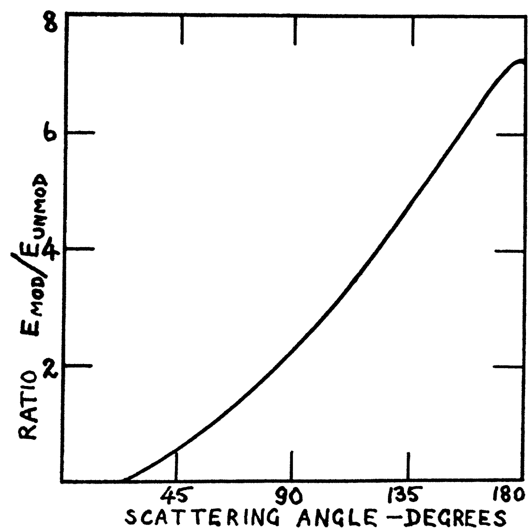

Fig. 2. Ratio of the energies in the modified and unmodified lines for varying angles.

mentally if the x-ray spectrometer had infinite resolving power, and also if the reflecting power of the crystal were the same for all wave-lengths. With an actual spectrometer the experimental curve would be more like the broken curve in Fig. 3. It is interesting to note that the theoretical curve begins at a change of wave-length equal to $\lambda_0^2/(\lambda_s-\lambda_0)$, which has been noted experimentally by Clark, Duane and Stifler[7] and has been explained by Duane on the basis of tertiary radiation.

In a previous paper[6] the formula for the width of the modified spectrum line was given as $\delta\lambda_\phi=4\lambda_0\sqrt{2a_s}\sin\frac{1}{2}\phi$. This formula is valid for small values of a_s if $l_2>l_3$. If however $l_3>l_2$ then the width is smaller and is given by

$$\delta\lambda_\phi=(l_1-l_3)\lambda_0\ . \tag{19}$$

[7] Clark, Duane and Stifler, Proc. Nat. Acad. Sci. **10**, 148 (1924)

The theoretical width of the modified line when MoKα x-rays are scattered at 90° by the K electrons of carbon is seen from Fig. 3 to be 0.045 A.

In conclusion it should be noted that the formulas developed in this paper must be taken only as approximations. Experimental curves of I against $\lambda_\phi - \lambda_0$ are made up of curves for the scattering from the L, M etc., electrons as well as the K electrons. Then, as regards the L, M etc., electrons, some of these move in elliptic orbits, whereas we have considered only circular orbits. Then again for assumptions I and II to have any basis in fact, the time of action between the primary x-ray

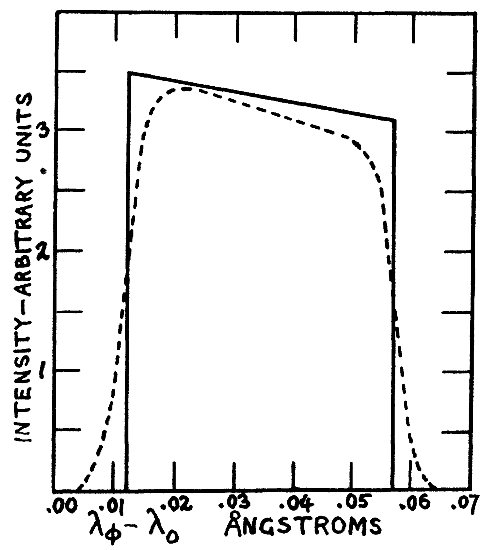

Fig. 3. Scattering of Mo Kα x-rays by K electrons of C at 90°.

corpuscle and the electron must be small compared with the period of revolution of the electron in its Bohr orbit. It would not be surprising therefore if the formulas deviated considerably from the experimental facts when applied to scattering from other than light elements. However in spite of the approximate nature of the paper the writer has been able to show the possibility of explaining the existence of the unmodified x-rays on a corpuscular theory and without having to resort to interference produced by one quantum being scattered by a group of electrons as suggested by Compton.[5]

WASHINGTON UNIVERSITY, ST. LOUIS,
November 29, 1924.