THE SETTLING OF SMALL PARTICLES IN A FLUID

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Abstract

Settling of small particles in a fluid; mathematical theory.—Small particles immersed in a liquid experience a motion which is the combination of a steady gravitational drift and a Brownian movement. If there are space variations in the density of distribution of particles, the Brownian movement produces a diffusion which tends to equalize the density. In the steady state the density n of particles is an exponential function of x, the distance below the surface of the liquid. This paper investigates the *manner in which the steady state is established.* A consideration of the combined effect of fall and diffusion leads to a partial differential equation for the number density of particles as a function of depth and time. A set of special solutions is obtained in terms of which a solution satisfying initial and boundary conditions can be expressed. (1) *Liquid of finite depth.* The solution is obtained for a liquid of finite depth with an arbitrary initial distribution $n_0=f(x)$. For the case of uniform initial distribution a reduced form of the solution is obtained which contains a single parameter. This one parameter family of curves is plotted, and from these curves, either directly or by interpolation, may be obtained the density distribution at any time for a solution of any depth, density, and viscosity, and for particles of any size and density. For small values of t, since the solution obtained converges slowly, an image method is used to obtain an integral formula for the density. (2) *Liquid of semi-infinite or infinite depth.* In the case of a liquid of infinite depth the solution for an arbitrary initial distribution is expressed by the Fourier integral identity. The case of zero initial density for negative x, and constant initial density for positive x is calculated, as is also the case of particles initially uniformly distributed over a layer of depth h. In the case of a liquid extending from $x=0$ to $x=\infty$, the boundary conditions are satisfied by assuming a suitable fictitious initial distribution over the range from $x=-\infty$ to $x=0$. The cases of uniform initial distribution, and initial distribution over a layer, are calculated. The latter case, while derived for a liquid of semi-infinite depth, gives approximately the distribution of density during the settling of a layer of particles initially distributed uniformly over a depth h at the upper end of a very long column of liquid.

SMALL particles immersed in a liquid experience a motion which is the combination of a steady gravitational drift, the velocity of which is given by Stoke's law of fall, and a Brownian movement due to molecular bombardment. If there are variations of the density of distribution of particles, the Brownian movement produces a diffusion tending to equalize the density. If a liquid containing such particles stand undisturbed, a steady state is reached in which the density n of particles is an exponential function of x, the distance below the surface of the liquid, as has been verified experimentally by Perrin. It is the purpose of this paper to investigate the manner in which the steady state is established. A consideration of the combined effect of fall and diffusion leads to a differential equation for the number-density of particles as a function of depth and time. The solution is obtained for an arbitrary initial distribution $n_0=f(x)$, both for a liquid of finite depth, and for a liquid of infinite depth.

1. The Differential Equation for the Non-Steady Case

The coefficient of diffusion can be obtained from a consideration of the steady state.[1] Suppose a gravitational force X to act on a spherical particle of radius a. Its velocity V is then given by Stoke's law of fall as

$$6\pi\mu a V = X,$$

where μ is the coefficient of viscosity of the liquid. Due to this velocity there will cross per second downward through unit horizontal area a number of particles given by

$$Vn = nX/6\mu\pi a.$$

The density distribution in the steady state is known to be given by

$$n = n' \; e^{(N/RT)X(x-x')}$$

where N is Avagrado's number, R the gas constant for a gram molecule, T the absolute temperature, and where n' is the density of particles at the position $x=x'$. It follows that the gradient of density in the steady state is given by

$$\partial n/\partial x = (N/RT)Xn.$$

The steady state, however, is characterized by the fact that the number drifting downward through unit horizontal area is equal to the number diffusing upward through the same area. The number of particles diffusing upward must therefore be given by

$$\frac{nX}{6\pi\mu a} = \frac{RT}{N6\pi\mu a}\,\frac{\partial n}{\partial x}.$$

Thus the factor by which the rate of change of density must be multiplied to give the flow of particles due to diffusion is $RT/N6\pi\mu a$. This quantity is therefore the desired coefficient of diffusion.

In the non-steady state there is, then, a net flow, per unit area, in the positive direction of x (that is, downward) given by

$$F(x,\,t) = \frac{n(x,\,t)X}{6\pi\mu a} - \frac{RT}{N6\pi\mu a}\,\frac{\partial n(x,t)}{\partial x},$$

where $n(x,\,t)$ is the number-density of particles at the place x and the time t. The net downward flow at the same time through a unit area a

[1] Einstein, Ann. der Phys. p. 554, 1916

distance dx lower is $F+(\partial F/\partial x)dx$; so that the net gain, per unit time, of the layer of thickness dx would be $-(\partial F/\partial x)dx$. That is

$$-\frac{\partial}{\partial x}\frac{1}{6\pi\mu a}\left[nX-\frac{RT}{N}\frac{\partial n}{\partial x}\right]=\frac{\partial n}{\partial t},$$

or

$$\frac{\partial n}{\partial t}=A\frac{\partial^2 n}{\partial x^2}-B\frac{\partial n}{\partial x}, \tag{1}$$

with

$$A=\frac{RT}{N6\pi\mu a};\ B=\frac{X}{6\pi\mu a}=\frac{\frac{4}{3}\pi a^3 g\delta}{6\pi\mu a}=\frac{2g\delta a^2}{9\mu}, \tag{2}$$

where δ is the effective density of the immersed particle; that is, its density minus the density of the liquid.

No particles cross the planes $x=0$ or $x=l$ where l is the depth of the liquid. Thus the net flow into the layer between $x=0$ and $x=dx$ is given by the net flow across the plane $x=dx$. Therefore

$$dx(\overline{\partial n/\partial t})=(A\ \partial n/\partial x-Bn)_{dx},$$

where $\overline{\partial n/\partial t}$ is the average value of $\partial n/\partial t$ over the layer, and where the subscript dx on the parenthesis indicates that its value is to be given at $x=dx$. If dx is now allowed to approach zero, the equation

$$A\ \partial n/\partial x=Bn,\quad \text{for } x=0,$$

is obtained as one boundary condition. In the same manner it follows that

$$A\ \partial n/\partial x=Bn,\quad \text{for } x=l,$$

To these must be added the condition that the density reduce to the arbitrary distribution $n_0=f(x)$ for $t=0$. *The problem thus consists of finding a solution of the partial differential equation* (1)

$$\frac{\partial n}{\partial t}=A\frac{\partial^2 n}{\partial x^2}-B\frac{\partial n}{\partial x},$$

under the boundary conditions

$$A\ \partial n/\partial x=Bn,\quad \text{for } x=0 \text{ and } l, \tag{3}$$

and the initial condition[2]

$$n_0=f(x),\quad \text{for } t=0. \tag{4}$$

The problem of the temperature distribution along a rod, insulated along its sides but radiating from its ends, differs analytically from the diffusion-fall problem here considered in the absence of the second term on the right side of (1). The differential equation (1) might be reduced to

[2] The differential equation here given was obtained by Th. De Coudres in an article, Ann. der Phys. 1894, which seems to be the only one previously published on the question. He did not attempt an exact solution under the boundary conditions, but limited himself from the beginning to approximate methods, studying in this manner the case of a liquid of finite depth.

the heat equation by the substitution $x'=x-Bt$. This substitution, however, would introduce a complexity in the boundary conditions, offsetting the gain in simplicity in the differential equation itself.

2. The Solution of the Differential Equation

If a solution of (1) be assumed in the form of a product of a function X of x alone by a function T of t alone, one obtains at once

$$T'/T=(AX''-BX')/X=\text{constant}=-r,$$

or the two ordinary equations

$$T'=-Tr;\quad AX''-BX'+rX=0,$$

where r is as yet unrestricted. A solution of the first of these equations is

$$T=e^{-rt}. \tag{5}$$

The solution of the second is

$$X=e^{Bx/2A}\ (C_1 \sin \omega x+C_2 \cos \omega x), \tag{6}$$

where

$$\omega=\sqrt{4rA-B^2}/2A.$$

Solving this last equation for r in terms of the new constant ω, substituting this in (5), and taking the product of (5) and (6) one obtains

$$n(x,t)=e^{-(4A^2\omega^2+B^2)t/4A}\ e^{(B/2A)x}\ (C_1 \sin \omega x+C_2 \cos \omega x). \tag{7}$$

3. The Boundary Conditions

The substitution of (6) in the boundary conditions leads to the equations

$$C_1\ \omega A - C_2\ \tfrac{1}{2}B=0,$$

$$C_1\ (\omega A\ \cos \omega l-\tfrac{1}{2}B\ \sin \omega l)-C_2\ (\tfrac{1}{2}B \cos \omega l+\omega A\ \sin \omega l)=0. \tag{8}$$

In order that these homogeneous equations possess solutions for C_1 and C_2, other than the trivial solution $C_1=C_2=0$, it is necessary that the determinant of the coefficients vanish. That is,

$$\sin \omega l\ (\omega^2A^2+B^2/4)=0,$$

so that

$$\omega=m\pi/l,\quad m=0,\ 1,\ 2,\ \ldots, \tag{9}$$

or

$$\omega=\pm i\ B/2A. \tag{10}$$

The constant ω having one of the values given by (9) or (10), the ratio of C_1 to C_2 is given by either equation (8). If one chooses

$$C_1=C_mB,$$

then

$$C_2=C_m\ 2m\pi A/l.$$

When ω has the value (10), r is zero so that T is unity, and the corresponding special solution of (1) is

$$C_0\ e^{Bx/A}.$$

It follows that the expression

$$n = e^{-(4A^2m^2\pi^2+B^2l^2)t/4Al^2}\, e^{Bx/2A} \sum_1^\infty C_m\,[B \sin (m\pi x/l) + (2m\pi A/l)\cos(m\pi x/l)] + C_0\, e^{Bx/A}, \quad (11)$$

satisfies (1) and the boundary conditions, C_m and C_0 being arbitrary.

It remains to satisfy the initial condition. For $t=0$ Eq. (11) gives

$$n_0\, e^{-Bx/2A} = f(x)\, e^{-Bx/2A} = C_0\, e^{Bx/2A} + \sum_1^\infty C_m\,[B \sin (m\pi x/l) + (2m\pi A/l) \cos (m\pi x/l)], \quad (12)$$

and it is necessary to determine the constants C_0 and C_m so that this equation will be satisfied. That is to say, it is necessary to expand an arbitrary function

$$\psi\,(x) = f(x)\, e^{-Bx/2A},$$

in a series of characteristic functions u_m, where

$$u_m = B \sin \omega_m x + 2A\omega_m \cos \omega_m x, \quad \omega_m = m\pi/l;\ m \neq 0$$
$$u_0 = e^{Bx/2A}, \qquad w_0^2 = -B^2/4A^2.$$

It is interesting to note that in order to obtain a complete set of characteristic functions, in terms of which an arbitrary function can be expanded, it is necessary to include the function u_0, which is so radically different in character from the other characteristic functions u_m.

The functions u_i satisfy the differential equation

$$u_i'' + \omega_i^2\, u_i = 0 \qquad i = 0, 1, 2, \ldots, \quad (13)$$

and the boundary conditions

$$u_i' = B/2A\ u_i, \qquad x = 0 \text{ and } l.$$

Hence, integrating by parts,

$$\int_0^l (u_i''u_j - u_j''u_i)\,dx = [u_i'u_j - u_j'u_i]_0^l - \int_0^l (u_i'u_j' - u_j'u_i')dx = 0.$$

From equation (13) it follows however that

$$\int_0^l (u_i''u_j - u_j''u_i)dx = (\omega_j^2 - \omega_i^2)\int_0^l u_iu_j\, dx.$$

Hence the integral from 0 to l of the product u_iu_j of two characteristic functions vanishes when $i \neq j$. When $i = j$

$$\int_0^l u_i^2 dx = \tfrac{1}{2}(B^2 + 4A^2\omega_i^2), \quad i \neq 0.$$

Multiplying both sides of (12) by u_0 and integrating from 0 to l

$$C_0 \int_0^l e^{Bx/A} dx = \int_0^l f(x)dx,$$

and for the special case that the initial distribution is uniform, i.e. that $f(x) = \text{constant} = n_0$,

$$C_0 = Bn_0l/A\,(e^{Bl/A} - 1).$$

Multiplying both sides of (12) by u_m, where $m \neq 0$, and integrating from 0 to l,

$$\frac{4m^2\pi^2A^2+l^2B^2}{2l^2}C_m=\int_0^l e^{-Bx/2A}f(x)\left\{B\sin(m\pi x/l)+(2m\pi A/l)\cos(m\pi x/l\right\}dx.$$

For the case $f(x)=\text{constant}=n_0$ this equation gives

$$C_m=\frac{16A^2Bm\pi l^3}{[4A^2m^2\pi^2+B^2l^2]^2}(1\mp e^{-Bl/2A}),$$

with the upper sign holding for m even, the lower for m odd. The solution of (1) which satisfies the boundary and initial conditions is thus

$$n(x,t)=\frac{n_0\, Ble^{Bx/A}}{A(e^{Bl/A}-1)}+ \qquad (14)$$

$$+n_0\,16\,A^2B\pi l^3e^{\frac{Bx}{2A}}\sum_1^\infty\frac{e^{-\frac{(4A^2m^2\pi^2+B^2l^2)t}{4Al^2}}m(1\mp e^{-\frac{Bl}{2A}})\left[B\sin\frac{m\pi x}{l}+\frac{2m\pi A}{l}\cos\frac{m\pi x}{l}\right]}{[4A^2m^2\pi^2+B^2l^2]^2}$$

4. Reduction of the Solution

In (14) the density is expressed as a function of x and t, and of the three parameters A, B, and l. By making suitable substitutions in the differential equations and boundary conditions it is possible to obtain a reduced form of the solution, which contains a single parameter. This one parameter family of curves then furnishes the solution for any given A, B, and l. The Eqs. (1) and (3) through the substitutions

$$lx=y,\quad \frac{A}{Bl}=\alpha,\quad \frac{l}{B}=\beta,\quad \beta t'=t \qquad (15)$$

are reduced to the form

$$\frac{\partial n}{\partial t'}=\alpha\frac{\partial^2 n}{\partial y^2}-\frac{\partial n}{\partial y},$$

$$\alpha\,\partial n/\partial y=n,\qquad y=0,\qquad y=1,$$

the solution of which, corresponding to (14), is

$$\frac{n}{n_0}=\frac{e^{y/\alpha}}{\alpha(e^{1/\alpha}-1)} \qquad (14')$$

$$+16\,\alpha^2\pi e^{(2y-t')/4\alpha}\sum_1^\infty\frac{e^{-\alpha m^2\pi^2t'}\,m(1\mp e^{-1/2\alpha})\,[\sin m\pi y+2\pi m\alpha\cos m\pi y]}{(1+4\pi^2m^2\alpha^2)^2}$$

In Figs. 1, 2, 3, 4, and 5 this equation is plotted for the values $\alpha=.025$, 0.1, 0.3, 0.5, and 2.0. For each value of α, curves are drawn for $t'=0.05$ 0.25, 0.50, 1.0, and ∞, except that such of these curves as practically coincide with the steady state curve $t'=\infty$ are omitted. The ratio, in the steady state, of the density at the bottom of the liquid to the density at the top is given by $e^{\frac{1}{\alpha}}$, the values of α plotted covering a range in this ratio from 1.65 to 2.35×10^{17}.

To illustrate the use of these curves a definite example will be considered. Perrin, in his observations of the variation with height of the concentrations of Brownian particles, experimented with gamboge grains of radius 0.212μ and density 1.194. These particles were immersed in

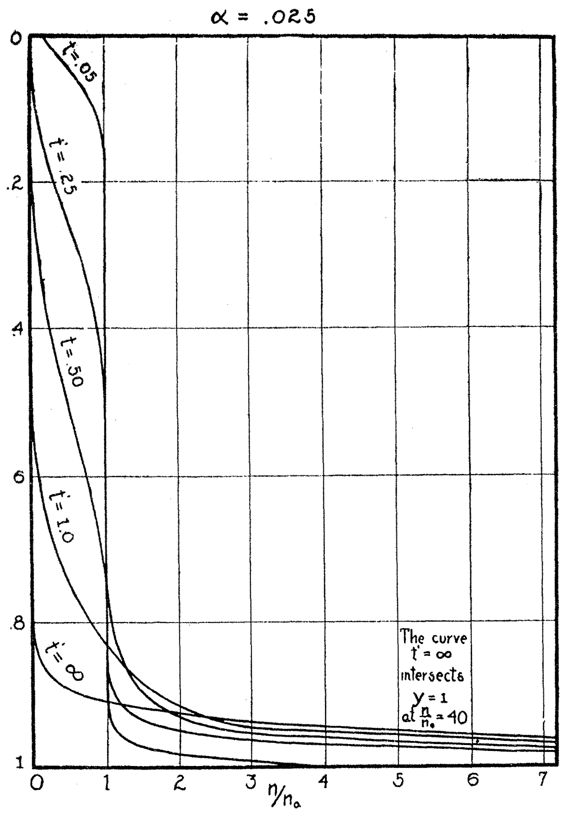

Fig. 1

water in a cell of depth 100μ. For these data it follows, from (2) and (15), that

$$A = 9.03 \times 10^{-9}, \quad B = 1.902 \times 10^{-6}, \quad l = 10,^{-2} \quad \beta = 5260, \quad \alpha = .475.$$

The curves for $\alpha = .475$ may be obtained by tracing on a single sheet the curves for $\alpha = .3$ and $\alpha = .5$ and interpolating. The result is shown in Fig. 6. In order to return to the original variables x and t it is necessary to multiply the ordinates by $l = 100\mu$, and the time t' by β. The curve $t' = 1$ or $t = 5260$ sec. sensibly coincides with the curve $t = \infty$ so that the steady state of distribution is practically established after about one and one half hours. Perrin remarks, "A few minutes suffice for the lower layers to become manifestly richer in granules than the upper layers. . . .

With the emulsions I have used, three hours are sufficient for the attainment of a well-defined limiting distribution in an emulsion left at rest, for practically the same values are found after three hours as after fifteen days." The information to be read from the curves is thus consistent with the observations of Perrin.

5. Liquid of Infinite Depth

In the case of a liquid which extends from $x=-\infty$ to $x=+\infty$ there are no boundary conditions, so that (7) is a solution for any value of ω.

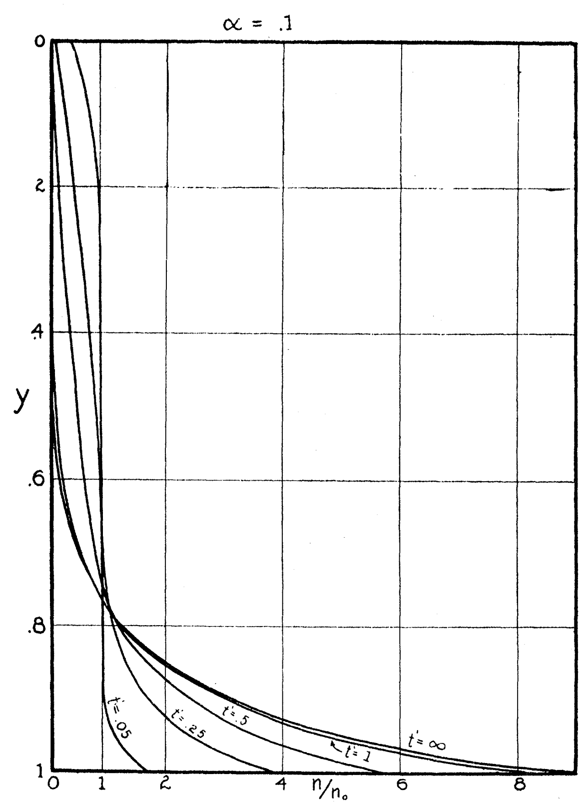

Fig. 2

The addition of all these solutions gives the solution

$$n=e^{-(4A^2\omega^2+B^2)t/4A}\ e^{Bx/2A}\int_0^\infty(C_1\sin\omega x+C_2\cos\omega x)d\omega,$$

where C_1 and C_2 are arbitrary. When $t=0$ this reduces to

$$e^{Bx/2A}\int_0^\infty(C_1\sin\omega x+C_2\cos\omega x)d\omega,$$

which, if the initial condition is to be satisfied, must be identical with the function $f(x)$. The comparion of this expression with the Fourier integral identity

$$\phi(x) = \int_0^{\infty} d\omega \int_{-\infty}^{+\infty} \phi(\gamma) \cos \omega(\gamma - x) d\gamma,$$

shows at once that the equation

$$f(x)e^{-Bx/2A} = \int_0^{\infty} (C_1 \sin \omega x + C_2 \cos \omega x) d\omega$$

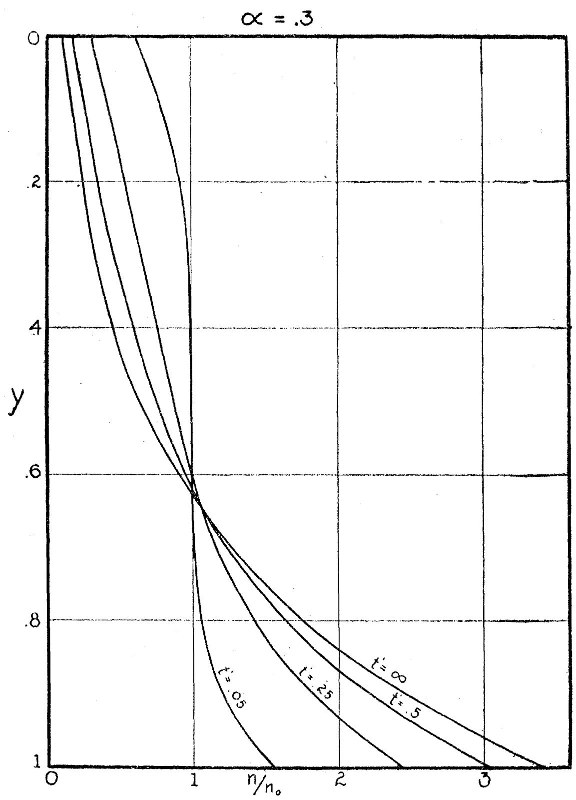

Fig. 3

is satisfied provided the constants C_1 and C_2 have the values

$$\pi C_1 = \int_{-\infty}^{+\infty} f(\gamma) e^{-B\gamma/2A} \sin \gamma\omega \, d\gamma,$$

$$\pi C_2 = \int_{-\infty}^{+\infty} f(\gamma) \, e^{-B\gamma/2A} \cos \gamma\omega \, d\gamma.$$

Thus

$$\pi n = \int_{-\infty}^{+\infty} f(\gamma) \, e^{-B\gamma/2A} d\gamma \int_0^{\infty} e^{-(4A^2\omega^2 + B^2) \cdot t/4A} \, e^{Bx/2A} \cos \omega(\gamma - x) d\omega,$$

satisfies the differential equation and the initial condition. If the integration with respect to ω be carried out this reduces to

$$n=(1/2\sqrt{A\pi t})\int_{-\infty}^{+\infty} f(\gamma)\ e^{-(\gamma-x+Bt)^2/4At}\ d\gamma. \tag{16}$$

If the initial density be zero for $x<0$, and a constant value n_0 for positive values of x, this expression takes the special form

$$n=(n_0/2\sqrt{A\pi t})\int_0^{\infty} e^{-(\gamma-x+Bt)^2/4At}\ d\gamma=\tfrac{1}{2}n_0\ [1-\Theta[Bt-x)/\sqrt{4At}], \tag{17}$$

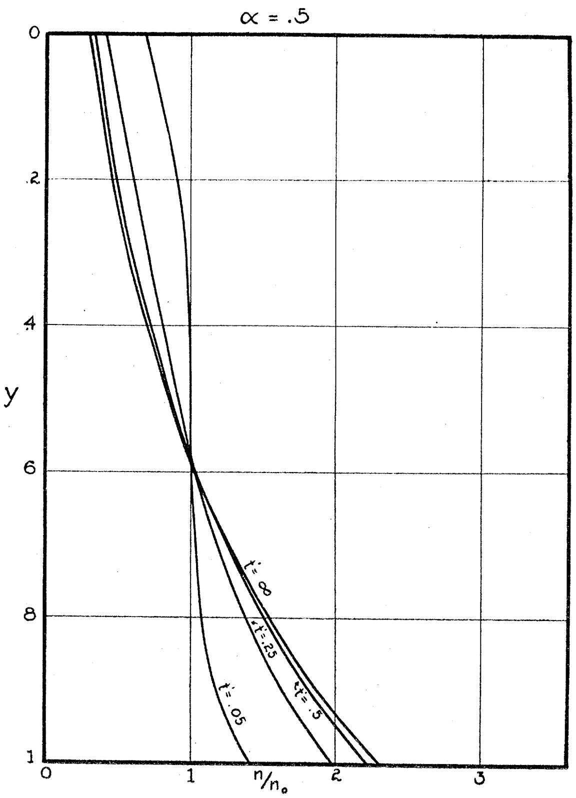

Fig. 4

where $\Theta(x)$ stands for the probability integral

$$(2/\sqrt{\pi})\int_0^x e^{-x^2}\,dx.$$

If the particles be initially uniformly distributed over a layer of depth h, so that

$$f(x)=0,\quad x<0 \text{ or } x>h;\quad f(x)=n_0,\quad 0<x<h,$$

equation (16) reduces to

$$n=\tfrac{1}{2}n_0\{\Theta[(h-x+Bt)/\sqrt{4At}]-\Theta[(Bt-x)/\sqrt{4At}]\}. \tag{18}$$

In terms of the running coordinate $x - Bt$, expressions (17) and (18) are of the same form as the expression for the one-dimensional diffusion of heat in an unlimited region, the initial distribution of temperature being, in the first case, constant for $x > 0$, and in the second, constant over a layer of thickness h. Thus when the liquid extends indefinitely in both the positive and negative directions of x there is superimposed upon the steady fall whose velocity is B a diffusion entirely similar to the one-dimensional diffusion of heat. This statement also follows from the remarks at the end of §(1).

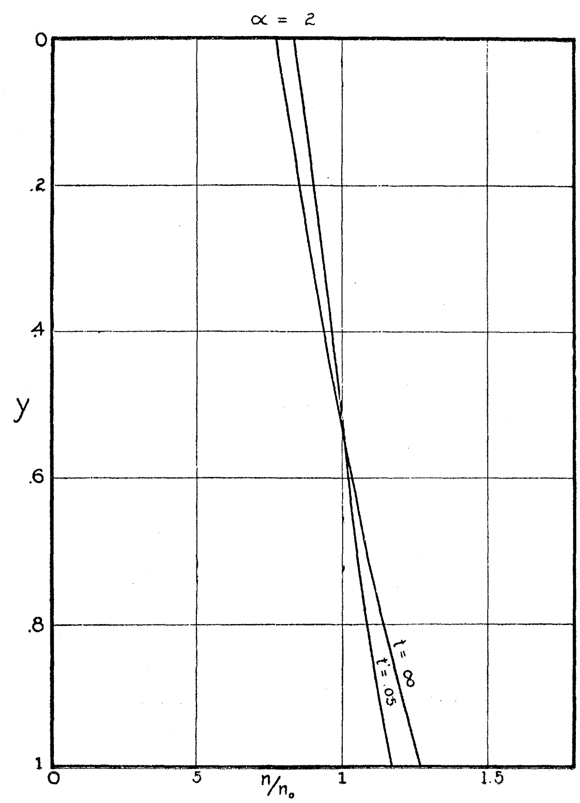

Fig. 5

If the liquid extends from $x = 0$ to $x = \infty$ the initial distribution $f(\gamma)$ is given for positive γ only, and the boundary condition

$$A\ \partial n/\partial x = Bn, \text{ for } x = 0, \tag{19}$$

must be satisfied. The problem can be reduced to the case just preceding by supposing that the liquid extend from $-\infty$ to $+\infty$, and that the region $x < 0$ have an initial distribution of density of such nature that the condition (19) will be satisfied for any t. If the boundary condition at $x=0$ is always satisfied it is obviously immaterial whether or not the liquid above $x=0$ actually exists. The solution (16) is thus available for this case provided $f(\gamma)$ for negative γ be determined in such a way that (16) satisfy (19).

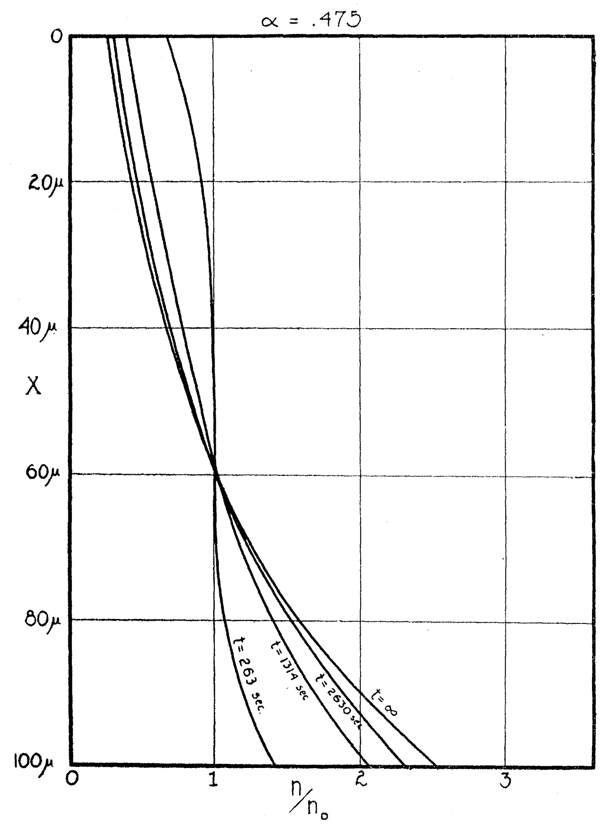

Fig. 6

If (16) be differentiated with respect to x and the result simplified by integration by parts the equation

$$\frac{\partial n}{\partial x}=\frac{1}{2\sqrt{A\pi t}}\int_{-\infty}^{+\infty} f'(\gamma)\ e^{-(\gamma-x+Bt)^2/4At}\ d\gamma,$$

results, it being assumed that $f(\gamma)$ is continuous at $\gamma=0$. Then to satisfy condition (19) the relation

$$\int_{-\infty}^{+\infty}[A\ f'(\gamma)-B\ f(\gamma)]\ e^{-(\gamma+Bt)^2/4At}\ d\gamma=0,$$

or

$$\int_{-\infty}^{+\infty} e^{-B\gamma/2A}\ [A\ f'(\gamma)-B\ f(\gamma)]\ e^{-\gamma^2/4At}\ d\gamma=0,$$

must hold identically in t. The factor $e^{-\gamma^2/4At}$ is even in γ, so that the condition will be satisfied if f be defined for negative values of γ so that

$$e^{-B\gamma/2A}\,[Af'(\gamma)-Bf(\gamma)] \tag{20}$$

is an odd function of γ. This condition gives a differential equation of the first order for $f(\gamma)$, $\gamma<0$, which, in combination with the assumed continuity of f at $\gamma=0$ uniquely determines f for negative γ in terms of the known f for positive γ. For example, if $f(\gamma)=\text{constant}=n_0$ for $\gamma>0$ then

$$f'(\gamma)-(B/A)\,f(\gamma)=(Bn_0/A)e^{B\gamma/A},$$

and

$$f(\gamma)=n_0\,e^{B\gamma/A}(1+B\gamma/A),\ \gamma<0,$$

so that the solution is

$$n/n_0=(1/2\sqrt{A\pi t})\left\{\int_{-\infty}^{0}(1+B\gamma/A)\,e^{B\gamma/A}\,e^{-(\gamma-x+Bt)^2/4At}\,d\gamma + \int_0^{\infty}e^{-(\gamma-x+Bt)^2/4At}d\gamma\right\} \tag{21}$$

which reduces to

$$\frac{n}{n_0}=-\frac{B\sqrt{t}}{\sqrt{A\pi}}\,e^{-(Bt-x)^2/4At}+\tfrac{1}{2}\left[1-\Theta\left(\frac{Bt-x}{\sqrt{4At}}\right)\right] +\tfrac{1}{2}\,e^{Bx/A}\left[1+\frac{B}{A}\,(Bt+x)\right]\left[1-\Theta\left(\frac{Bt+x}{\sqrt{4At}}\right)\right]. \tag{22}$$

As a second example suppose that the particles are initially distributed uniformly over a layer extending from $x=0$ to $x=h$; that is,

$$f(\gamma)=n_0,\qquad 0<\gamma<h. \tag{23}$$

Then, as before

$$f(\gamma)=n_0\,e^{B\gamma/A}(1+B\gamma/A),\quad -h<\gamma<0. \tag{24}$$

For values of γ less than $-h$, however, the equation

$$f'(\gamma)-(B/A)\,f(\gamma)=0,$$

gives

$$f(\gamma)=C\,e^{B\gamma/A}.$$

Evaluating the constant by means of the condition that $f(\gamma)$ is a continuous function, one has

$$C\,e^{-Bh/A}=n_0\,e^{-Bh/A}\,(1-Bh/A).$$

or

$$f(\gamma)=n_0\,e^{B\gamma/A}\,(1-Bh/A),\quad \gamma<-h. \tag{25}$$

If (23), (24), and (25) are substituted in (16) and the integrals evaluated the result is

$$\frac{n}{n_0}=\tfrac{1}{2}\Theta\left(\frac{h-x+Bt}{\sqrt{4At}}\right)-\tfrac{1}{2}\,e^{Bx/A}\left[1+\frac{B}{A}(x-Bt)\right]\Theta\left(\frac{Bt-x-h}{\sqrt{A\pi t}}\right) -\frac{B}{A}\,e^{Bx/A}\sqrt{At/\pi}\,(1-e^{-(Bt-h-x)^2/4At})+\tfrac{1}{2}(1-Bh/A)\left[1+\Theta\left(\frac{Bt-x-h}{\sqrt{4At}}\right)\right]. \tag{26}$$

This expression, while derived for a liquid of semi-infinite depth, approximately gives the distribution of density during the settling of a layer of particles, initially distributed uniformly over a depth h at the upper end of a very long column of liquid.

6. Integral Solution for the Finite Case

Eq. (14′) converges very slowly for small values of t, especially if α be small, so that it is unsuitable for calculation. An application of the image method sometimes used in heat problems furnishes an integral formula from which the density may be easily calculated when t is small.

If Eq. (16) be differentiated with respect to x, integrated by parts (assuming $f(\gamma)$ continuous), and substituted in the boundary conditions

$$A\ \partial n/\partial x = Bn, \qquad x=l,$$
$$A\ \partial n/\partial x = Bn, \qquad x=0,$$

the result is the pair of equations

$$\int_{-\infty}^{+\infty}[f'-(B/A)f]\ e^{-(\gamma-l+Bt)^2/4At}\ d\gamma=0,$$
$$\int_{-\infty}^{+\infty}[f'-(B/A)f]\ e^{-(\gamma+Bt)^2/4At}\ d\gamma=0,$$

or

$$\int_{-\infty}^{+\infty}e^{-(B/A)(\gamma-l)}[f'-(B/A)f]\ e^{-(\gamma-l)^2/4At}\ d\gamma=0,$$
$$\int_{-\infty}^{+\infty}e^{-B\gamma/2A}\ [f'-(B/A)f)]\ e^{-(\gamma^2/4At)}\ d\gamma=0.$$

These two equations will be satisfied identically in t if the function

$$e^{-(B/2A)(\gamma-l)}\ [f'-(B/A)f] \tag{27}$$

is skew-symmetric about $\gamma=l$, and the function

$$e^{-B\gamma/2A}\ [f'-(B/A)f] \tag{28}$$

skew-symmetric about the origin $\gamma=0$.

These two conditions suffice to determine $f(\gamma)$ for any γ provided the function is given over the range from 0 to l. Condition (27) "reflects" this given strip of the function, so that it is then given over the strip from 0 to $2l$: condition (28) then reflects this given strip about the origin, so that it is then given from $-2l$ to $2l$; and so on. If, for example, $f(\gamma)=n$ from 0 to l, the function from l to $2l$ is given by the equation

$$e^{-(B/2A)(\gamma-l)}\ [f'(\gamma)-(B/A)f(\gamma)]=(B/A)\ e^{-(B/2A)(l-\gamma)},$$

the solution of which is

$$f(\gamma)=(B\gamma/A)\ e^{(B/A)(\gamma-l)}+C\ e^{B\gamma/A}.$$

If the constant is evaluated from the condition that $f(\gamma)$ is continuous the result is

$$f(\gamma)=e^{(B/A)(\gamma-l)}\ \{1+(B/A)\ (\gamma-l)\},\ l<\gamma<2l.$$

In the same way the function in the range from $-l$ to 0 is determined from condition (28), reflecting about the origin. Thus

$$(\gamma)=ef^{(B/A)(\gamma+l)}\left\{1+(B/A)\ (\gamma+l)\right\},\quad -l<\gamma<0.$$

If it were desired to establish a formula valid for any time it would be necessary to repeat this process indefinitely, but it is obvious that for t small the distribution in the range 0 to l will be affected only by the initial distribution in the immediate neighborhood. If the values here determined for the initial distribution in the range from $-l$ to $2l$ be substituted in (16) and the integration carried out the following formula results:

$$\frac{n}{n_0}=\tfrac{1}{2}e^{(y-1)/\alpha}\left[1+\frac{y+t'-1}{\alpha}\right]\left[\Theta\left(\frac{2-y-t'}{\sqrt{4\alpha t'}}\right)-\Theta\left(\frac{1-y-t'}{\sqrt{4\alpha t'}}\right)\right]$$
$$-\sqrt{\frac{t'}{\alpha\pi}}\,e^{\frac{(y-1)}{\alpha}}\left[e^{-\frac{(2-y-t')^2}{4\alpha t'}}-e^{-\frac{(1-y-t')^2}{4\alpha t'}}\right]+\tfrac{1}{2}e^{\frac{y}{\alpha}}\left[1+\frac{y+t'}{\alpha}\right]\left[\Theta\left(\frac{1+y+t'}{\sqrt{4\alpha t'}}\right)\right.$$
$$\left.-\Theta\left(\frac{y+t'}{\sqrt{4\alpha t'}}\right)\right]+\sqrt{\frac{t'}{\alpha\pi}}\,e^{\frac{y}{\alpha}}\left[e^{-\frac{(1+y+t')^2}{4\alpha t'}}-e^{-\frac{(t'+y)^2}{4\alpha t'}}\right]+\tfrac{1}{2}\left[\Theta\left(\frac{1-y+t'}{\sqrt{4\alpha t'}}\right)\right.$$
$$\left.-\Theta\left(\frac{t'-y}{\sqrt{4\alpha t'}}\right)\right], \qquad (29)$$

where y, t', and α are given by (15).

This formula was used in calculating the curves for $t'=0.05$ and $t'=0.25$ in Fig. 1. One of these curves was also calculated from (14) and the two checked.

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