A NEW APPLICATION OF THE BAR METHOD FOR THE MEASUREMENT OF THERMAL CONDUCTIVITY.

By Marcus D. O'Day

Abstract

Thermal conductivity.—(1) *Electrically heated bar method of measurement.* Equations are derived for the case of a long thermally insulated bar, with ends kept at the same constant temperature, and carrying an alternating current such as to make the temperature distribution parabolic. This is possible only with conductors for which $(\alpha-3\beta)>0$, where α and β are the temperature coefficients of electrical and thermal conductivity, respectively. It is shown that the conductivity at each end $K_\theta=jI^2R_0L^2/2A\theta_m=jI^2R_0L/Ap_0$, where jI^2R_0 is the heat generated per unit length at the ends, $2L$ is the length, A is the cross section, θ_m is the mean temperature above that at the ends, and p_0 is the temperature gradient at the ends. (2) *Conductivity of lead and tin.* A long test bar was used, with ends fastened into heavy copper blocks, insulated from each other but both in the same thermostatic oil bath. The temperature distribution was determined by thermo-junctions and the critical current found by interpolation. For lead, $K_0=.0877$, $\beta=.000138$; for tin $K_0=.1575$, $\beta=.00067$.

Thomson effect.—(1) *Bar method.* In connection with thermal conductivity measurements, the coefficient may be determined from observations of the effect at a point $.42L$ from one end, of reversing a direct current of the critical magnitude. (2) *The coefficient for tin* at 38° came at 9.7×10^{-7} cal. per coulomb per sec.

THE writer restricts the term "bar method" to mean one of the electrical methods such as used by Jaeger and Diesselhorst,[1] and

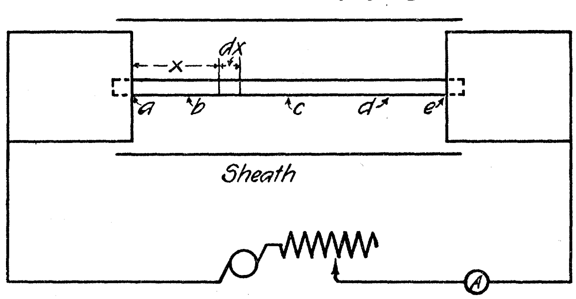

Fig. 1. Diagram of method.

Callendar.[2] The principle of such a method is illustrated in Fig. 1. A bar of metal AC is placed between two heavy copper blocks kept at a

[1] Jaeger and Diesselhorst, Preusse. Akad. Wiss., Berlin, **38**, 711, 1899

[2] Callendar, Article on Conduction of Heat in Encyclopedia Britannica

constant temperature. It is covered with some fairly good heat insulating substance and then by a metal sheath, electrically insulated from the two copper blocks. Thermo-couples are placed along the bar for the measurement of the temperature distribution when an electric current is flowing through it. The whole is immersed in a constant temperature bath, the temperature of which is taken as a zero of reference. We can measure directly the following quantities

The dimensions of the bar; the length $2L$ and the cross section A;

The resistance of the bar $R_0(1+\alpha\theta)$ where R_0 is the resistance of the bar at the zero of temperature.

The temperature gradient at the ends p_0;

The temperature at the middle θ_m;

The electric current passing through the bar I.

The quantities which we cannot measure directly are:

The heat conductivity of the bar, $K=K_0(1-\beta\theta)$;

The heat conductivity of the insulating substance, $k=k_0(1-\gamma\theta)$;

The coefficients β and γ;

The emissivity of the bar E.

Theory

The amount of heat crossing a plane section of the bar at a distance x due to conduction along in time dt is,

$$Q_1=AK\ (d\theta/dx)dt$$

The amount of heat accumulating in the element in time dt is,

$$Q_1-Q_2=A\frac{d}{dx}\left(K\frac{d\theta}{dx}\right)\ dxdt$$

The amount of heat accumulating in the element in time dt due to the Joule heating of the electric current is $jI^2R_0(1+\alpha\theta)dxdt$ where $j=.239$, the number of calories in a joule. The amount of heat accumulating in the element in time dt due to the Thomson effect, "Specific heat of Electricity," which is directly proportional to the current and to the temperature gradient is $-Is\ (d\theta/dx)dxdt$ where s is the Thomson coefficient.

For the steady state the sum of these three terms must be equal to the loss of heat through the surface, which is, assuming Newton's law of cooling $h\theta dxdt$ where h is Ep, the emissivity times the perimeter of the bar.

We thus get as the differential equation expressing the flow of heat in a rod carrying an electric current

$$A\frac{d}{dx}\left(K\frac{d\theta}{dx}\right)-Is\ \frac{d\theta}{dx}+(jI^2R_0\alpha-h)\ \theta=-jI^2R_0 \tag{1}$$

This equation is of the form

$$\frac{d}{dx}\left(K\frac{d\theta}{dx}\right)-a\frac{d\theta}{dx}+b\theta=-c \tag{1'}$$

The integration of this equation in the general case is not feasible. There are certain special cases, however, of great interest from the experimental standpoint, where the equation reduces to one with a simple algebraic integral.

Case I. In this case we use an alternating current so as to make a zero and so adjust its value that the quantity b is also made zero. This requires that

$$h = jI^2R_0\alpha \tag{2}$$

This was first suggested by Callendar.[2] Putting for K its value $K_0(1-\beta\theta)$, the second integral of the reduced equation is seen to be

$$\theta K_0(1-\tfrac{1}{2}\beta\theta) = -\tfrac{1}{2}cx^2 + Dx + E$$

We have as our boundary conditions, $\theta = 0$ when $x = 0$ and when $x = 2L$, whence

$$E = 0, \text{ and } D = cL \tag{3}$$

The quantity $K_0(1-\frac{1}{2}\beta\theta_m)$ is easily seen to be the heat conductivity at the temperature $\frac{1}{2}\theta_m$ and is the mean conductivity which we shall denote by K_m. We have the relation that when $\theta = \theta_m$, $x = L$ (at the middle). Hence

$$K_m = cL^2/2\theta_m = jI^2R_0L^2/2A\theta_m \tag{4}$$

We can determine the constant D by the relation that $K = K_0$ and $d\theta/dx = p_0$ when $x = 0$. We thus get $D = K_0p_0$. Combining this with Eq. (3)

$$K_0 = cL/p_0 = jI^2R_0L/Ap_0 \tag{5}$$

From Eqs. (4) and (5) we can calculate the value of β, the temperature coefficient.

The question naturally arises how the experimenter knows how to adjust the current to such a value that Eq. (2) is satisfied. Probably the best way to do this is to measure the quantity h directly as was done by King.[3] If we use a very small current, the value of θ will not be large and the temperature gradient near the middle of the bar will be zero. In this case all of the heat generated at this portion of the bar will be lost and we have the relation

$$h\theta_m = jI^2R_m \tag{6}$$

Case II. In this case we adjust the current until we have a perfect parabolic distribution of temperature. We will use alternating current so as to make the second term zero. We can write Eq. (1′)

$$K_0\frac{d^2\theta}{dx^2} - \beta K_0\theta\frac{d^2\theta}{dx^2} - K_0\beta\left(\frac{d\theta}{dx}\right)^2 + b\theta + c = 0 \tag{7}$$

and so adjust the current that the sum of the three middle terms is equal

[3] King, Amer. Acad. Proc. **33**, 353, 1898

to zero. To do this it is necessary to solve the following equations and eliminate θ.

$$K_0\ (d^2\theta/dx^2)+c=0 \tag{8}$$

$$\beta K_0\theta\ (d^2\theta/dx^2)+K_0\beta\ (d\theta/dx)^2-b\theta=0 \tag{9}$$

Solving the last one first we get

$$\tfrac{1}{2}\beta K_0\ (\theta\ d\theta/dx)^2=\tfrac{1}{3}\ b\theta^3+D \tag{10}$$

Since $d\theta/dx$ is finite when $\theta=0$ we have $D=0$. Eq. (10) then integrates into

$$\sqrt{3\beta\ K_0\theta}=\sqrt{\tfrac{1}{2}b}\ x+E$$

and since $x=0$ when $\theta=0$, we have $E=0$. Squaring we get

$$3\beta\ K_0\theta=\tfrac{1}{2}\ bx^2 \text{ and } \theta=\theta_m \text{ when } x=L \tag{11}$$

It will be shown later from Eq. (8) that $2K_0\theta_m=cL^2$ which substituted into Eq. (11) gives

$$b=3\beta c \tag{12}$$

Replacing b and c by their values from Eq. (1) we get

$$jI^2R_0\alpha-h=3\beta jI^2R_0, \quad I^2=h/jR_0(\alpha-3\beta) \tag{13}$$

The larger the difference, $\alpha-3\beta$, the better the method and it fails when this difference is very small or negative.

The first integral of Eq. (8) for the critical value of the current is $K_0\ (d\theta/dx)=-cx+D$ and the second integral is

$$K_0\theta=-\tfrac{1}{2}\ cx^2+Dx+D' \tag{15}$$

which reduces to

$$K_0=jI^2R_0L^2/2a\theta_m \tag{16}$$

if we apply the same boundary conditions as before. This replaces Eq. (4) and instead of Eq. (5) we get

$$K_0=jI^2R_0L/A\ p_0 \tag{17}$$

These two equations are compatible only when

$$\theta_m=\tfrac{1}{2}L\ p_0 \tag{18}$$

The above relation shows how we may realize the conditions of Eq. (13) without knowing either h or β. If the bar under investigation is of reasonable length (25 cm or over) a thermo-couple placed one centimeter from the ends of the bar will give the value of p_0. A thermo-couple is placed at the end of the bar as in Fig. (1) to get the temperature of the end point. The experimenter need only to try several different currents, plot the variations from Eq. (18) and note at what current the relation is satisfied. If he does this for two different temperatures of the constant temperature bath, he can calculate β, and then from Eq. (16) he may obtain h.

If h is known and the inner and outer radii R' and R of the insulating layer, the heat conductivity of the insulating substance may be calculated from the formula

$$K=(h/2\pi)\ \log\ (R/R') \tag{18'}$$

Both methods were used by the writer, the first on lead and the second on tin. However, he prefers the latter method especially when $\alpha-3\beta$ is fairly large. The difficulties usually met with, viz., the measurement of small temperatures and the loss of heat through the insulating substance, have to a large extent been overcome. We have a very accurate method of measuring h, and the temperature differences are very large, the smallest being about five centigrade degrees. The measurement of so large a temperature presents no great experimental difficulties.

After the experimenter has found the correct value of the current he may use direct current of the same value and measure the Thomson effect. Instead of Eq. (8) we have

$$K_0\frac{d^2\theta}{dx^2}-a\frac{d\theta}{dx}=-c \tag{19}$$

which has for its integral

$$\theta=\mathrm{B}e^{mx}-D+cx/m \tag{20}$$

where $m=a/K_0$.

Using the same boundary conditions we get

$$\theta=\frac{c}{m}\{x-2L\ (e^{mx}-1)/e^{2mL}-1)\} \tag{21}$$

and

$$B=D=-2cL/m(e^{2mL}-1) \tag{22}$$

The second term of the right hand side of Eq. (21) can be expanded by means of Bernoulli's functions, and neglecting m^2 in comparison with m, since it is less than 10^{-5}, we get

$$\theta=c\{x(2L-x)+\tfrac{1}{6}\ mx(2L-x)\ (L-X)\} \tag{23}$$

The value of m changes sign with the direction of the current and calling $d\theta$ the change in temperature obtained upon reversal of the current, we get

$$d\theta=\tfrac{1}{3}\ cmx(2L-X)\ (L-X)$$

This is a maximum when $x=.42L$. Therefore if we place a thermoelement at this point and measure $d\theta$, we get for the Thomson coefficient

$$s=A^2K_0d\theta/0.128\ jI^3L^3R_0 \tag{25}$$

Since the value of the current is determined by Eq. (13), the only arbitrary quantity in the above equation is the length $2L$ and the equation shows that increasing the length very greatly increases the sensibility. The Thomson coefficient is measured in calories per coulomb per sec. and is of the order of 10^{-7}. It is thus seen that $d\theta$ is necessarily a very small difference in temperature. Ordinarily currents of from three to ten amperes are used, while in this method the currents are well above fifty amperes. Since $d\theta$ is proportional to I^3 in the latter case it will be very materially increased. If the loss of heat from the surface is not compensated, the error caused thereby is proportional to the length of the bar.

In this case where it is compensated, the length can be as large as mechanically feasible. Eq. (25) shows that increasing the length has the same effect as increasing the current.

The method differs from that used by the majority of investigators in that the ends are kept at the same temperature and the heating of

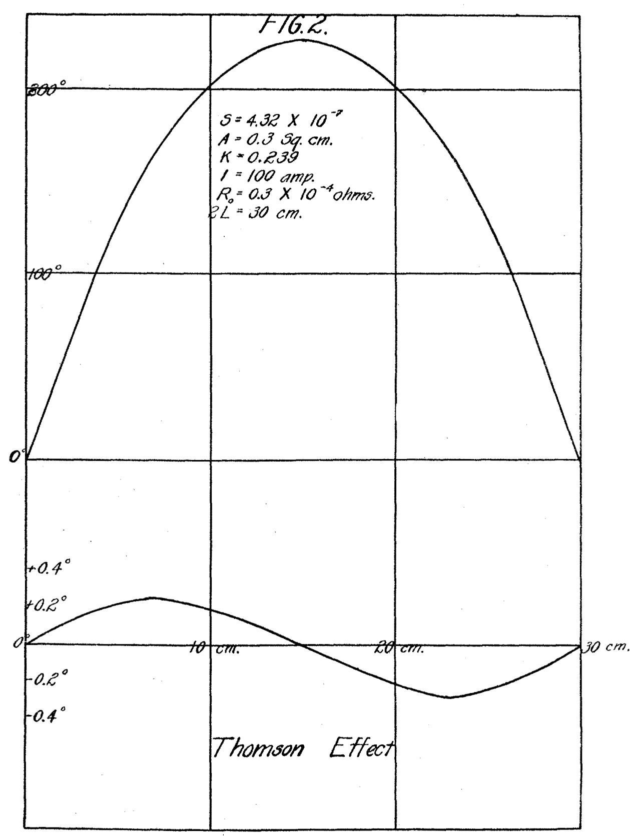

Fig. 2. Upper curve: Temperature distribution for ideal case.
Lower curve: Effect of Thomson effect on the distribution.

the current causes its own temperature gradient. In Fig. 2, the writer has plotted the temperature distribution for an ideal case which is approximately that of tin. It is seen that for a given maximum temperature the method outlined above gives a much larger temperature gradient. Since the value of a in Eq. (19) is directly proportional to the temperature gradient, it is seen that $d\theta$ is much greater in the method suggested. The bottom curve in Fig. (2) (drawn on a scale one hundred times as large) shows the effect of the Thomson heating upon the temperature distribution.

Description of Apparatus

The bar of metal to be investigated was screwed into two large copper blocks, two inches in diameter and one foot long. The bar was bent in the shape of a U, as in Fig. 3. The copper blocks were separated by thin sheets of mica. The bar was covered with wool yarn and placed in an inverted Dewar flask. The copper blocks were then placed in an oil

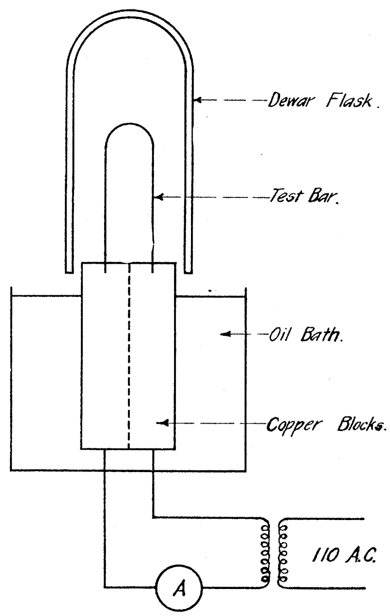

Fig. 3. Apparatus.

bath kept in motion by a stirrer. The alternating current was supplied by a low voltage transformer and measured by a current transformer. Five thermo-couples were attached to the bar and from these it was easy to get p_0 and θ_m. For the measurement of p_0 the thermo-couples were placed two centimeters apart. The thermo-couples were then brought to a special selector switch which enabled a series of thermo-couples to be balanced very rapidly against one standard thermo-couple. This switch was so designed as to eliminate spurious e.m.fs. at the contact points. A Leeds and Northrup type K potentiometer was used for measurement of temperature with the standard cell replaced by a standard thermo-couple with one junction in ice and the other in steam. This proved to be very satisfactory and enabled the potentiometer to read directly in degrees as thermo-junctions were chosen to all have the same e.m.f. Thermo-couples soldered without twisting proved to be much more uniform than twisted ones. Isolated thermo-couples were not used in the case of lead and the Thomson effect was not measured. However, in the case of tin, thermo-couples coated with lacquer and then glued into the bar gave good results and enabled a determination of the Thomson effect.

Data on Lead

A bar of lead, supposed to be Kahlbaum's, 25 cm long, was used. Only K and β were determined in these early experiments as the apparatus was designed before the complete theory of the method was attempted. Also the Thomson effect was not measured, it being difficult to eliminate leaks in the potentiometer circuit. The method used was that of Case I. The data taken with the bath at room temperature are as follows: $I=50$ amp; $L=12.5$ cm; $A=0.439$ cm^2; $R_0=0.000478$ ohms (resistance at 20°C); $c=0.6504$.

Table I

Results for lead at room temperature

Temp. (ends)	p_0	θ_m	K/c	K_0/c	K_m/c
25.97	8.72	52.34	0.1370	0.1430	0.1490
26.67	8.91	53.14	0.1350	0.1400	0.1450
27.14	8.90	53.76	0.1355	0.1406	0.1455
26.47	8.87	52.93	0.1345	0.1409	0.1475
Means: 26.56	8.85	53.17	0.1355	0.1411	0.1467

From this the heat conductivity (in c.g.s. units) was found to be

At 0° C	0.0877
At 26.5°C	0.0920
At 53.1°C	0.0955

By using values of heat conductivity for wool given in the tables, h was calculated by use of Eq. (18′). From this it was estimated that 50 amperes would approximately compensate for the loss of heat through the surface. In Fig. 4 are given the values of the conductivity at these three points and the straight line drawn. It is to be noted that the value of the conductivity at these three points is from one set of readings at just one temperature of the bath. The bath was then heated to varying temperatures and the conductivity at these temperatures noted. In this case the constant c was not equal to jI^2R_0 but to $(jI^2R_0+h\theta_0)$ where θ_0 is the temperature of the surrounding atmosphere. Since it had a large negative value, measurements were difficult and values of the temperature gradients were not consistent so the heat conductivity was calculated from the formula

$$AK_m=(c_1-c_2)^2L^2/\Delta\theta_m=jI^2R_0/\Delta\theta_m \qquad (28)$$

where $\Delta\theta_m$ is the difference in the values of θ_m when the current is flowing and when it is shut off. Readings were taken with the current off for thirty minutes, on for thrity minutes, and off again for thirty minutes. This gave a good average to substitute in the above formula. Since the resistivity of the bar checked exactly with that given by the Bureau of Standards, their value of the temperature coefficient was used.

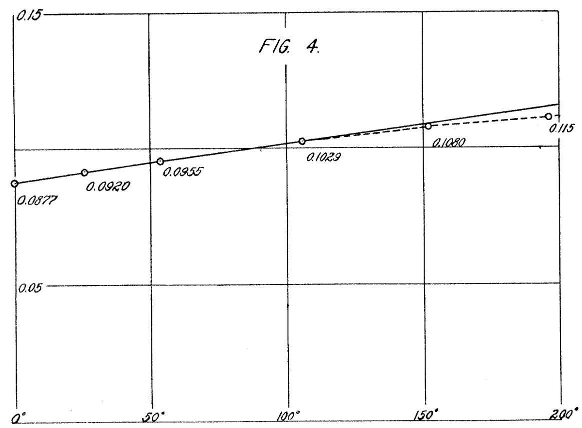

Fig. 4. Thermal conductivity of lead as a function of temperature.

DATA ON TIN

The method of Case II was used. In the following table let V be the per cent variation from the true parabolic distribution of temperature. $2L=28$ cm; $A=0.211$ cm^2; $R_0=0.000528$ ohms per cm (room temp.); $\alpha=0.0042$; and $c=0.584\ I^2\times10^{-3}$.

I	Temp. (ends)	p_0	θ_m	$\theta_m-\frac{1}{2}Lp_0$	V	$\sqrt{V}$
15 amp.	20.52	0.970	6.15	−0.640	10.40	3.20
20	20.21	1.705	10.90	−1.035	9.50	3.08
25	21.25	2.775	17.48	−1.945	10.90	3.30
30	22.24	3.855	24.51	−2.490	10.05	3.16
35	22.43	5.475	35.24	−3.069	8.69	2.95
40	23.57	7.630	49.97	−3.430	6.78	2.60
45	26.00	10.270	68.50	−3.090	4.50	2.12
50	29.14	13.780	96.46	+1.040	−1.07	1.03

In Fig. 5 two curves are plotted giving the manner in which $\sqrt{V}$ and p_0 depend upon the current. It is evident from the figure that the balancing current is 49.2 amperes and that the value of p_0 is 13.00. This gives a value of the conductivity equal to 0.1575 c.g.s. units. The peculiar behavior of the curve in the neighborhood of 20 amperes may be due to a recalescence phenomenon which the writer hopes to investigate at a later date. In computing the value of the conductivity it is to be noted that R_0 has increased, owing to the fact that the ends have been raised to a higher temperature. This is also to be noted in the calculation of the temperature coefficient β. The value of h was determined from Eq. (6) and found to be $h=(2.64)^2\times 5.28\times 10^{-5}/0.448\times 4.184=1.96\times 10^{-4}$ cal./sec., which combined with Eq. (16) gives for β the value -0.000670.

A direct current of more than 36 amperes which was sufficiently steady for the measurement of the Thomson effect was not available. However with a current of 36.1 amperes, there was a deflection corresponding to 0.4° C upon reversal of the current. This value of $d\theta$ substituted in Eq. (25) gives $s=9.7\times 10^{-7}$ calories per coulomb per sec. This would be the value of the Thomson effect at 38°.

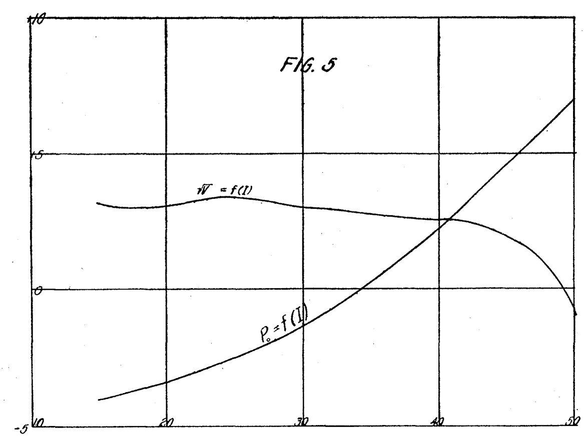

Fig. 5. Case II. Variation of $\sqrt{V}$ and p_0 with current I.

The results show that the samples were not of the utmost purity as the temperature coefficients are positive. The writer hopes to investigate this point further in later researches on the effect of a systematic alloying of the lower melting point metals upon the thermal conductivity. He wishes to express his gratitude to Professor E. E. Hall, of this university whose interest and suggestions have been the inspiration of this work. He is also indebted to Professors Boynton and Caswell of the University of Oregon for valuable suggestions concerning the Thomson effect.

University of California,
Dept. of Physics,
August 13, 1923.