THE OPACITY OF AN IONIZED GAS

By John Q. Stewart

Abstract

Theory of the absorption of radiant energy by free electrons in an ionized gas.—The classical theory of the optical properties of metals, supposed to contain free electrons, is applied to an ionized gas. As a result of collisions with molecules, vibrational energy supplied to the electrons by the electromagnetic field is transformed into energy of thermal agitation. The coefficient of absorption comes out proportional to $\lambda^2 p^2$, where λ and p are the wavelength and pressure. There is also a scattering effect proportional to p and independent of λ, but this is relatively unimportant except at pressures of 10^{-4}atm. or less. The equations developed are tentatively applied to a discussion of (1) *opacity of the vapors of exploded wires.* However Anderson reports an opacity 200 times greater than that roughly computed from theory, and varying inversely as λ or λ^2. Evidently the subject requires further study. (2) *Opacity of the solar atmosphere.* Employing Saha's theory to calculate the ionization, coefficients are computed which indicate that electronic absorption may be an important cause of the opacity of the solar photosphere. It is concluded that light from regions where the pressure is greater than .01 atm. is cut off completely, so that all we see comes from a spherical shell of rarefied gas. (3) *Opacity of giant stars.* Computations made, assuming the relation between temperature and pressure given by Eddington's theory of stellar constitution, are found in rough agreement with Eddington's values of opacity.

YEARS ago J. J. Thomson showed that, owing to scattering of radiation, an atmosphere containing free electrons should possess marked general opacity; and he suggested that this cause of opacity might be effective in the solar atmosphere.[1] Now that Saha's theory is available for calculation of the ionization, this suggestion can be quantitatively applied.

By the process of scattering, radiation is diffused in direction but not lessened in amount nor much changed in frequency. The term absorption should be reserved for those processes which result in the disappearance of radiant energy. The present paper points out that in an ionized gas collisions between free electrons and molecules theoretically result in such an absorption. These collisions transform into thermal energy of translation the vibrational energy supplied to the electron by the electromagnetic radiation field. Formulas describing this process have been worked out by Drude, Lorentz, and others in connection with the

[1] J. J. Thomson, Phil. Mag. **4**, 253, 1902

free electron theory of the optical properties of metals.[2] In the present paper these formulas are applied to conditions in an ionized gas.[3]

Theory

The standpoint is that of classical electromagnetic theory. How far this theory represents the facts is doubtful. For this and other reasons the results of this paper are to be regarded as only tentative.

When plane waves of polarized light are advancing through an ionized gas which is not too dense, the following well-known equation applies statistically to the motion of each free electron in a layer parallel to the wave-front and thin with reference to the wave-length,

$$m\frac{du}{dt} - \frac{ma}{c}\frac{d^2u}{dt^2} + 2mru = eE. \tag{1}$$

The mass of an electron is m, or 8.93×10^{-28}; its charge is e, or 4.774×10^{-10}; its radius is a, $=2e^2/3mc^2$, or 1.88×10^{-13}; c is the velocity of light, or 3.00×10^{10}. (In this paper c.g.s. and electrostatic units are employed throughout, except that pressures are expressed in atmospheres.) The magnitude of the alternating electric vector of the radiation field is E; t represents time; and u is that part of the component of the electron's velocity in the direction of E which is due to the action of the radiation field. The number of collisions with atoms or ions per unit time per electron, due to the thermal agitation, is r.

The electric field of the radiation exerts a force eE on each electron; the direction of this force lies in the plane of the wave-front. There is an additional force due to the polarization of the medium; but this is negligibly small unless the density of the ionized gas is very great or the frequency of the radiation low compared with the frequency of light. The magnetic field exerts a very small force (the radiation pressure) in the direction of propagation of the radiation; and this force also is negligible in the present discussion.

Eq. (1) states that the impressed force eE is balanced by three reactions. The existence of the first is obvious. The second is the well-known reaction on an electron of its own radiation, and may also be written $-(2e^2/3c^3)\ (d^2u/dt^2)$. The third is statistical; its existence is indicated by the following argument.

If a steady electrostatic field E' acts on an ionized gas, each free electron is subject to an acceleration eE'/m in the direction of the field. The effect of collisions between electrons will be neglected. Since the

[2] For example, O. W. Richardson, Electron Theory (1916), pp. 410, 432

[3] A preliminary account of the work has appeared: J. Q. Stewart, Nature **111**, 186, 1923.

mass of an electron is relatively small, the magnitude of its velocity in a given direction after a collision with an atom may be assumed uninfluenced by its velocity before the collision. Between successive collisions, then, each electron moves on the average in the direction of the field the distance $eE'/2mr^2$; so that the average velocity of drift due to the electrostatic field is $u=eE'/2mr$. Thus the average momentum of each electron in the direction of the field is $mu=eE'/2r$. Statistically the effect of collisions, therefore, can be interpreted as a frictional force on each electron of amount $2mru$. Since the velocity of drift is small compared with the average velocity of thermal agitation of the electrons, r is independent of u.

When E' is equal to unity, the magnitude of the electric current in a column of unit cross-section, or the electrical conductivity, obviously is, if there are n free electrons per unit volume,

$$\sigma=ne^2/2mr. \tag{2}$$

The current carried by the comparatively massive ions is negligible.

Returning to (1), write $E=E_0\cos 2\pi\nu t$, where ν is the frequency of the radiation of wave-length λ. The average rate s at which energy is diverted by each electron from the primary radiation is $(1/t)\int_0^t eEudt$. The intensity I of the radiation is given by $cE_0^2/8\pi$. The quantity a/λ is so small that its square may be neglected in comparison with unity. If r/ν likewise is small, as is true in the cases to be considered, it follows from (1) that

$$s=(3/\pi)\ (2\pi^2 a/\lambda+r/\nu)a\lambda I. \tag{3}$$

If $r=0$, as for an isolated electron, all of the diverted energy is reradiated—"scattered"—with frequency unchanged (to the first order of small quantities); and no energy of the radiation is transformed to heat. The energy scattered per unit time per electron is[4]

$$s_1=6\pi a^2 I=(8\pi e^4/3m^2c^4)I=(6.7\times 10^{-25})I, \tag{4}$$

when s_1 and I are expressed in c.g.s. units.

When r is not zero, the average rate at which heat is developed per unit volume is, from (3), $3nrac^2E_0^2/8\pi^2\nu^2$. This may be written $\frac{1}{2}\sigma_\nu E_0$, where σ_ν, thus defined, is the electrical conductivity of the ionized gas for alternating current of frequency ν. Accordingly, provided r/ν is small, and making use of (2),

$$\sigma_\nu=n^2e^4/4\pi^2m^2\nu^2\sigma, \tag{5}$$

nearly.

The physical significance of (3) is better brought out by writing it in the form,

$$s/r=(4\pi^2 a\nu/\lambda r)\phi+2\phi. \tag{6}$$

[4] L. Page, Astrophys. J. **52**, 67, 1920

Obviously s/r is the energy diverted per electron per collision interval. The quantity ϕ is the maximum kinetic energy of vibration of an electron due to the radiation; its value is easily shown from (1), when r/ν is small, to be $(3a\lambda^2/2\pi c)I$, since the maximum velocity attained in the simple harmonic vibration is, nearly, $eE_0/2\pi m\nu$. According to (4) the amount of energy scattered per electron per vibration is

$$(6\pi a^2/\nu)I = (4\pi^2 a/\lambda)\phi.$$

Thus the first term on the right hand side of (6) represents the energy scattered by an electron during the time between two successive collisions. (Collisions are assumed instantaneous). Evidently, then, (6) implies that at the average collision, twice the maximum vibrational kinetic energy of an electron in the radiation field is transformed into disorganized heat-molecular energy. The appropriate assumptions about the nature of collisions should immediately give this result, without going through (1) with its statistical term $2mru$. Another supporting line of argument has been given by Lorentz, in connection with the effect of molecular collisions on bound electrons. He has shown[5] theoretically that if an electron is given r random blows per unit time the resultant term due to such "impact-damping" in the equation corresponding to (1) is $2mru$.

Eq. (3) can also be written

$$s = (8\pi e^4/3m^2c^4)I + (2e^2r\lambda^2/\pi mc^3)I, \tag{7}$$

or, in c.g.s. units,

$$s = (6.7\times 10^{-25})I + (16.0\times 10^{-24})r\lambda^2 I.$$

The first term, or s_1, expresses the amount of energy scattered, the second term s_2, the amount of energy absorbed, per unit time per electron. Write

$$K = K_1 + K_2 = ns_1/I + ns_2/I.$$

Then K_1 is that part of the opacity coefficient K which is due to scattering, and K_2 is the part due to absorption.

It may be shown that the intensity I of radiation in the direct transmitted beam emergent from a layer of ionized gas of thickness z, when the intensity in the incident beam is I_0, is given by

$$I = I_0\epsilon^{-Kz}, \tag{8}$$

where ϵ is the base of natural logarithms. By (3) the ratio between the energy scattered in unit time per electron and the energy absorbed is $2\pi^2 a\nu/\lambda r$, a ratio which, owing to the decrease in r, increases with decreasing pressure of the gas. If scattering is appreciable the total amount

[5] H. A. Lorentz, Proc. Amsterdam Acad. **8**, 591, 1905, and other papers in the same Proceedings.

of light transmitted is greater than that which gets through in the direct beam.

Before (7) can be applied it is necessary to obtain an expression for r in terms of known quantities. As the kinetic theory of an ionized gas is an unexplored field, the following argument may be greatly in error. Assuming equipartition of energy between molecules and free electrons, the root-mean-square velocity of a free electron in a gas is given by $\sqrt{(3RT/m)}$, where R is the gas constant per molecule, or 1.37×10^{-16}, and T is the absolute temperature. This reduces to $(6.8\times10^5)\sqrt{T}$. The length of the electronic free path is $1/\pi NA^2$, where N is the number of atoms in unit volume, whether ionized or not, and A is the radius of an atom (assumed independent of ionization, an assumption doubtless incorrect). In most cases of astrophysical interest the ionized gas is presumably monatomic. Thus (nearly enough),

$$r=\pi NA^2\sqrt{(3RT/m)}. \tag{9}$$

Write $n=Ni$, so that i is a numerical factor expressing the average amount of ionization. Also write $N=273\ N_0p/T(1+i)$, where p is the total gas pressure in atmospheres, including the pressure $ip/(1+i)$ of the free electrons, and N_0 is the number (2.705×10^{19}) of molecules per unit volume of gas when $p=1$ and $T=273$.

Then, in the units adopted, since $273\pi N_0\sqrt{(3R/m)}=1.6\times10^{28}$,

$$r=(1.6\times10^{28})A^2p/(1+i)\sqrt{T}; \tag{10}$$

$$\sigma=e^2i/2\pi A^2\sqrt{(3mRT)}=60i/A^2\sqrt{T}; \tag{11}$$

$$K_2=\frac{2\sqrt{3}(273)^2N_0^2e^2\sqrt{R}}{c^3m^{3/2}}\cdot\frac{A^2\lambda^2ip^2,}{T^{3/2}(1+i)^2} \tag{12}$$

$$=(6.9\times10^{26})A^2\lambda^2ip^2/T^{3/2}(1+i)^2;$$

$$K_1=8\pi ne^4/3m^2c^4=(4.9\times10^{-3})ip/T(1+i), \tag{13}$$

since $$8\pi(273)N_0e^4/3m^2c^4=4.9\times10^{-3}.$$

When the mass coefficient of opacity k is employed, K in the above equations is replaced by $k\rho$ where ρ is the density. Thus the mass coefficient of absorption corresponding to K_2 is, for a monatomic gas,

$$k_2=2e^2\lambda^2ri/\pi mc^3M_0W=4.4\lambda^2ri/W \tag{14}$$

$$=(5.7\times10^{28})A^2\lambda^2ip/W(1+i)\sqrt{T},$$

where W is the atomic weight, and M_0 is the mass of the unit of atomic weight, or 1.65×10^{-24}.

In order to apply to physical problems certain of the above equations it is necessary to know the approximate value of A, the collision radius of atoms or ions of the gas. In view of the tentative nature of the theory presented in this paper the determination of A need not be precise. The collision radii of the molecules of the permanent gases at ordinary temperatures are known to be of the order 10^{-8} cm. The atomic volumes indicate that this is the order of magnitude of the radii of the uncharged atoms of metals. A crude theoretical formula for the collision radii, with regard to electrons, of metallic ions may be derived by the following method.

The ions, each carrying a positive change of average magnitude ie, may be regarded as centers of electrostatic forces which conform to Couloumb's inverse square law. Accordingly, the energy equation of an electron moving in a hyperbolic orbit with velocity u at distance x from an ion is, neglecting the forces due to other ions,

$$\tfrac{1}{2}mu^2=\tfrac{1}{2}mu'^2+ie^2/x,$$

where u' is the velocity at infinity. Assuming equipartition of energy between electrons and other molecules, $mu'^2=3RT$, on the average. It can easily be shown that the distance of closest approach between electron and ion, when the former is deflected through an angle 2θ by the encounter, is $(2ie^2/3RT)\cot^2\theta$. Thus, if θ is taken as $\pi/4$ for the average collision,

$$A=(1.1\times10^{-3})i/T. \tag{15}$$

As the ion is a structure as well as a center of force this value of A is to be regarded only as a rough approximation.

Astrophysical Applications

The opacity of the vapor of an exploded wire. The experiments of Anderson[6] have shown that a layer a few centimeters thick of the vapor of an exploded iron wire is sufficiently opaque to result in the production of a continuous spectrum, and to prevent the transmission of light. In a letter to the writer Dr. Anderson states that his most recent experimental results indicate that the value of K for iron vapor at 20,000° absolute and a total pressure of about 2 atmospheres, is of the order unity (presumably for λ near 4×10^{-5}); and, further, that K varies inversely as λ or λ^2.

The observed spectrum indicates that the iron atoms are for the most part doubly ionized, so that the value of i is about 2. Under these circumstances, (13) gives $K_1=3.3\times10^{-7}$; (10), (11), (12), respectively, give $r=(7.5\times10^{25})A^2$, $\sigma=0.85/A^2$, and $K_2=(3.5\times10^{11})A^2$. The small size of K_1 shows that scattering is not important here.

When $i=2$ and $T=20{,}000$, (15) gives $A=10^{-7}$ cm. Accordingly, $\sigma=8\times10^{13}$ e.s.u./cm^3, a conductivity seven thousand times less than that of metallic copper; and $K_2=0.004$, which is 250 times smaller than

[6] J. A. Anderson, Astrophys. J. **51**, 37, 1920; Proc. Nat. Acad. Sci. **8**, 231, 1922

Anderson's experimental value. If Anderson's published estimate of the pressure, namely $p/(1+i)=20$ atmospheres, is employed, the calculated value of K_2 comes out about 4. Thus, much depends upon the accuracy of the determination of the pressure. Perhaps a more serious disagreement between the theoretical K_2 and the experimentally observed K is that the former varies as λ^2 while the latter, it is thought, varies inversely as λ. Obviously the subject wants further study. The value of the "radius" of the iron ion, especially, is also very uncertain.

Good electrical conductors are good reflectors, as well as good absorbers and emitters of light. If the boundary of the vapor of an exploded wire were sufficiently sharply defined a considerable portion of incident light might be reflected. Thus the "Heaviside layer" of ionized gas in the upper atmosphere of the earth has been supposed by radio engineers to reflect Hertzian waves.

Pressures in the solar atmosphere. The following cursory examination of conditions in the outer regions of the sun indicates that the thermal absorption described by (12) may be an important cause of the opacity of the sun's photosphere.

The temperature of the sun's photosphere and lower reversing layer is not greatly different from 6000°. Substituting in (13) gives

$$10^{-5}/K_1=12(1+i)/ip \qquad (16)$$

for the distance in kilometers which would be required to reduce the intensity of the direct beam in the ratio of $1/\epsilon$ if scattering by free electrons were the only cause of opacity.

Ionization in the photosphere may not be sufficiently intense to make (15) applicable; but for purposes of argument A may be assumed as 5×10^{-8}. Then, for $\lambda=6\times10^{-5}$, (12) gives

$$10^{-5}/K_2=(75\times10^{-5})\,(1+i)^2/ip^2 \qquad (17)$$

as the distance in kilometers which would be required to reduce the intensity of the direct beam in the ratio of $1/\epsilon$ if the thermal absorption by free electrons were the only cause of opacity.

Saha's well-known theory permits calculation of i, and consequently of $10^{-5}/K_1$ and $10^{-5}/K_2$ in terms of p and T. Table I presents such calculated results for the case of sodium vapor at 6000°. The second ionization of sodium is neglected and Russell's values of i are used.[7] Values of r, the number of collisions per second per electron, are included from (10). This table may be regarded as illustrating, on a simplified scale, conditions in the solar atmosphere. The actual conditions are,

[7] H. N. Russell, Astrophy. J. **55**, 134, 1922

of course, much more complex, and the free electrons which are present have been liberated from all sorts of ions.

TABLE I

The opacity of ionized sodium at 6000°K

Pressure in atmospheres, of sodium vapor and free electrons p	Fraction of ionization i	Collisions per second per electron r	Kilometers distance to reduce intensity to $1/\epsilon$th Scattering $10^{-5}/K_1$	Absorption $10^{-5}/K_2$
1	0.21	4.2×10^{11}	69	0.0052
10^{-1}	0.56	3.3×10^{10}	330	0.33
10^{-2}	0.90	2.7×10^{9}	2,500	30.
10^{-3}	0.99	2.5×10^{8}	24,000	3000.
10^{-4}	1.00	2.5×10^{7}	240,000	300000.

Table I brings out the predominating influence at moderate and high pressures of thermal absorption by free electrons in producing general opacity, as compared with scattering by free electrons. The latter process is relatively important only at extremely low pressures. It is worth recalling that electromagnetic theory shows that general, as distinct from sharply selective, scattering by bound electrons is not enormously greater (and may be much less) than scattering by free electrons in an appreciably ionized gas.

Since K_2 varies as A^2p^2, a reduction in the estimate of the radius of the sodium ion would increase in the same proportion the values of pressure in the table. In a crystal of sodium chloride the shortest distance between sodium and chlorine ions is known to be 2.8×10^{-8} cm. It seems unlikely that the collision radius of the sodium ion should be much less than half this. If thermal absorption by free electrons in the solar atmosphere really is as intense as the numbers in the last column of Table I indicate, it is at once apparent that at a depth where the pressure is as great as 0.01 atmosphere the opacity is sufficient to cut off light from lower levels. This figure, then, is indicated as an upper limit to the pressure in the visible regions in the solar atmosphere. Photospheric absorption may be supposed to set in at, or above, this depth.

When observations are made of the flash spectrum, the grazing line of sight runs for tens of thousands of kilometers through the solar atmosphere. The faintness of the continuous spectrum under these conditions indicates, in connection with Table I, a pressure of less than 0.001 atmosphere at these levels. A detailed examination of the conditions is outside the province of the present paper. Other lines of evidence, for example the sharpness of the Fraunhofer lines, indicate that the pressures are very low.

The opacity deep in stars. In his well-known theoretical study of conditions in the interior of stars Eddington[8] has found that the mass absorption coefficient has a value of the order 20. Recently he has dealt with this matter by a kind of quantum theory.[9]

It is of interest to consider the application of (14) to conditions in the interior of stars. If the wave-length considered is that of maximum energy radiation, the value of λ to be substituted in (14) is $\lambda = 0.29/T$. Making use also of (15),

$$k_2 = (5.8 \times 10^{22}) i^3 p / W T^{9/2} (1+i). \tag{18}$$

By Eddington's theory of stellar constitution,

$$(1.0 \times 10^6) p = \beta a T^4 / 3(1+\beta), \tag{19}$$

where a is Stefan's constant, or 7.6×10^{-15}, and the factor 1.0×10^6 is approximately the number of c.g.s. units of pressure in one atmosphere. For purposes of illustration the value of β may be taken as 0.83, which makes (19) apply to a giant star of mass 1.5 times that of the sun. Substituting in (18),

$$k_2 = 700 i^3 / W(1+i)\sqrt{T}.$$

Since by definition there are i free electrons for every atom, W may be written as $W'/(1+i)$, where W' is the average molecular weight of the positive ions. Accordingly,

$$k_2 = 700 i^3 / W' \sqrt{T} = 30 i^3 / \sqrt{T}, \tag{20}$$

if W' is 23.

At the center of the star, T is of the order 4×10^6, and i may be taken as 12, say. Thus k_2 at the center is given as 25. Near the surface, say, where T is 10,000 and i is 2, k_2 is given as 2.4 Thus the opacity described by (20) is of the order of magnitude of the Eddington opacity, and also is roughly consistent with Eddington's original assumption that k does not vary with depth in the star.

In conclusion, the tentative character of the results presented scarcely needs to be emphasized. The subject of the opacity of gases to radiation is a broad and important one, which can be approached from many angles. In dealing with it the possibilities of classical electromagnetic theory appear by no means to have been exhausted.

I wish to thank Professor Russell for his very helpful interest in the preparation of this paper.

PRINCETON UNIVERSITY OBSERVATORY,
March 7, 1923.[10]

[8] A. S. Eddington, Astrophys. J. **48**, 212, 1918

[9] A. S. Eddington, Monthly Notices, **83**, pp. 32, 98, 1923

[10] Received April 16, 1923