ON THE PARTIALS OF A PIANOFORTE STRING STRUCK BY AN ELASTIC HAMMER

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Abstract

Dynamics of a stretched string struck by an elastic hammer.—Neglecting small terms in the partial differential equation of motion, a solution is obtained involving the mass, elasticity and impact speed of the hammer and the distance of impact a from the nearer fixed end. Also approximate expressions are obtained for the pressure exerted on the string and for the duration of impact. The final solution for the displacement contains besides the Helmholtz term, three others. For a given a, the Helmholtz term disappears for partials with periods 2, 2/3, 2/5, etc., but not the other terms; this explains the observed existence of these partials with feeble intensity. The equation shows how the relative strengths of the partials depend on the various factors. If the duration of impact is half the period of vibration of a component, that component will become very prominent. For a given velocity of impact, the resulting amplitude increases with a.

Introduction

HELMHOLTZ built up his classical theory[1] supposing that the duration of impact was small compared to the period of vibration of the string, that it was not displaced appreciably from its equilibrium position, and that the hammer was elastic. Consequently he found that the duration of impact was independent of the striking point. Kaufmann in the year 1895 showed however that the duration of impact was not independent of the striking point, and gave an approximate theory for the calculation of the duration of impact when the striking distance was small, on the supposition (1) that the string was appreciably displaced from its equilibrium position, and (2) that the hammer was perfectly hard. The theory has since been extended by Raman and Banerji,[3] and also by P. Das[4] for the calculation of the duration of impact at any point of the string.

Recently it has been shown[5] that the observed values of the duration of impact do not agree with those calculated from Kaufmann's theory, at least in the middle octaves of the piano where the hammer is elastic. In the higher octaves where the hammer is almost hard Kaufmann's theory is partly true. Further it has been shown that his theory can be easily extended to the case of an elastic hammer.

[1] Helmholtz, Sensation of Tone (Ellis trans.) page 380
[2] Kaufmann, Ann. der Physik, **54**, 675 (1895)
[3] Raman and Banerji, Proc. Roy Soc. **97**, 99
[4] P. Das, Proc. Indian Asso. Calcutta **7**, 13
[5] Ghosh, Indian Science Congress Jan. 1922

In the present paper an attempt is made to point out the differences in the relative strength of the partials in the cases stated above. The main results are, (1) that the partials whose periods are 2/3, 2/5, 2/7, etc. of the duration of impact do not completely disappear; (2) that the strengths depend upon the striking point; and (3) that certain undesirable partials may become prominent under some circumstances—depending upon the striking point, the elasticity and the mass of the hammer.

Law of Motion of the String

Let: l = length of the string;

a = the striking distance from the nearer fixed end;

M = the effective mass of hammer;

μ = the strength of the elastic spring of the hammer felt;

$c=\sqrt{T/\rho}$ = velocity of transverse wave in the string;

y = the displacement of the string;

y_0 = the displacement of the center of mass of the hammer;

ξ = the compression of the hammer felt.

The equation of motion of the hammer is given by

$$Md^2y_0/dt^2=-p \tag{1}$$

where p represents the pressure upon the hammer.

$$y_0=y+\xi\,,\quad \text{and} \quad p=\mu\xi \tag{2}$$

and

$$T\left\{\frac{y}{a}+\frac{1}{c}\frac{dy}{dt}\right\}-p=\rho\frac{d^2y}{dt^2}\delta s \ \text{(approx.)}. \tag{3}$$

Eq. (3) is derived on the assumption that the small part a of the string included between the striking point and the fixed end, follows the motion of the hammer, i.e. the wave takes very little time to reach the end from the striking point. Eliminating p from (1) (2) and (3) we obtain

$$\frac{M\rho\delta s}{\mu}\frac{d^2y}{dt^2}+\frac{T}{\mu c}\frac{d^3y}{dt^3}+\frac{d^4y}{dt^4}\left\{M+\frac{MT}{\mu a}-\rho\delta s\right\}+\frac{T}{c}\frac{dy}{dt}+\frac{Ty}{a}=0\,. \tag{4}$$

A complete solution of (4) is difficult and even if it were obtained, it would be too complicated to be of any practical value; hence we shall neglect the first two terms which are very small compared to the others; then (4) reduces to

$$M\left(1+\frac{T}{\mu a}\right)\frac{d^2y}{dt^2}+\frac{T}{c}\frac{dy}{dt}+\frac{Ty}{a}=0\,. \tag{5}$$

Since T/c is small, the solution of (5) is obtained in the form

$$y = Ae^{-\frac{1}{2}kt}\sin qt \tag{6}$$

$$k = \frac{T}{Mc}\left(\frac{\mu a}{T+\mu a}\right) \tag{7}$$

$$q^2 = \frac{T}{aM}\left(\frac{\mu a}{T+\mu a}\right) - \tfrac{1}{4}k^2 . \tag{8}$$

If v_0 is the velocity of impact of the hammer at $t=0$, then A is given by

$$A = \frac{v_0}{q}\left\{1+\frac{T}{\mu}\left(\frac{1}{a}-\frac{k}{2c}\right)\right\}^{-1} . \tag{9}$$

When the hammer is perfectly hard, μ becomes infinite; then k becomes independent of the striking distance and q^2 comes out

$$q_1^2 = T/aM - \tfrac{1}{4}(T/Mc)^2. \tag{10}$$

The duration of impact is approximatly equal to π/q. It is observed from (8) and (10) that the duration of impact in the case of an elastic hammer is greater than with a hard hammer. In both cases however it increases with increasing distance of the striking point from an end.

The amplitude of the resulting motion of the string is given by Eq. (9). It is proportional to the impinging velocity of the hammer and increases as a increases.

The Force on the String

From (3) and (6) p is easily found to be

$$p = ATe^{-\frac{1}{2}kt}[\sin qt/a + (R/c)\cos(qt+\phi)] \tag{11}$$

where $R = \sqrt{q^2+k^2/4}$ and $\tan\phi = k/2q$.

From (11) it is observed that the pressure on the string is not zero at the beginning of the impact but is equal to TAq/c, and at the end of the interval π/q its magnitude is again the same; it becomes zero after a short interval δt obtained by putting $t = (\pi/q + \delta t)$ in Eq. (11). The fact that the pressure is not zero at the beginning of the impact is also confirmed by Raman's curves[6] where the hammer is considered to be perfectly hard. The interval δt however is very small; it is approximately given by a/c. Hence we shall for simplicity consider the interval π/q to be the duration of impact. This amounts to neglecting the second term in Eq. (11). Then p is given by

$$p = (AT/a)e^{-\frac{1}{2}kt}\sin qt. \tag{12}$$

[6] Raman and Banerji, loc. cit.[3], Figs. 1 and 2

Physically the neglect of the term $(R/c)\cos(qt+\phi)$ amounts to the assumption that the disturbance has traveled a long distance along the string but has not yet been reflected from the farther end, i.e. the duration of impact is smaller than the time taken by the wave to reach the farther end of the string; so that if we continually increase α, our approximations will be sufficient as long as the above condition is satisfied.

Calculation of the Partials

The equation of motion of a point on the string in terms of normal co-ordinates is given by[7]

$$\frac{d^2\phi_s}{dt^2}+\left(\frac{s\pi c}{l}\right)^2\phi_s=\frac{2}{l\rho}\Phi_s \tag{13}$$

where Φ is the generalised component of force, and

$$\Phi_s=A\sin(s\pi a/l)e^{-\frac{1}{2}kt'}\sin qt' .$$

If ϕ_s and $d\phi_s/dt$ are both zero initially then for $t>\pi/q$

$$\phi_s=\frac{2A}{nl\rho}\sin\frac{\pi s a}{l}\int_0^{\pi/q}e^{-\frac{1}{2}kt'}\sin n(t-t')\sin qt'dt$$

$$=\frac{2A}{nl\rho}\sin\frac{s\pi a}{l}\left[\frac{2q\cos(nq/2\pi)\sin n(t-\pi/2q)}{q^2-n^2}\right. \tag{14}$$

$$-\frac{\frac{1}{2}k\sin(n\pi/2q)\cos n(t+\pi/2q)}{(q-n)2}+\frac{k\pi\sin n(t+\pi/q)}{4q(q-n)}$$

$$\left.-\frac{k\pi\sin n(t-\pi/q)}{4q(q+n)}\right]\text{(approx.)} \tag{15}$$

where $n=s\pi c/l$.

The final solution for the displacement y becomes, when we substitute the value of n, to the same order of approximation,

$$y=\frac{2Al^2}{\rho\pi c}\sum_1^\infty\frac{\sin(s\pi a/l)\sin(s\pi c/l)}{s(lq-s\pi c)}\left\{\frac{2q\cos(s\pi^2c/2lq)\sin(s\pi c/l)(t-\pi/2q)}{(lq+s\pi c)}\right.$$

$$-\frac{\frac{1}{2}k\sin(s\pi^2c/2lq)\cos\{s\pi c/l(t+\pi/2q)\}}{(lq-s\pi c)}+\frac{\pi k}{4ql}\sin\frac{s\pi c}{l}(t+\pi/q)$$

$$\left.-\frac{k\pi\sin(s\pi c/l)(t-\pi/q)}{4ql(lq+s\pi c)/(lq-s\pi c)}\right\} \tag{16}$$

[7] Rayleigh, Theory of Sound, Vol 1, Art. 128

Equation (15) shows besides the Helmholtz term three other terms which appear on account of the finite displacement of the string, but on account of the presence of k their magnitudes are small. The convergence of the first two terms is of the order s^{-3}, but the last two terms converge slowly. They increase if k increases, that is when the striking distance increases. This partly explains the choice of the striking point which is about 1/7th the length of the string. Whenever the striking distance is large, the higher partials become prominent and disturb the musical quality of the note.

For a given striking distance the first term disappears for those partials whose periods are 2/1, 2/3, 2/5, 2/7 etc. of the duration of impact, but the other three terms do not, hence their existence with feeble intensity is explained. When $\sin(s\pi\alpha/l)=0$ (i.e. the point struck is the first node of the sth component) then $\alpha=l/s$, and the component will be absent; but if at the same time the duration of impact $\pi/q=l/cs$, then $\sin(s\pi^2c/2lq=1$ and $\cos(s\pi^2c/2lq)=0$, and the first term and the last two terms disappear, but the second term does not and its magnitude is then approximately given by $k/2s$ which is small and decreases with the order of the component. So we find that a component having a node at the point struck is not entirely absent if the duration of impact is half the period of vibration of the component. Hipkins and others have found this to be the case experimentally.[8]

A given component will converge more rapidly the greater the value of q, which increases as α decreases (see Eq. 8), hence α must be small so that the higher undesirable partial may be very feeble. It is further observed that the magnitude of the second term is inversely proportional to $(lq-s\pi c)^2$ hence when $q=s\pi c/l$, that is when the duration of impact is equal to half the period of vibration of that component, it will become very prominent.

A is not constant throughout all the length of the string for the same velocity of impact of the hammer. It changes with q and therefore increases as α the striking distance increases.

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[8] Helmholtz, loc. cit.[1] pp. 76, 77