A CORPUSCULAR QUANTUM THEORY OF THE SCATTERING OF X-RAYS BY LIGHT ELEMENTS

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Abstract

Corpuscular quantum theory of the scattering of x-rays.—In A. H. Compton's recent theory of the scattering of x-rays in separate quanta, a definite change in wave-length due to scattering is predicted; but in order to calculate the intensity of the scattered beam, he reasons in a not quite rigorous manner from analogy with the Doppler effect. In the present paper it is shown that the energy removed from the primary beam is of the order of magnitude of the energy falling on a sphere of the radius of the electron. It is therefore assumed that quanta of x-rays in the form of corpuscles are deflected by the electrons according to a law of force such that for corpuscles of small momentum (low frequency quanta), the distribution of the scattered rays is that expressed by the classical theory. It is found that for corpuscles of large momentum (high frequency quanta) the scattering electron recoils on collision in such a manner that the distribution of the energy of the scattered rays is modified. Curves and formulas are given showing for different radiation frequencies the theoretical values of the total energy removed from the primary beam by scattering, the energy which reappears in the scattered beam, and the energy of recoil in the scattering electrons. The formula expressing the distribution of the scattered x-rays (Eq. 17) is similiar in form to that obtained by Compton, but gives appreciably different results for very high frequency radiation such as hard γ-rays. Comparison with experimental results for the scattering of hard γ-rays shows an agreement which is probably within experimental error, and which is as good as that obtained with Compton's equations. By slightly modifying the assumptions, however, it is possible to obtain Compton's expression exactly, or to obtain other expressions differing slightly from it.

1. Introduction

In order to account for the change in wave-length which occurs when a beam of x-rays is scattered, A. H. Compton has proposed the view[1] that each quantum of radiation is scattered by a single electron. The change in momentum of the x-ray, due to its change in direction, results in a recoil of the scattering electron, so that the energy and hence the frequency of the scattered quantum is less than that of the incident ray. The change in wave-length thus predicted is in good accord with the experimental observations. In extending his theory to express the intensity of the radiation scattered in different directions, Compton calculates the intensity of the rays that would be scattered according to

[1] A. H. Compton, Phys. Rev. **21**, 483, 1923.

classical theory if the electrons were moving in the direction of the primary beam at a certain velocity. He takes this velocity as that which, due to the Doppler effect, would give a change in wave-length of the scattered beam equal to that predicted by his quantum theory. While this device gives an expression for the intensity which is in close accord with experiment, the method is not rigorous, as Compton himself points out, since he has not proved that the two scattering processes giving the same wave-length change will necessarily give the same distribution of scattered energy.

In order to avoid the uncertainty of this method of attack, the writer has developed a form of corpuscular quantum theory which does not involve any use of the Doppler effect. This point of view is encouraged by a consideration of the dimensions of the scattering coefficient. According to Thomson's classical theory of x-ray scattering,[2] the energy removed from the primary beam by a single electron is $8\pi e^4 I_0/3m^2c^4$, where e and m are the charge and mass of the electron, c is the velocity of light, and I_0 is the intensity of the incident beam. This is, however, the energy which falls on an area of $(8\pi e^4/3m^2c^4)cm^2$, or on a sphere of $\sqrt{(8/3)}e^2/mc^2=4.6\times 10^{-13}$ cm radius. According to the classical theory, therefore, radiation is scattered as if rebounding from surfaces whose radii are approximately those of the electrons. The problem of x-ray scattering is thus somewhat similar to the scattering of alpha particles by atomic nuclei. This view is similar to that proposed several years ago by W. H. Bragg[3] to explain the asymmetrical distribution of the photo-electrons ejected from matter by x-rays.

If the energy $h\nu$ of a quantum is entirely kinetic, and if the quantum has a momentum $h\nu/c=h/\lambda$, then the quantum cannot be distinguished from a corpuscle except in the matter of frequency. It will accordingly be assumed that the quantum is a corpuscle which in some way gives rise to a frequency expressed by the relation $w=h\nu$, where w is the kinetic energy. Further it will be supposed for the time being that these corpuscles are mathematical points moving with the speed of light. In accord with the considerations of the last paragraph, we must suppose that whenever a corpuscle approaches within a certain distance of the center of an electron it is scattered or deflected. For great wave-lengths, such that the momentum h/λ of the corpuscular ray is small, the mass of the scattering electron is large compared with the effective mass of the quantum, and we may suppose, in accord with experiments, that the scattering is as expressed by the classical theory. But for small wave-

[2] J. J. Thomson, "Conduction of Electricity through Gases," 2nd ed., p. 325.

[3] W. H. Bragg, Nature, Jan. 23, 1908; Phil. Mag. **16**, 918, 1908.

lengths the effective mass of the quantum $h/\lambda c$ approaches that of the electron, and the recoil of the scattering electron after impact with the corpuscular ray, will affect the distribution and the energy of the scattered rays.

2. Scattering of Long Wave-Length X-Rays

In this case an x-ray corpuscle comes within the field of action of the electron and is deflected, the electron having negligible momentum imparted to it, and the corpuscles are scattered to give an energy distribution according to Thomson's classical theory. Let the primary x-rays consist of a beam of corpuscles traveling in a direction parallel to the axis of x. Let there be n_0 corpuscles crossing unit area placed perpendicular to the beam in unit time, so that the intensity of the primary rays is

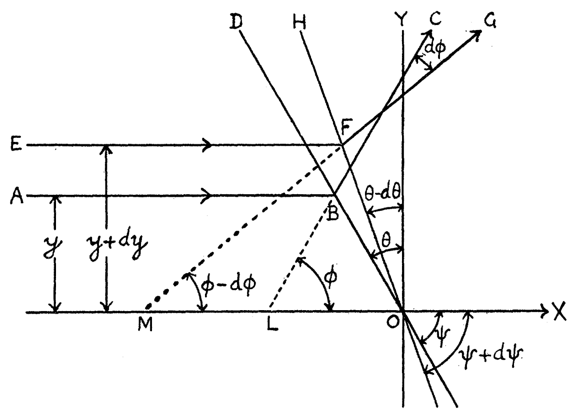

Fig. 1

$n_0 h\nu_0$ ergs/cm^2 sec. Since in an unpolarized beam of x-rays the scattering is symmetrical about the axis of x, it will be assumed that the field of the electron is symmetrical about this axis. Let the center of the electron be at O on the axis of x in Fig. 1. If the corpuscle approaches the electron along this axis, there is a head-on collision and the corpuscle is scattered directly backwards. If the line of flight of the impinging corpuscle is at a distance y from the x-axis, then the corpuscle is scattered in a direction ϕ, provided that y is not greater than a, where a is the maximum value of y at which scattering takes place. This distance a will be called the "radius" of the electron.

The problem now is to discover the relation between y and ϕ. The number of corpuscles approaching the electron between the perpendicular distances y and $(y+dy)$ is $2\pi n_0 y dy$. These corpuscles are scattered between the angles ϕ and $\phi - d\phi$. Taking a sphere about O as center with a radius R, where R is very great compared with a, these scattered corpuscles cross an area $-2\pi R^2 \sin\phi d\phi$ on the sphere. The number of corpuscles crossing the sphere per unit area in a direction ϕ is therefore:

$$n_\phi = -\frac{2\pi n_0 y}{2\pi R^2 \sin\phi} \cdot \frac{dy}{d\phi} = -\frac{n_0 y}{R^2 \sin\phi} \cdot \frac{dy}{d\phi} \tag{1}$$

and the intensity of the beam scattered in a direction ϕ is

$$-\frac{n_0 h\nu_0 y}{R^2 \sin\phi} \cdot \frac{dy}{d\phi} = -\frac{I_0 y}{R^2 \sin\phi} \cdot \frac{dy}{d\phi} \tag{2}$$

where I_0 is the intensity of the primary rays. By our hypotheses, this must be equal to the intensity scattered by a single electron according to the classical theory developed by Thomson,[2] i.e.,

$$-\frac{I_0 y}{R^2 \sin\phi} \cdot \frac{dy}{d\phi} = \frac{I_0 e^4}{R^2 m^2 c^4} \cdot \frac{(1+\cos^2\phi)}{2} \tag{3}$$

Integrating this we obtain

$$y^2 = (e^4/3m^2c^4)\ (4+3\cos\phi+\cos^3\phi) \tag{4}$$

since $y=0$ when $\phi=180°$. The greatest value of y occurs when $\phi=0$. This is the radius a. Hence

$$a = \sqrt{(8/3)} \times (e^2/mc^2) = 4.57 \times 10^{-13} \text{ cm.} \tag{5}$$

Relation (4) may thus be written

$$y^2 = (a^2/8)\ (4+3\cos\phi+\cos^3\phi) \tag{6}$$

Since Thomson's theory is so nearly true for long wave-length x-rays it follows that the radius given in (5) can be taken as the experimental value on the present theory. The total energy scattered by an electron is thus equal to its cross sectional area multiplied by I_0, i.e.,

$$\pi a^2 I_0 = (8\pi/3)\ (e^4/m^2c^4) I_0 \tag{7}$$

This is, as it should be, the same as the total energy scattered by an electron according to Thomson's theory.

3. Scattering of Short Wave-Length X-rays

In Compton's paper the equations expressing the conservation of energy and the conservation of momentum when a quantum is scattered by an electron are, referring to Fig. 2,

$$h\nu_0 - h\nu_\phi = [(1-\beta^2)^{-\frac{1}{2}} - 1]mc^2 \tag{8}$$

and

$$m^2\beta^2c^2/(1-\beta^2) = (h^2/c^2)\ (\nu_0^2 + \nu_\phi^2 - 2\nu_0\nu_\phi \cos\phi) \tag{9}$$

where β is the ratio of the velocity of the recoiling electron to that of light. It is from these relations that he obtains the formula for the change of frequency

$$\nu_\phi = \nu_0/(1+2\alpha \sin^2 \tfrac{1}{2}\phi), \text{ where } \alpha = h\nu_0/mc^2 \tag{10}$$

The angle of recoil ψ of the electron may be shown from these equations to be

$$\tan\,\psi = \frac{\cot\,\frac{1}{2}\phi}{(1+\alpha)}. \tag{11}$$

Referring again to Fig. 1, it will be assumed that when the impinging corpuscle arrives at B there is a sudden interaction between the corpuscle and the electron, the corpuscle being deflected so as to travel in the direction ϕ. It will be further assumed that the impulse given to the electron will be along the line BO. The problem therefore, is somewhat

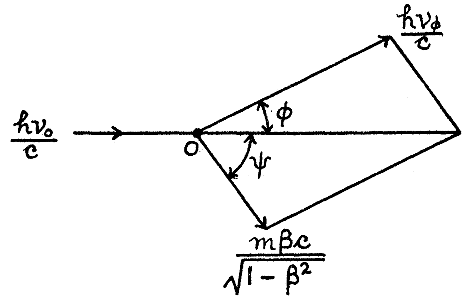

Fig. 2

similar to the oblique impact of two billiard balls of different masses. In the case of the momentum of the corpuscle being small, the angle $\phi = 2\theta$, where θ = angle DOY, OY being perpendicular to the axis of x. The relation between y and θ is therefore given by (6) when 2θ is substituted for ϕ, thus

$$y^2 = (a^2/8)\ (4+3\cos 2\theta+\cos^3 2\theta). \tag{12}$$

In this form the relation holds whether the momentum of the corpuscle is small or large. From Eqs. (1) and (12) we find that the number of corpuscles per cm^2 scattered at an angle ϕ is

$$n_\phi = \frac{3n_0 a^2}{8R^2}\ \frac{(1+\cos^2 2\theta)\sin 2\theta}{\sin\phi}\ \frac{d\theta}{d\phi}. \tag{13}$$

This may be expressed in terms of ϕ if we notice from Fig. 1 and Eq. (11) that

$$\tan\,\theta = \cot\,\psi = (1+\alpha)\tan\,\tfrac{1}{2}\phi \tag{14}$$

But the intensity of the x-rays scattered in a direction ϕ by N electrons per unit volume is $I_\phi = N\,n_\phi\,h\nu_\phi$. Using the values of ν_ϕ and n_ϕ given by (10) and (13) respectively, this becomes

$$I\phi = \frac{NI_0}{2R^2}\cdot\frac{3a^2}{8}\cdot\frac{(1+\alpha)^2\{1+\cos^2\phi+2\alpha(4+6\alpha+4\alpha^2+\alpha^3)\sin^4\frac{1}{2}\phi\}}{\{1+(2\alpha+\alpha^2)\sin^2\frac{1}{2}\phi\}^4\ \{(1+2\alpha\sin^2+\frac{1}{2}\phi\}} \tag{15}$$

The radius a can be determined as before by placing the value of I_ϕ when $\phi=0$ equal to the intensity scattered in the forward direction according to Thomson's formula. We thus obtain

$$a=\sqrt{(8/3)}\times(e^2/mc^2)/(1+\alpha) \tag{16}$$

Whence

$$I_\phi=\frac{I_0}{2R^2}\cdot\frac{Ne^4}{m^2c^4}\cdot\frac{\{1+\cos^2\phi+2\alpha(4+6\alpha+4\alpha^2+\alpha^3)\sin^4\tfrac{1}{2}\phi\}}{\{1+(2\alpha+\alpha^2)\sin^2\tfrac{1}{2}\phi\}^4\cdot\{1+2\alpha\sin^2\tfrac{1}{2}\phi\}} \tag{17}$$

The total energy removed from the primary beam due to the scattering process is therefore

$$N\times\pi a^2 I_0=\frac{8\pi}{3}\;\frac{Ne^4}{m^2c^4}\,I_0\cdot\frac{1}{(1+\alpha)^2}. \tag{18}$$

Both Eqs. (17) and (18) reduce to Thomson's classical form when $\alpha=0$, i.e., for great wave-lengths.

In Fig. 3, curve I shows for different values of α the total energy removed from the primary beam as expressed by Eq. (18). This energy is divided into two portions, as Compton has pointed out. One part,

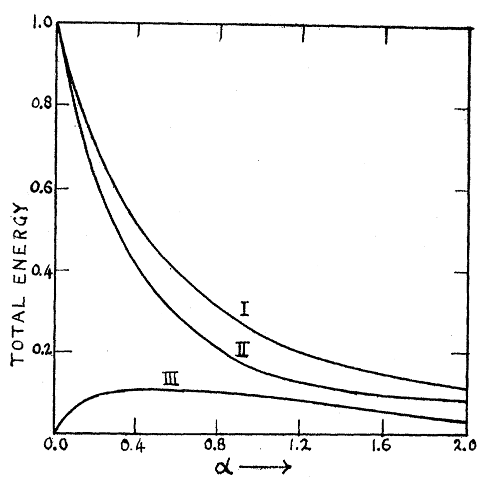

Fig. 3

represented by curve II, reappears as scattered rays, and has been calculated graphically by integrating expression (17) over the surface of a sphere. The second part, shown in curve III, is the energy of recoil of the scattering electrons. It is calculated by taking the difference between curves I and II. In all three curves the unit ordinate is the total energy removed according to the classical theory.

4. Comparison with Experiment

These equations expressing the intensity of the scattered x-rays differ only by second and higher powers of α from the equations derived by A. H. Compton from analogy with the Doppler effect. The differences amount at most to only a few per cent in the region of x-rays, so that in this region the present results are in equally satisfactory agreement with the experiments. For hard γ-rays, however, α is about unity, and in this case appreciable differences arise between these results and those of Compton.

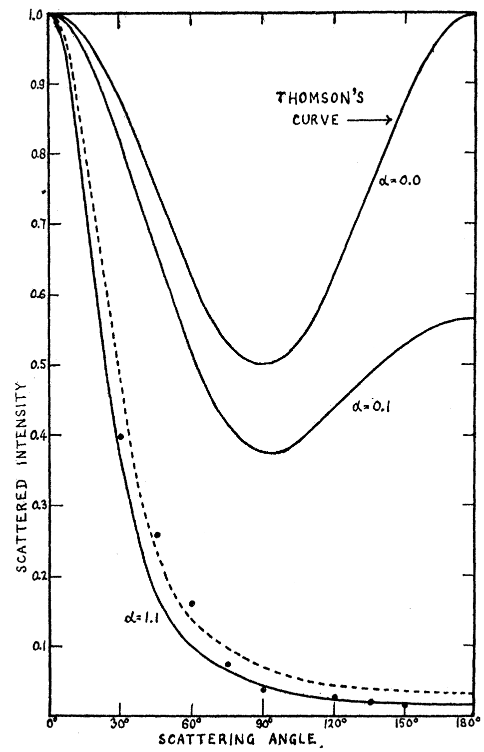

Fig. 4

Ishino's measurement of the total absorption of γ-rays by aluminium and iron[4] and A. H. Compton's measurements of the scattering of hard gamma rays at different angles[5] are perhaps as reliable as any that are available. On the present view, all the absorption of hard γ-rays by the lighter elements is due to the scattering process. Ishino's values of the total mass absorption coefficients for the hard γ-rays from RaC for

[4] M. Ishino, Phil. Mag. **33**, 140 (1917).

[5] A. H. Compton, Phil. Mag. **41**, 758 (1921).

aluminium and iron are 0.066 and 0.063 per gram respectively. According to Eq. (18), taking Compton's estimate of 0.022 A for the wave-length of the γ-rays, these values should be 0.044 for aluminium and 0.043 for iron, whereas Compton's theory gives the corresponding values as 0.061 and 0.059. This agreement is not very satisfactory.

Better agreement is obtained with the scattering experiments on hard γ-rays. Assuming that the ionization measures the intensity of the scattered rays, and taking the unit of scattered intensity as that at $\phi=0°$ according to Thomson's theory, the circles in Fig. 4 are obtained. These points are taken from Compton's paper.[1] Using as before $\lambda=0.022$ A, or $\alpha=1.1$, the lowest solid curve of Fig. 4 is obtained from Eq. (17). The broken curve is plotted from Compton's formula, using the same value of α. The experimental points fall somewhat more closely on my curve than on Compton's. Taking these tests as a whole, we must conclude that the accuracy of the experiments is not sufficient to decide which formula is the more nearly accurate.

5. Discussion

It has been shown that a corpuscular quantum theory which will give scattering according to Thomson's theory when long wave-length x-rays are used will also give a formula representing very well the experimental values of the scattering when short wave-length x-rays are used. Such differences as do occur may be explained as being due either to experimental error or to some unwarranted approximation in the theory. In the derivation of formula (17) it should be remembered that the value of the scattering at $\phi=0°$ has been assumed equal to that given by Thomson's theory at $\phi=0°$. The effect of this has been to cause the "radius" a of the electron to be a function of α and therefore of the wave-length. Also, the corpuscles have been considered as mathematical points. The "radius" may be the "radius" of the electron plus the "radius" of cross section of the quantum. Further, referring to Fig. 1, it has been assumed that for the same value of y/a, the angle BOY or θ remains the same no matter what the wave-length of the incident quantum may be, even though a changes with the wave-length. That is, although the "size" of the electron changes, yet the "shape" as represented by the relation between y/a and θ in (12) remains the same. If the "shape" varies with α, then expression (12) no longer holds.

Let us introduce a new variable A defined by the relation

$$\tan A=[\sqrt{(1+F)}/(1+\alpha)]\tan\theta=\sqrt{(1+F)}\tan\tfrac{1}{2}\phi \qquad (19)$$

where F is any function of α which becomes zero when $\alpha=0$. In order to bring about distortion of the "shape" of the electron, we shall replace θ by A in (12) and we obtain

$$y^2=(a^2/8)\ (4+3\cos 2A+\cos^3 2A). \tag{20}$$

This leads then to

$$I_\phi=\frac{I_0}{2R^2}\cdot\frac{Ne^4}{m^2c^4}\cdot\frac{\{1+\cos^2\phi+4F(1+\frac{1}{2}F)\sin^4\frac{1}{2}\phi\}}{\{1+F\sin^2\frac{1}{2}\phi\}^4.\{1+2\alpha\sin^2\frac{1}{2}\phi\}} \tag{21}$$

which takes the place of (17), and

$$N\pi a^2 I_0=\frac{1}{(1+F)}\cdot\frac{8\pi}{3}\cdot\frac{Ne^4}{m^2c^4}\cdot I_0 \tag{22}$$

which replaces (18). Putting $F=2\alpha$, Compton's formulas for I_ϕ and the scattering absorption coefficient are obtained from Eqs. (21) and (22) respectively. Putting $F=2\alpha+\alpha^2$, my formulas (17) and (18) are obtained. Hence, while the corpuscular theory gives the results stated in the earlier parts of this paper if no distortion of the electron is assumed, by postulating the appropriate distortion Compton's formula or any approximately similar formula may be derived.

In spite of these somewhat uncertain assumptions, formulas have, however, been derived from the corpuscular hypothesis which agree well with experiment. If a law of force between the quantum and the electron could be found such as to give Thomson's scattering formula for small values of h/λ, then possibly the true form of F in (21) could be attained.

It is perhaps worth noticing that the quantity α in these formulas is equal to the ratio of the equivalent mass of the quantum to the rest-mass of the electron.

In conclusion, the writer wishes to thank Professor A. H. Compton for much interesting discussion in regard to the subject of this paper.

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