MAGNETIC AND NATURAL ROTATORY DISPERSION IN ABSORBING MEDIA

By E. O. Hulburt

Abstract

Electron Theory of Magnetic and Natural Rotatory Dispersion in Absorbing Media.—On the basis of the electron theory of Lorentz, theoretical formulas have previously been derived for rotatory dispersion in perfectly transparent liquids, but on comparison with experimental results, a discrepancy was found which seemed to be due to the absorption. In the present paper, the theory has been extended to take account of absorption, and by making certain approximations, simplified formulas are obtained which give the rotation for wave-lengths sufficiently removed from the critical wave-length, provided the refractive index and extinction coefficient conform to the Lorentz dispersion equations with one resonance frequency. The equation of magnetic rotation in isotropic media agrees closely with experimental results for CS_2 and α-monobrom-naphthalene. It is suggested that the theoretical equation for natural rotation: $\theta/l=r_1+(2\pi^2\gamma_1/\lambda^2)\,[\mu_0{}^2-1/(1+\kappa_0{}^2)]$, (where μ_0 and κ_0 are refractive index and extinction coefficient, respectively, and r_1 and γ_1 are constants), may be assumed to describe the phenomenon in any medium, whether μ_0 and κ_0 satisfy the Lorentz dispersion equations or not, but this has not yet been tested experimentally.

Introduction

In two recent communications[1] formulas for the natural and magnetic rotatory dispersion in transparent liquids were developed from the electron theory of H. A. Lorentz and were submitted to experimental test by means of data obtained for the purpose. It was found that the values of the magnetic rotation angles calculated from the theoretical formulas increased with decrease of wave-length more rapidly than the observed angles and it was pointed out that the discrepancy might be attributed, in part at least, to the neglect in the theory of the effect of absorption. In the present paper absorption of the radiation in the medium has been considered from a theoretical standpoint and more complete rotatory dispersion formulas have been derived.

The theory of magnetic rotatory dispersion, i.e., the Faraday effect, and of natural rotatory dispersion, i.e., the Biot effect, has been excellently stated by Drude[2] and Voigt.[3] H. A. Lorentz[4] has further con-

[1] Hulburt, Astrophysical Journal, **54**, 45 and 116, 1921.
[2] Drude, Lehrbuch der Optik, 1906.
[3] Voigt, Magneto- und Electrooptik, 1908.
[4] H. A. Lorentz, Theory of Electrons 1916, p. 132.

tributed to the electron theory of the Faraday phenomenon, obtaining formulas similar to those of Drude and of Voigt but with a different interpretation of the constants of the equations. The formulas when reduced by simplifying assumptions were found to be in accord with the observed magnetic rotation in the case of the sodium flame and other experiments.[5] It seems, however, that the general formulas have not been sufficiently developed to be useful for the consideration of absorbing media.

Magnetic Rotatory Dispersion

We use the electron theory of dispersion as given by H. A. Lorentz and restrict the discussion to isotropic media in which the temperature remains constant. Let ξ and E_x be the X-components of the displacement of the electron from its equilibrium position and the electric force, respectively, and η, ζ, E_y, and E_z be the Y and Z components of the quantities. The components of the "restoring force" and of the "frictional force" with which the medium acts upon the electron are $f\xi$, $f\eta$, $f\zeta$ and $\beta\dot{\xi}$, $\beta\dot{\eta}$, $\beta\dot{\zeta}$, respectively, where f and β are positive constants. The charge on the electron is denoted by e, its mass by m; N is the number of such electrons per unit volume. σ is a constant which Lorentz has shown to be approximately one-third for isotropic media. We shall denote the external magnetic field by H and shall suppose it to have the direction of the axis of Z, which is also the direction of the propagation of the light. The magnetic permeability of the medium is taken to be unity. All quantities are expressed in c.g.s. e.m. units. In Newtonian notation the equations of motion of the dispersion electrons of a single type are, then,

$$\left.\begin{aligned} m\ddot{\xi} &= e(E_x+4\pi c^2\sigma Ne\xi)-f\xi-\beta\dot{\xi}+He\eta, \\ m\ddot{\eta} &= e(E_y+4\pi c^2\sigma Ne\eta)-fn-\beta\dot{\eta}-He\dot{\xi}, \\ m\ddot{\zeta} &= e(E_z+4\pi c^2\sigma Ne\zeta)-f\zeta-\beta\dot{\zeta} \end{aligned}\right\} \qquad (1)$$

Let e be the base of natural logarithms, and let all dependent variables of (1) contain the time only in the factor $e^{i2\pi tc/\lambda}$, where c/λ is the frequency, λ the wave-length of the vibration in vacuum, c is the velocity of light in vacuum, and i is $\sqrt{-1}$. The solution of (1) yields two vibrations, circularly polarized in opposite directions, whose refractive indices μ_1 and μ_2 and extinction coefficients κ_1 and κ_2 are given by the relations

$$\frac{1}{\sigma+\dfrac{1}{[\mu(1-i\kappa)]^2-1}}=\frac{C_s}{1/\lambda^2_s-1/\lambda^2+ib_s/\lambda\pm Hhs/\lambda}, \qquad (2)$$

where $C_s=N_se^2/\pi m_s,\ b_s=\beta_s/2\pi cm_s,\ h_s=e/2\pi cm_s.$

[5] Lorentz, loc. cit., p. 164.

The details of this solution are here omitted for they are essentially the same as those given by Lorentz. The subscript s is used to denote the s'th type of electron. c/λ_s is the frequency of the natural undamped vibration of the electron, and is equal to $1/2\pi\sqrt{f_s/m_s}$. When the plus sign is used in the right hand term of (2), μ is μ_1 and κ is κ_1. When the minus sign is used μ is μ_2 and κ is κ_2. We see that under the influence of the magnetic field the medium is doubly refracting and possesses two different degrees of transparency.

The *absorption coefficient* K of the nonmagnetized medium is defined by the relation

$$K=(1/2l)\ \log_e I_0/I, \tag{3}$$

where I_0 is the intensity of the incident radiation and I is the intensity, after passing through a layer of the medium of thickness l. In obtaining (2) from (1) we have made the substitution

$$\kappa=\lambda K/2\pi\mu. \tag{4}$$

From (3) and (4) the extinction coefficient κ is expressed by

$$\kappa=(\lambda/4\pi\mu l)\ \log_e I_0/I. \tag{5}$$

By means of (5), κ may be determined from measurements of the transmitted radiation.

Eq. (2) describes the refractive indices and extinction coefficients in terms of the constants of a single type of dispersion electron. There may be other types of dispersion electrons in the medium with constants peculiar to the type, so that in the more general case the right hand member of (2) may be assumed to be a summation of similar terms, one term for each type. For this case the complete dispersion formula is

$$\frac{1}{\sigma+\dfrac{1}{[\mu(1-i\kappa)]^2-1}}=\sum\frac{C_s}{1/\lambda_s^2-1/\lambda^2+ib_s/\lambda\pm Hh_s/\lambda}. \tag{6}$$

We assume we are dealing with a region of the spectrum in which the change of the refractive index with wave-length is determined by the electrons of a single type, denoted by the subscript 1, so that in the summation of (6) all the terms except one may be replaced by a quantity q_1, which is independent of λ and H. Then (6) becomes

$$\frac{1}{\sigma+\dfrac{1}{[\mu(1-i\kappa)]^2-1}}=q_1+\frac{C_1}{1/\lambda_1^2-1/\lambda^2+i\,b_1/\lambda\pm Hh_1/\lambda}. \tag{7}$$

Separating the real and imaginary parts of (7) by a transformation similar to one used by Havelock,[6] gives

[6] Havelock, Proc. Roy. Soc., **86**, 1, 1912.

$$\mu^2(1-\kappa^2)/q_1' = 1 + g\lambda^2(\lambda^2-\lambda_2'^2)/[(\lambda^2-\lambda_2'^2)^2+g'^2\lambda^2],$$
$$2\mu^2\kappa/q_1' = gg'\lambda^3/[(\lambda^2-\lambda_2'^2)^2+g'^2\lambda^2], \tag{8}$$

where q_1', λ_2', g and g' are defined by the following equations;

$$q_1' = [1+q_1(1-\sigma)]/(1-q_1\sigma), \quad \lambda_2'^2 = \lambda_1'^2(1 \mp Hh_1\lambda),$$
$$1/\lambda_1'^2 = 1/\lambda_1^2 - C_1\sigma/(1-q_1\sigma), \quad g = C_1\lambda_1'^2/(1-q_1\sigma)^2 q_1', \; g' = b_1\lambda_1'^2. \tag{9}$$

Again, just as in the case of Eq. (2), Eqs. (8) are each two formulas, giving μ_1 and κ_1 when the minus sign is used in the expression for λ_2' in (9), and μ_2 and κ_2 when the plus sign is used. It will abbreviate our expressions to replace the right hand terms of (8) by A_1, A_2, B_1 and B_2, obtaining

$$\mu_1^2(1-\kappa_1^2)/q_1' = A_1, \quad \mu_2^2(1-\kappa_2^2)/q_1' = A_2,$$
$$2\mu_1^2\kappa_1/q_1' = B_1, \quad 2\mu_2^2\kappa^2/q_1' = B_2. \tag{10}$$

We shall be concerned with the dispersion formulas for the nonmagnetized medium. These are obtained by placing $H=0$ in Eqs. (8), which leads to

$$\mu^2(1-\kappa^2)/q_1' = 1 + g\lambda^2(\lambda^2-\lambda_1'^2)/[(\lambda^2-\lambda_1'^2)^2+g'^2\lambda^2],$$
$$2\mu^2\kappa/q_1' = gg'\lambda^3/[(\lambda^2-\lambda_1'^2)^2+g'^2\lambda^2], \tag{11}$$

where q_1', λ_1', g and g' are defined by (9) and μ and κ, now physically single valued, refer to the nonmagnetized medium.

Let θ be the angle in radians of the rotation of the plane of polarization produced by a thickness l of the magnetized, doubly refracting, medium. It may be shown[7] that

$$\theta = (\pi l/\lambda)\,(\mu_2-\mu_1). \tag{12}$$

Solving (10) for μ_1 and μ_2 and substituting these values in (12) gives

$$\theta = (\pi l/\lambda)\sqrt{q_1'/2}\left\{\sqrt{A_2+\sqrt{A_2^2+B_2^2}} - \sqrt{A_1+\sqrt{A_1^2+B_1^2}}\right\} \tag{13}$$

This expression is rather cumbersome, so we change it into a more convenient form by availing ourselves of the fact that in most instances $Hh_1\lambda$ is small compared to unity. When the terms containing the square and higher powers of $Hh_1\lambda$ are neglected, (13) becomes

$$\theta = \pi l H h_1 \lambda_1'^2 \{[q_1' g\lambda^2/\mu(1+\kappa^2)]\,[(\lambda^2-\lambda_1'^2)^2 - g'^2\lambda^2]/[(\lambda^2-\lambda_1'^2)^2+g'^2\lambda^2]^2$$
$$+[8\mu^3\kappa^3/(1+\kappa^2)q_1']\,[(\lambda^2-\lambda_1'^2)/gg'\lambda^3]\}. \tag{14}$$

We may obtain a less expansive value for θ by assuming that the changes in κ occasioned by the magnetic field produce negligibly small changes in θ, or in other words, by assuming that $\kappa=\kappa_1=\kappa_2$. Making use of this assumption as well as of the smallness of $Hh_1\lambda$ the approximate values of μ_1 and μ_2 were obtained from the two Eqs. (10) and were introduced into (12). This gave

$$\theta = [\pi l H h_1\, q_1'\, g\lambda^2\lambda_1'^2/\mu(1-\kappa^2)]\,[(\lambda^2-\lambda_1'^2)^2 - g'^2\lambda^2]/[(\lambda^2-\lambda_1'^2)^2+g'^2\lambda^2]^2 \tag{15}$$

[7] Drude, loc. cit., p. 396.

Because of the approximations, formulas (14) and (15) are in close accord with the exact formula (13) for wave-lengths sufficiently removed from the critical wave-length λ_1' but are invalid for wave lengths in the region of λ_1'. Furthermore, all the formulas are permissible only when the refractive index and the extinction coefficient of the medium conform to the dispersion equations (11).

For zero absorption (14) and (15) each reduce to

$$\theta = \pi l H h_1 \, [q_1' g/\mu] \, [\lambda_1'^2 \lambda^2/(\lambda^2 - \lambda_1'^2)^2], \tag{16}$$

and this in turn is changed by means of (9) to

$$\theta = \pi l H h_1 C_1 \, \{[\sigma(\mu^2 - 1) + 1]^2/\mu\} \, \{\lambda_1^4 \lambda^2/(\lambda^2 - \lambda_1^2)^2\}. \tag{17}$$

This is the same as formula (12) of a former paper,[8] and if σ is placed equal to zero, (17) agrees with a formula of Voigt when his quantities are expressed in c.g.s. e.m. units.

In the paper just cited the measured magnetic rotation angles of a number of liquids for wave-lengths of light in the visible region of the spectrum were found to be less than the angles calculated by means of (17) for the shorter wave-lengths. In order to apply the more complete formulas (14) or (15) the extinction coefficients must be known. These have not been measured directly for the substances concerned, but in the cases of carbon disulphide and α-monobromnaphthalene, values of g' are available which were derived from measurements of the ultraviolet reflecting powers[9] of the liquids. With these values of g', formula (15) gave close agreement with the observations throughout the visible spectrum. For illustration, the approximate calculations for carbon disulphide are given in Table I. From refractive index data in the visible spectrum, considering absorption negligible, it is found that $q_1' = 1.7544$, $g = 0.434$, and $\lambda_1' = 228.4\mu\mu$. From ultraviolet reflection data $g' = 0.5136 \times 10^{-5}$.

TABLE I

Wave-length	θ Magnetic Rotation Angles		
	from observation	from (16)	from (15)
$430\mu\mu$	22.1°	23.4°	22.0°
500	15.3	15.6	15.3
640	8.3	8.3	8.3

Natural Rotatory Dispersion

We postulate an optically active medium in which the temperature remains constant, free from the influence of an external magnetic field.

[8] Astrophysical Journal, 54, 45, 1921.
[9] Astrophysical Journal, 44, 1, 1917.

If the axis of Z be the direction of propagation of the light we find for the equations of motion of the dispersion electrons of a single type in Newtonian notation and c.g.s. e.m. units,

$$\left.\begin{aligned} m\ddot{\xi} &= e[E_x+4\pi c^2\sigma Ne\xi+\gamma\ \mathrm{curl}_x\bar{E}]-f\xi-\beta\dot{\xi}, \\ m\ddot{\eta} &= e[E_y+4\pi c^2\sigma Ne\eta+\gamma\ \mathrm{curl}_y\bar{E}]-f\eta-\beta\dot{\eta}, \\ m\ddot{\zeta} &= e[E_z+4\pi c^2\sigma Ne\zeta+\gamma\ \mathrm{curl}_z\bar{E}]-f\zeta-\beta\dot{\zeta}. \end{aligned}\right\} \tag{18}$$

All the quantities in the expressions except γ have been defined in the preceding section. γ is a constant which depends upon the anisotropy of of the medium and may be positive or negative. The explanation of the formation of these equations is given in the communication referred to in the introduction.[1]

The solution of (18) is carried out in the same way as in the case of equations (1) and is found to yield two vibrations circularly polarized in opposite directions, whose refractive indices μ_1 and μ_2 and extinction coefficients κ_1 and κ_2 are expressed by the relations

$$\frac{1}{\sigma+\dfrac{1-[2\pi\gamma/\lambda][\mu_1(1-i\kappa_1)]}{[\mu_1(1-i\kappa_1)]^2-1}}=\frac{C_s}{1/\lambda_s^2-1/\lambda^2+i\,b_s/\lambda}, \tag{19}$$

$$\frac{1}{\sigma+\dfrac{1+[2\pi\gamma/\lambda][\mu_2(1-i\kappa_2)]}{[\mu_2(1-i\kappa_2)]^2-1}}=\frac{C_s}{1/\lambda_s^2-1/\lambda^2+i\,b_s/\lambda}, \tag{20}$$

where C_s, b_s, and λ_s have the same meanings as before. Equations (19) and (20) describe the refractive indices and extinction coefficients in terms of the constants of a single type of dispersion electron. Since there may be other types of dispersion electrons in the medium, in the more general case the right hand members of (19) and (20) may be assumed to be summations of similar terms, one term for each type. However, if we deal with a region of the spectrum in which the changes of the refractive index and extinction coefficient with wave-length are determined by the electrons of a single type, denoted by the subscript 1, in the summations just mentioned all the terms except one may be replaced by a quantity q_1 which is independent of λ. Then (19) and (20) become, respectively,

$$\frac{1}{\sigma+\dfrac{1-[2\pi\gamma/\lambda][\mu_1(1-i\kappa_1)]}{[\mu_1(1-i\kappa_1)]^2-1}}=q_1+\frac{C_1}{1/\lambda_1^2-1/\lambda^2+ib_1/\lambda}, \tag{21}$$

$$\frac{1}{\sigma+\dfrac{1+[2\pi\gamma/\lambda][\mu_2(1-i\kappa_2)]}{[\mu_2(1-i\kappa_2)]^2-1}}=q_1+\frac{C_1}{1/\lambda_1^2-1/\lambda^2+i\,b_1/\lambda}. \tag{22}$$

If γ is zero in (21) or (22) we obtain quantities μ_0 and κ_0 defined by the expression

$$\frac{1}{\sigma+\dfrac{1}{[\mu_0(1-i\kappa_0)]^2-1}}=q_1+\frac{C_1}{1/\lambda_1^2-1/\lambda^2+ib_1/\lambda}. \tag{23}$$

Comparing (21), (22), and (23) we find

$$[\mu_0(1-i\kappa_0)]^2=\mu_1(1-i\kappa_1).\mu_2(1-i\kappa_2). \tag{24}$$

The real and imaginary parts of (21) and (22) may be separated, κ_1 and κ_2 may be eliminated, and μ_1 and μ_2 determined in terms of the dispersion constants. These values of μ_1 and μ_2 may be then introduced into equation (12) to determine θ. The expressions are unwieldy and in the present instance we choose to avoid such a straight-forward procedure and to simplify the equations by means of an approximation. We leave out of consideration the difference in transparency of the anistropic medium for the two kinds of circularly polarized light and assume that $\kappa_0=\kappa_1=\kappa_2$. This assumption is not permissible for wave-lengths near the critical region λ_1, but is closely valid for spectral regions removed from this. Eq. (24) then reduces to $\mu_0^2=\mu_1\mu_2$, which shows that μ_0 is the geometrical mean of μ_1 and μ_2. If the refractive index of the optically active medium be determined directly by a refractometer, the value obtained may be interpreted to be almost exactly μ_0. In like manner a measurement of the absorption coefficient by the usual transmission methods gives a number which may be taken to be K_0. The extinction coefficient κ_0 is obtained from this by means of the relation $\kappa_0=\lambda K_0/2\pi\mu_0$.

Introducing (21) and (22) into (12) and making use of the above assumption, we obtain

$$\theta=(2\pi^2\gamma l/\lambda^2)\ \{[\mu_0(1-i\kappa_0)]^2-1\}/(1-i\kappa_0,) \tag{25}$$

where θ is now a complex quantity. The real part of θ is given by

$$\theta=(2\pi^2\gamma l/\lambda^2)\ [\mu_0^2(1+\kappa_0^2)-1]/(1+\kappa_0^2). \tag{26}$$

By the same reasoning as before we note that several types of electrons may contribute to the rotation and therefore (26) may be written in the general case,

$$\theta=\Sigma(2\pi^2\gamma l/\lambda^2)\ [\mu_0^2(1+\kappa_0^2)-1]/(1+\kappa_0^2). \tag{27}$$

When κ is zero formula (27) agrees with one obtained by Drude. Again, considering that only a single type of electron is important in determining the variation of θ with λ for the region of the spectrum under investigation, (27) simplifies to

$$\theta=r_1l+(2\pi^2\gamma_1l/\lambda^2)\ [\mu_0^2(1+\kappa_0^2)-1]/(1+\kappa_0^2). \tag{28}$$

The quantity r_1 is independent of λ and may be positive or negative.

Because the refractive index μ_0 and the extinction coefficient κ_0 occur explicitly in (28) and the constants of the Lorentz dispersion equation (23) do not occur therein, we may proceed as is often done in theoretical physics. We may discard the scaffolding by means of which the equation (28) has been built up and, without troubling ourselves any more about the theory of electrons we may postulate equation (28) as a concise description of the phenomenon in any medium, no matter whether the refractive index and extinction coefficient of the medium do or do not satisfy the Lorentz dispersion equation. It remains to substantiate this by experiment.

UNIVERSITY OF IOWA,
IOWA CITY,
November, 1922