CURRENTS LIMITED BY SPACE CHARGE BETWEEN COAXIAL CYLINDERS

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Abstract

Limiting current between coaxial cylinders; calculation of the function $\beta = f(r/r_o)$ in the space charge equation $i=(2\sqrt{2}/9)\sqrt{(e/m)}\,V^{3/2}/(r\beta^2)$. Two different infinite series were obtained for β and the coefficients for fourteen terms of each were determined. These two sets were checked against each other. Thus values of β^2 were obtained accurate to 1 in 10,000. These were checked by an integration method which was also used to calculate values in the region where the series failed. For a cathode of radius r_0 inside an anode of radius r, β^2 increases from 1 at $r/r_0=11.2$ to a maximum value 1.0946 at $r/r_0=42$, decreases to a minimum value 0.9990 at $r/r_0=30{,}000$, and becomes 1 at $r/r_0=\infty$. The customary assumption $\beta^2=1$ has therefore led to errors up to 9.5 per cent in previous calculations, but this error is usually about equal and opposite to that introduced by neglecting the effect of initial velocities. For the cathode outside the anode, β^2 is given very closely by the equation $\beta^2=4.6712\ (r_0/r)\ [\log_{10}(r_0/r)-\log_{10}\sqrt{2}]^{3/2}$, for $r_0/r>10$. The empirical constant $\sqrt{2}$ is interpreted to mean that the potential distribution near the anode is unaltered if the hot cathode is replaced by a cold cylinder having one half the cross-section of the original cathode. The correction for initial velocities is less for a cylindrical cathode inside an anode than for parallel planes. In the inverted case it is much greater than in the case for parallel planes, and the effect of the tangential component of the initial velocity may greatly decrease the current that flows.

In PREVIOUS papers[1] it has been shown that in a very high vacuum the space charge of the electrons limits the current that can flow from a hot cathode to a positively charged anode so that beyond a certain point an increase in the temperature of the cathode causes no further increase in current. When the cathode and anode are long coaxial cylinders the current limited by space charge is given by

$$i=\frac{2\sqrt{2}}{9}\sqrt{\frac{e}{m}}\frac{V^{3/2}}{(r\beta^2)} \qquad (1)$$

Here i is the electron current per unit length along the axis, V the voltage at any point P, r the radius at P, e and m the charge and mass respectively of an electron.

Expressing i, r, and V in amperes per cm, cm, and volts, respectively, the equation becomes

$$i=14.68\times10^{-6}V^{3/2}/(r\beta^2) \qquad (1a)$$

[1] Langmuir, Phys. Rev. **2**, 450, 1913; Phys. Zeits. **15**, 348, 1914

$\beta = f(r/r_0)$ is a quantity of zero dimensions, r_0 being the radius of the cathode. Values for β^2 were given in a short table, but these were derived from a rough method of calculation and many of them have an error of several per cent. The object of the present paper is to correct and extend this table and to give values of β for the inverted case where the cathode is the outside cylinder.

It was shown in the derivation of Eq. (1) that

$$r(d^2V/dr^2) + dV/dr = i\sqrt{\frac{2m}{eV}} \tag{2}$$

Substitution of Eq. (1) in Eq. (2) gives

$$3\beta r^2(d^2\beta/dr^2) + r^2(d\beta/dr)^2 + 7\beta r(d\beta/dr) + \beta^2 - 1 = 0 \tag{3}$$

Placing

$$\gamma = \log\,(r/r_0) \tag{4}$$

we obtain

$$3\beta(d^2\beta/d\gamma^2) + (d\beta/d\gamma)^2 + 4\beta(d\beta/d\gamma) + \beta^2 - 1 = 0 \tag{5}$$

The solution of this equation gives the series[2]

$$\beta = \gamma - (2/5)\gamma^2 + (11/120)\gamma^3 - (47/3300)\gamma^4 + \;.\;.\;.\;. \tag{6}$$

This was the series used to calculate β^2 but is accurate only for small values of γ. The values obtained are all too low, the error at $r/r_0 = 5$ amounting to $-.011$.

The value $\beta = 1$ is a solution of Eq. (3) and corresponds to the case that $r/r_0 = \infty$. A second method[3] for calculating β was based on a substitution

$$\beta = 1 - \epsilon^{-\mu}$$

and a series was obtained for μ. The values of β obtained in this way converged so rapidly to unity for $r/r_0 = 11$ to 20 that the accuracy of the method was held to be doubtful. This series was therefore abandoned in favor of a method of extending the series in Eq. (6) to give accurate values of β for large values of γ.

Derivation of Series for β

The four terms of the series in Eq. (6) fail to give values of β accurate to five places of decimals at $r/r_0 = 1.7$. A much improved method suggested by H. M. Mott-Smith has made it possible to determine ten more terms of this series. This method consists in expanding β, which was given as a function of γ in Eq. (5), in the form

$$\beta = A_0 + A_1\gamma + A_2\gamma^2 + A_3\gamma^3 + \;.\;.\;.\;. + A_n\gamma^n + \;.\;.\;.\;. \tag{7}$$

[2] Due to a printer's error the coefficient of the second term was incorrectly printed $-2/3$ in the Phys. Rev. paper, but it was correctly given in a paper in the Phys. Zeits. J. A. Fleming pointed out this error in the Radio Review **2**, 133, 1921.

[3] Langmuir, Phys. Zeits. **15**, 351, 1914

From Maclaurin's series we have

$$\beta=\beta_0+\gamma\left(\frac{d\beta}{d\gamma}\right)_0+\frac{\gamma^2}{2!}\left(\frac{d^2\beta}{d\gamma^2}\right)_0+\;\cdot\;\cdot\;\cdot\;\cdot\;+\frac{\gamma^n}{n!}\left(\frac{d^n\beta}{d\gamma^n}\right)_0+\;\cdot\;\cdot \tag{8}$$

where the terms with the subscript 0 have values corresponding to $\gamma=0$. The values of the successive derivatives can be conveniently calculated as follows:

Let $\beta=a,\ d\beta/d\gamma=b,\ d^2\beta/d\gamma^2=c$, etc.

Then Eq. (5) takes the form

$$3ac+b^2+4ab+a^2-1=0 \tag{9}$$

Differentiating (9) we have

$$3ad+5bc+4b^2+4ac+2ab=0 \tag{10}$$

Differentiating (10) we have

$$3ae+8bd+4ad+12bc+5c^2+2ac+2b^2=0 \tag{11}$$

Differentiating (11) we have

$$3af+11be+4ae+18cd+16bd+2ad+12c^2+6bc=0 \tag{12}$$

From Eq. (6) we see that $\beta=0$ when $\gamma=0$. Therefore in the present expansion for β in the neighborhood of $\gamma=0$ we may place $a_0=0$ and Eq. (9) reduces to

$$b_0^2-1=0,\ b_0=1 \tag{13}$$

the positive sign being chosen since β increases with γ and hence the first derivative is positive. Substituting $a_0=0$, $b_0=1$ in Eq. (10) we have

$$c_0=-4/5 \tag{14}$$

Substituting the values of a_0, b_0, and c_0 in Eq. (11) we have

$$d_0=11/20 \tag{15}$$

Similarly from Eq. (12) we have

$$e_0=-376/1100 \tag{16}$$

By comparing (7) and (8) it is seen that

$$A_n=\frac{1}{n!}\left(\frac{d^n\beta}{d\gamma^n}\right)_0 \tag{17}$$

Therefore dividing the derived values of b_0, c_0, d_0 and e_0 by 1, 2, 6 and 24 respectively we obtain

$$A_1=1,\ A_2=-2/5,\ A_3=11/120,\ A_4=-47/3300$$

and these coefficients agree with those in Eq. (6).

It is seen that in this process when each successive equation is differentiated the new equation thus formed can be solved for a value of $(d^n\beta/d\gamma^n)_0$ by a knowledge of the values of all the preceding derivatives. Thus the coefficients of any number of terms of the series in Eq. (7) can be calculated. The 2nd column of Table I gives the first fourteen of these coefficients calculated by this method.

H. M. Mott-Smith also suggested a second series, usually more rapidly convergent than the first, which he obtained by putting

$$\beta = \theta \epsilon^{-\gamma/2}, \tag{18}$$

which reduced Eq. (5) to

$$3\theta \frac{d^2\theta}{d\gamma^2} + \left(\frac{d\theta}{d\gamma}\right)^2 - \epsilon^{\gamma} = 0. \tag{19}$$

This equation was also expanded by Maclaurin's series, using the same method as that described above.

TABLE I

Coefficients of terms in series for β

n	$\beta = \Sigma A_n \gamma^n$ A_n	$\beta = \epsilon^{-\gamma/2} \Sigma B_n \gamma^n$ B_n
0	0.0	0.0
1	+1.0	+1.0
2	−0.40	+0.10
3	+ .09166667	+ .01666667
4	− .01424242	+ .002424242
5	+ .001679275	+ .0002872294
6	− .0001612219	+ .00002658476
7	+ .00001293486	$+1.766124 \times 10^{-6}$
8	-8.87693×10^{-7}	$+6.332946 \times 10^{-8}$
9	$+5.46192 \times 10^{-8}$	-8.73852×10^{-10}
10	-2.94843×10^{-9}	-1.93844×10^{-11}
11	$+1.36026 \times 10^{-10}$	$+5.77287 \times 10^{-11}$
12	-7.1101×10^{-12}	$+9.4502 \times 10^{-12}$
13	$+2.6644 \times 10^{-13}$	$+4.7012 \times 10^{-13}$
14	$+1.2526 \times 10^{-15}$	-6.5539×10^{-14}

When combined with Eq. (18) it takes the form

$$\beta = \epsilon^{-\gamma/2}(B_0 + B_1\gamma + B_2\gamma^2 + \ldots\ldots + B_n\gamma^n + \ldots) \tag{20}$$

where

$$B_n = \frac{1}{n!}\left(\frac{d^n\theta}{d\gamma^n}\right)_0. \tag{21}$$

The third column of Table I gives the first fourteen coefficients and shows that this series which is at first more convergent than the A series becomes less convergent after the eleventh term.

Each coefficient of the series was checked against the corresponding coefficient of the A series by the following method. Differentiating Eq. (18) successively and placing $\gamma = 0$ and hence $\beta = 0$ and $\theta = 0$ we obtain a series of equations for $d^n\beta/d\gamma^n$ in terms of $d\theta/d\gamma$, $d^2\theta/d\gamma^2$, ... $d^n\theta/d\gamma^n$. From Eqs. (21) and (17), substituting $n!B_n = d^n\theta/d\gamma^n$ for each derivative of θ, and $n!A_n = d^n\beta/d\gamma^n$ we obtain $A_1 = B_1$; $2A_2 = 2B_2 - B_1$; $6A_3 = 6B_3 - 3B_2 + \frac{3}{4}B_1$; etc. Each coefficient was tested by means of these equations using one or two more decimal places than are given in Table I, and all were shown to be correct within the degree of accuracy recorded in the table.

Calculation of β

Case where $r/r_0 > 1$. (*Cathode inside of anode*). The B series was used for all calculations of the value of β when $r/r_0 > 1$ because it is the more convergent series over the first part of the range, while over the latter part it has the advantage that the error of the neglected final terms is multiplied by the increasingly small quantity $\epsilon^{-\gamma/2}$ which becomes 0.1 at $r/r_0 = 100$. The fourteen terms were sufficient for an accurate calculation of the values of β up to $\gamma = 4.2$, $r/r_0 = 66.7$, and were useful as an approximation as far as $\gamma = 7.5$, $r/r_0 = 1808$.

In order to check the results obtained, the following method was adopted. For $\gamma = 3.0$ to $\gamma = 4.2$, β was calculated for equal intervals $\Delta\gamma = 0.2$, and in addition the value of $d\beta/d\gamma$ was calculated at each point from the equation obtained by differentiating (20). The values of β and $d\beta/d\gamma$ for each value of γ were then substituted in (5) and each equation solved for $d^2\beta/d\gamma^2$. The successive values of $d^2\beta/d\gamma^2$ thus obtained were integrated by Simpson's rule to give $d\beta/d\gamma$ and again integrated to give β, the integration in each case being checked by Weddle's rule. It was found that the values of β obtained at the end of this process as a result of integration were exactly the same as those substituted in the equation at the beginning of the process. It was therefore shown that the series calculations for β (requiring all fourteen terms at $\gamma = 4.2$) had yielded correct values which were solutions of the differential equation for β.

When the series began to fail at $\gamma = 4.2$ the integration process proved to be a practicable method of determining the correct values. For if preliminary values of β and $d\beta/d\gamma$, which are approximately correct, be substituted in the equation to obtain $d^2\beta/d\gamma^2$ and the integration process carried through as before, the result of integration will be a new set of values of $d\beta/d\gamma$ and β slightly different from the first. These new values may in turn be substituted in the equation and the process repeated, and if the final values of β do not then agree with the values obtained in the preceding approximation the process may be carried through as many times as necessary. It is found that the values of β thus successively obtained converge toward a fixed set of values which are the desired solutions of the equation, so that the method provides a means of arriving at these solutions.

From $\gamma = 4.5$ to $\gamma = 7.5$ the series calculations were used as preliminary values for the integration method, and the results of the operation of the method showed that these values of β required at $\gamma = 4.5$ a correction of

-0.00003, and at $\gamma=7.5$ a correction of -0.00260. Beyond $\gamma=7.5$ the series was abandoned and the preliminary values obtained by extrapolation.

The results of these calculations for $\gamma=3.0$ to $\gamma=10.5$ are given in Table II, together with the corresponding values of $d\beta/d\gamma$.

TABLE II

γ	β	$d\beta/d\gamma$
3.0	1.03528	+0.03506
3.2	1.04098	+ .02239
3.4	1.04440	+ .01221
3.6	1.04600	+ .00415
3.8	1.04618	− .00211
4.0	1.04526	− .00686
4.2	1.04352	− .01035
4.5	1.03986	− .01369
4.8	1.03548	− .01527
5.1	1.03084	− .01562
5.4	1.02620	− .01509
5.7	1.02182	− .01402
6.0	1.01782	− .01263
6.3	1.01426	− .01107
6.6	1.01118	− .00949
6.9	1.00856	− .00797
7.2	1.00638	− .00656
7.5	1.00461	− .00530
8.0	1.00242	− .00352
8.5	1.00101	− .00221
9.0	1.00015	− .00126
9.5	0.99970	− .00061
10.0	.99951	− .00020
10.5	.99947	+ .00001

Table III gives the complete range of values of β^2, which were all calculated to at least one more decimal place than appears in the table. The values corresponding to $r/r_0>20$ were derived from interpolations for β from Table II, using Newton's interpolation formula. They show that β^2 passes through unity at $r/r_0=11.2$, increases to a maximum value 1.0946 at $r/r_0=42$, and again passes through unity close to $r/r_0=10{,}000$. A minimum is reached just before $\gamma=10.5$ ($r/r_0=36{,}316$) and its value differs from unity by less than 0.0012. Probably from this point onward the curve passes successively through an infinite number of maxima and minima which rapidly approach unity.

It has been customary to place $\beta^2=1$ in the space charge equation (1a), but the data in Table III indicate that in the range of diameters of cathode and anode most commonly used in electron tubes (r/r_0 between 20 and 500) this practice has involved an error up to about 9.5 per cent. The reasons that this error has not been detected by experiments are probably (1) that the "end convections" are usually not accurately known and (2) that this error is usually about equal and opposite to that introduced by neglecting the effect of initial velocities.[4]

TABLE III

β^2 as function of radius

r_0 = radius of cathode; r = radius at any point P

β^2 applies to case where P is outside cathode, $r > r_0$.

$(-\beta)^2$ applies to case where P is inside cathode, $r_0 > r$.

r/r_0 or r_0/r	β^2	$(-\beta)^2$	r/r_0 or r_0/r	β^2	$(-\beta)^2$
1.00	0.00000	0.00000	6.0	.8362	14.343
1.01	.00010	.00010	6.5	.8635	16.777
1.02	.00039	.00040	7.0	.8870	19.337
1.04	.00149	.00159	7.5	.9074	22.015
1.06	.00324	.00356	8.0	.9253	24.805
1.08	.00557	.00630	8.5	.9410	27.701
1.10	.00842	.00980	9.0	.9548	30.698
1.15	.01747	.02186	9.5	.9672	33.791
1.2	.02875	.03849	10.	.9782	36.976
1.3	.05589	.08504	12.	1.0122	50.559
1.4	.08672	.14856	14.	1.0352	65.352
1.5	.11934	.2282	16.	1.0513	81.203
1.6	.1525	.3233	18.	1.0630	97.997
1.7	.1854	.4332	20.	1.0715	115.64
1.8	.2177	.5572	30.	1.0908	214.42
1.9	.2491	.6947	40.	1.0946	327.01
2.0	.2793	.8454	50.	1.0936	450.23
2.1	.3083	1.0086	60.	1.0910	582.14
2.2	.3361	1.1840	70.	1.0878	721.43
2.3	.3626	1.3712	80.	1.0845	867.11
2.4	.3879	1.5697	90.	1.0813	1018.5
2.5	.4121	1.7792	100.	1.0782	1174.9
2.6	.4351	1.9995	120.	1.0726	1501.4
2.7	.4571	2.2301	140.	1.0677	1843.5
2.8	.4780	2.4708	160.	1.0634	2199.4
2.9	.4980	2.7214	180.	1.0596	2567.3
3.0	.5170	2.9814	200.	1.0562	2946.1
3.2	.5526	3.5293	250.	1.0494	3934.4
3.4	.5851	4.1126	300.	1.0440	4973.0
3.6	.6148	4.7298	350.	1.0397	6054.1
3.8	.6420	5.3795	400.	1.0362	7172.1
4.0	.6671	6.0601	500.	1.0307	9502.2
4.2	.6902	6.7705	600.	1.0266	
4.4	.7115	7.5096	800.	1.0209	
4.6	.7313	8.2763	1000.	1.0171	
4.8	.7496	9.0696	1500.	1.0114	
5.0	.7666	9.8887	2000.	1.0082	
5.2	.7825	10.733	5000.	1.0020	
5.4	.7973	11.601	10000.	.9999	
5.6	.8111	12.493	30000.	.9990	
5.8	.8241	13.407	∞	1.0000	∞

Case where $r/r_0 < 1$. (Cathode outside of anode). For the inverted case of electron current between concentric cylinders where the cathode is the outside cylinder, the same series may be used as in the preceding case, but γ is now negative so that alternate terms of the series have

[4] This has been discussed in a footnote to a recent paper, Langmuir, Phys. Rev. **21**, 435, (1923) and illustrated by Dushman's experimental data.

changed sign. This makes the values of β negative, but since only β^2 occurs in Eq. (1) the current remains positive.

The calculations of β were made from the B series as far as $r_0/r=20$. From this point they were carried through to the end, using the A series, for after the factor $\epsilon^{\gamma/2}$ in the B series reached the value $\sqrt{20}$ it multiplied the error of neglected terms by too large a quantity. Also the A series is more convergent in this region. The results are given in Table III and are believed to be accurate to the number of places given.

For high values of r_0/r a useful approximation formula for obtaining β^2 is given by

$$\beta^2=4.6712\ (r_0/r)\ [\log_{10}(r_0/r)-0.1505]^{3/2} \qquad (22)$$

This is accurate to one part in 10,000 for values of r_0/r greater than 80, and accurate within one per cent for values of r_0/r greater than 10. The equation is obtained from Eq. (19) by neglecting ϵ^{γ}, integrating twice, substituting the value of θ into Eq. (18) and eliminating γ by Eq. (4). The two integration constants are then so determined as to make the values of β^2 for large r_0/r agree with those calculated from the series as given in Table III.

Logarithmic differentiation of Eq. (1), considering i as constant, gives

$$3(r/V)/(dV/dr)=2+4(d\log\beta)/(d\log r) \qquad (23)$$

Logarithmic differentiation of Eq. (22) gives

$$4(d\log\beta)/(d\log r)=-2-3/[\log_\epsilon(r_0/r)-0.34654]$$

Combining this with Eq. (23) we have

$$dV/dr=-(V/r)/[\log_\epsilon(r_0/r)-0.34654] \qquad (24)$$

This equation for the potential gradient at any point is of course valid only in the same range as Eq. (22), that is, when r is less than say $0.1r_0$.

In the absence of electron current and the resulting space charge the field intensity between concentric cylinders is

$$dV/dr=(V/r)/\log(r_a/r) \qquad (25)$$

where r_a is the radius of one cylinder at zero potential and V is the potential at any point of radius r.

Comparing Eqs. (24) and (25) we see that the field intensities become identical for all values of r if we place

$$\log(r_0/r_a)=0.34654=\log\sqrt{2}$$

or

$$r_0=r_a\sqrt{2} \qquad (26)$$

The ratio $r_0:r_a$ is equal to $\sqrt{2}$ within about one part in 30,000, and therefore probably is rigorously equal to this number. This result may be interpreted as follows.

With current limited by space charge between an outer hot cathode and an inner concentric anode of small diameter, the potential distribution near the anode is unaltered if the cathode is replaced by a cold cylinder having one half the cross section of the original cathode. The effect of space charge is thus to increase the field intensity near the anode to the same extent as a 50 per cent reduction in the cross section of the outer cylinder in the case where space charge is absent.

From this it appears that the space charge close to the anode is without appreciable effect in limiting the current.

Effect of initial velocities of the electrons. In the derivation of Eq. (2) it was assumed that the electrons leave the cathode without initial velocities. In a previous paper[5] a full discussion has been given of the effects produced by these initial velocities in the case of parallel plane electrodes.

With a cylindrical cathode of small diameter in a large coaxial anode there is a relatively high potential gradient near the cathode so that the electrons move with high velocity through the large part of their paths. The correction for the initial velocities is thus considerably less than for parallel planes (about one fourth).

In the inverted case, with the cathode as outer cylinder, the electrons move with low velocity over most of their paths. The correction for initial radial velocities must therefore be much greater than for parallel planes.

In the inverted case the tangential velocities of the emitted electrons must be of very great importance in determining the current limitation by space charge. Consider an electron which is emitted from the cathode with a tangential velocity corresponding to a fall through the potential V_0. Under the influence of the central force the electron describes an orbit in which the angular momentum is constant. The tangential velocity thus increases in proportion to the reciprocal of the radius. Since voltages are proportional to the squares of the corresponding velocities, we have

$$V_t = V_0(r_0/r)^2$$

where V_t is the voltage equivalent to the tangential velocity at a point of radius r.

If V_1 is the voltage of the anode with respect to the cathode, it is clear that (V_t+V_0) cannot exceed V_1. Neglecting V_0 in comparison with V_1, we see that the only electrons which can reach the anode are those for which

$$V_0 < V_1(r/r_0)^2 \tag{27}$$

Taking the energy corresponding to the tangential velocity as $kT/2$ and placing this equal to V_0e we find

$$V_0 = kT/(2e) = T/23{,}200 \text{ (volts)}.$$

[5] Langmuir, Phys. Rev. **21**, 419, 1923

With adsorbed films of alkali metals[6] it is possible to obtain large electron currents at temperatures as low as 700°K under conditions particularly favorable for the use of a cathode outside of an anode. It is thus possible to make V_0 (for the average tangential velocity) as low as about .03 volt. With this value of V_0 and the anode voltage V_1 equal to 300, we see by Eq. (27) that $r_0/r < 100$. Thus if the diameter of the anode is less than 1/100th of that of the coaxial cathode, about half the emitted electrons cannot reach the anode because of their tangential velocities. Since these, however, contribute about equally to the space charge, the electron current flowing to the anode will be approximately half that calculated by Eq. (1). These considerations make it possible to estimate the magnitude of deviations from Eq. (1) resulting from tangential velocities and will serve as the basis for future theoretical and experimental investigations.

Lack of radial symmetry in the electrodes is another factor that can cause tangential velocity components and lead to a decrease in the space charge current.

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[6] Langmuir and Kingdon, Science, **57**, 58 (1923)