NOTE ON ELECTRON THEORY OF THE HALL EFFECT

By Leigh Page

Abstract

Eldridge's conclusion that a *constant-free-path electron theory* leads to a zero Hall effect is shown to be due to an error in approximation. The constant-free-path theory is found to lead to a normal (negative) Hall effect and to a decrease in resistance in both magnetic and electric fields. Therefore though it seems more rational than the usual theory, it is no more successful in explaining the experimental results.

In a recent paper[1] Eldridge has given a theory of the Hall effect in which he has supposed that the lengths of the free paths of the electrons in a metal remain unaltered when a magnetic field is applied. This assumption certainly seems more natural than the usual hypothesis that the free times between collisions are unaffected by the presence of a field. Eldridge concludes that if his assumption is accepted, together with the hypothesis that the initial velocities of electrons of the same free path are distributed uniformly in direction, the simple electron theory predicts a zero Hall coefficient in an isotropic conductor. Unfortunately he seems to have made an error in approximating which quite vitiates his conclusion. The error occurs in his expression (2) for the free time T in the presence of the field in terms of the free time T_0 existing when no electric or magnetic field is at hand. For this expression, when carried to a higher order of approximation, is found to contain a term involving the product of the electric and magnetic fields which, when multiplied by the components of the initial velocities of the electrons in the direction at right angles to the current, gives rise to terms of just the order under investigation and leads to a Hall effect of the same sign as that afforded by the simplest form of the electron theory.

In the following analysis of the motion of an electron in mutually perpendicular electric and magnetic fields we adopt Eldridge's assumption that the length of the curved path described in the presence of the fields is the same as that of the straight path which would have been followed in their absence. The analysis is carried through terms of the third order in the products of the field strengths to determine whether the present theory is any more successful than the constant-free-time theory in predicting changes in resistance.

[1] Eldridge, Phys. Rev. **21**, 131 (1923)

Let the magnetic field H have the direction of the Z axis, and the electric field E lie in the XY plane. Then if e_0 is the charge and m the mass of the electron, and $e\equiv(e_0/m)E$, $h\equiv(e_0/mc)H$, the equations of motion of the electron are

$$\frac{d^2x}{dt^2}=e_x+h\frac{dy}{dt}\,,\quad \frac{d^2y}{dt^2}=e_y-h\frac{dx}{dt}\,,\quad \frac{d^2z}{dt^2}=0\,,$$

of which the solutions are

$$x=\left(\frac{v_{0y}}{h}+\frac{e_x}{h^2}\right)(1-\cos ht)+\left(\frac{v_{0x}}{h}-\frac{e_y}{h^2}\right)\sin ht+\frac{e_y}{h}t\,, \tag{1}$$

$$y=-\left(\frac{v_{0x}}{h}-\frac{e_y}{h^2}\right)(1-\cos ht)+\left(\frac{v_{0y}}{h}+\frac{e_x}{h^2}\right)\sin ht-\frac{e_x}{h}t\,, \tag{2}$$

$$z=v_{0z}\,t\,, \tag{3}$$

where v_{0x}, v_{0y}, v_{0z} are the three components of the initial velocity v_0. Expanding the trigonometrical terms in series, and retaining all terms through the third order in e and h,

$$x=v_{0x}t+\tfrac{1}{2}(e_x+hv_{0y})t^2+\tfrac{1}{6}(e_yh-h^2v_{0x})t^3-\tfrac{1}{24}(e_xh^2+h^3v_{0y})t^4\,\ldots\,, \tag{1'}$$

$$y=v_{0y}t+\tfrac{1}{2}(e_y-hv_{0x})t^2-\tfrac{1}{6}(e_xh+h^2v_{0y})t^3-\tfrac{1}{24}(e_yh^2-h^3v_{0x})t^4\,\ldots, \tag{2'}$$

$$z=v_{0z}t. \tag{3'}$$

Differentiating and adding dx^2, dy^2 and dz^2, the element of path $d\lambda$ is found to be

$$d\lambda=v_0dt\left[1+e_p\frac{t}{v_0}+\tfrac{1}{2}(e^2-e_p{}^2+\mathbf{h}\times\mathbf{e}\cdot\mathbf{v}_0)\frac{t^2}{v_0{}^2}-\frac{1}{6}e_p\,(3e^2-3e_p{}^2+3\mathbf{h}\times\mathbf{e}\cdot\mathbf{v}_0+h^2v_0{}^2)\frac{t^3}{v_0{}^3}\cdots\right], \tag{4}$$

where e_p is the component of e parallel to v_0.

To find the free time this equation must be integrated and solved for t. Putting t_0 for the free time λ/v_0 which would exist in the absence of both fields, it is found that

$$t=t_0\left[1-\tfrac{1}{2}e_p\frac{t_0}{v_0}-\frac{1}{6}(e^2-4e_p{}^2+\mathbf{h}\times\mathbf{e}\cdot\mathbf{v}_0)\frac{t_0{}^2}{v_0{}^2}+\frac{1}{24}e_p\,(13e^2-28e_p{}^2+13\mathbf{h}\times\mathbf{e}\cdot\mathbf{v}_0+h^2v_0{}^2)\frac{t_0{}^3}{v_0{}^3}\cdots\right]. \tag{5}$$

The second term differs in sign from that in the corresponding expression obtained by Eldridge because he has denoted the charge on the electron by $-e$. The third term, however, Eldridge failed to include, although it is of the order of the effect which he is investigating.

Substituting the free time (5) in (1′) and (2′), and calculating the currents in the X and Y directions due to the n electrons of free path λ which start out per unit volume per unit time after suffering a collision,

$$i_x=\frac{ne_0^2t_0^2}{3m}\left[E_x+\frac{1}{6}\frac{e_0t_0}{mc}HE_y+\frac{2}{15}\frac{e_0^2t_0^2}{m^2v_0^2}E^2E_x\right], \tag{6}$$

$$i_y=\frac{ne_0^2t_0^2}{3m}\left[E_y-\frac{1}{6}\frac{e_0t_0}{mc}HE_x+\frac{2}{15}\frac{e_0^2t_0^2}{m^2v_0^2}E^2E_y\right]. \tag{7}$$

If the current is in the X direction, there will be a potential gradient in the Y direction given by

$$E_y=\frac{1}{6}\frac{e_0t_0}{mc}HE_x\ ,$$

which has the sign of the normal Hall effect given by the elementary theory. Putting this value of E_y back in (6), the increase in resistance per unit resistance is seen to be

$$\frac{\delta R}{R}=-\frac{1}{36}\frac{e_0^2t_0^2}{m^2c^2}H^2-\frac{2}{15}\frac{e_0^2t_0^2}{m^2v_0^2}E^2\ . \tag{8}$$

Hence the theory predicts a decrease of reistance in either a magnetic or an electric field. The experiments of Patterson[2] show that the resistance is *increased* by a magnetic field, and those of Bridgman[3] on the deviation from Ohm's law at high current densities show the same effect in the case of an electric field. Therefore the theory is as unsuccessful in accounting for resistance changes as in explaining the positive Hall coefficient which exists in the case of many metals.

The constant-free-time theory is equally unable to explain the positive Hall coefficient and as van Everdingen[4] has shown, unless extraneous assumptions are introduced, it leads to the wrong sign for the change of resistance when a conductor is placed in a transverse magnetic field. A simple calculation shows that it leads to a zero second order change in resistance in the case of an electric field.

If Eldridge's hypothesis is modified to make the *chord* joining the ends of the curved path described by the electrons in crossed electric and magnetic fields equal to the free path which would be followed in the absence of the fields, instead of the length of the *arc*, the theory is in no wise improved. While the numerical coefficients are somewhat changed, the signs of all terms remain the same, both in the expression for the Hall effect and in that for the change in resistance.

SLOANE PHYSICS LABORATORY,
YALE UNIVERSITY,
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[2] Patterson, Phil. Mag. **3,** 643 (1902)
[3] Bridgman, Am. Acad. Arts and Sci. **57,** 131 (1922)
[4] van Everdingen, Comm. Phys. Lab. Leiden, 1902, p. 72