LINES OF ELECTRIC FORCE OF A MOVING CHARGE ON THE EMISSION THEORY.

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Abstract.

Simplified method of determining the lines of force of a moving electric charge, on the emission theory of Page and Bateman.—This theory supposes that the charge emits streams of elements which move with the velocity of light, and constitute the lines of force. It leads to a vector equation for the directions of emission. By a transformation a simpler vector equation is derived from which the solutions previously given for variable rectilinear and uniform circular motion are more easily obtained. These solutions are both shown to conform, each in a special way, to the Einstein law of composition of velocities for uniform motion.

Leigh Page [1] has shown how the equations of the electromagnetic field may be derived on the assumption that an electric charge is a source continuously emitting non-homogeneities which may be termed moving elements. "The nature of these moving elements is immaterial for the purpose of the theory other than each must be susceptible of continuous identification and must move in a straight line with the velocity of light. *A line of electric force* is defined as the locus of a stream of moving elements emerging from a single source." [2]

The same representation has been used and developed by Bateman.[3, 4] By this means he obtained the equations for the lines of electric force for a moving electric charge from the expression for the electric field derived from the Lorentz retarded potentials.

Let l, m, n be the direction-cosines of emission of the *moving elements*, or, as Bateman has termed them, *light particles* for any given line of force at any time τ where ξ, η, ζ are the coördinates of the electric charge at any time τ, then the equations of the line of force at any subsequent time t are given by

$$x = \xi + c(t - \tau)l; \qquad y = \eta + c(t - \tau)m; \qquad z = \zeta + c(t - \tau)n. \tag{1}$$

The vector $d\mathbf{p}$ joining the positions of two elements emitted at τ and $\tau + d\tau$ gives the direction of the electric field at any subsequent time t

[1] Amer. Jour. Sci., 38, p. 169, 1914.

[2] Page, Phys. Rev., N.S., Vol. XVIII., p. 292, 1921.

[3] Bull. Amer. Math. Soc., March, 1915.

[4] Phil. Mag., 41, p. 107, 1921.

at the point (x, y, z) which may be denoted by the vector $\boldsymbol{r}$ $(x - \xi, y - \eta, z - \zeta)$ where $r = c(t - \tau)$.

Using the vector $\boldsymbol{c}$ to denote the velocity of the moving element emitted at the time τ and $\boldsymbol{v}$ the velocity of the charge, we have

$$\frac{d\boldsymbol{p}}{d\tau} = \boldsymbol{c} - \boldsymbol{v} - \frac{r}{c}\frac{d\boldsymbol{c}}{d\tau}. \tag{2}$$

Writing f for the vector acceleration of the electric charge e and $\boldsymbol{E}$ for the electric field, the expression for $\boldsymbol{E}$ may be written[1]

$$\boldsymbol{E} = \frac{e(1 - \beta^2)}{4\pi^{r^2}c\left(1 - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}\right)^3}\left[\boldsymbol{c} - \boldsymbol{v} + \frac{r}{c}\frac{(f \times (\boldsymbol{c} - \boldsymbol{v})) \times \boldsymbol{c}}{c^2(1 - \beta^2)}\right]. \tag{3}$$

Since $d\boldsymbol{p}/d\tau$ is parallel to $\boldsymbol{E}$ we have

$$\frac{d\boldsymbol{c}}{d\tau} = -\frac{[f \times (\boldsymbol{c} - \boldsymbol{v})] \times \boldsymbol{c}}{c^2(1 - \beta^2)}. \tag{4}$$

Bateman has given a complex transformation by which the vector equation (4) is replaced by a Riccatian equation. This equation has been discussed by Murnaghan[2] who has given solutions for some simple forms of motion of the charge.

It is shown in this paper that equation (4) can be thrown into a vector equation of simple form from which solutions can readily be obtained for all the simple types of motion treated by Murnaghan. The solutions are shown to be in accordance with a general assumption made by Page regarding the acceleration of a moving charge and the direction of emission of the moving elements. This assumption has been used by him to deduce equation (4) and so expression (3) for the electric force. The results in this paper also support the application made by Page of the Einstein law for the composition of velocities to accelerated motion, an application which would seem to require justification.

Vector Solution.

Writing

$$\boldsymbol{s} = (\boldsymbol{c} - \boldsymbol{v})\frac{\sqrt{1 - \beta^2}}{1 - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}} \tag{5}$$

we find that (4) becomes

$$\frac{\dot{\boldsymbol{s}}}{\boldsymbol{s}\cdot\boldsymbol{v}} = -\frac{\dot{\boldsymbol{v}}}{c^2 - v^2}. \tag{6}$$

[1] Page, Amer. Jour. Sci., 38, p. 169, 1914.

[2] Amer. Jour. of Math., Vol. 39, April, 1917.

This is a linear vector equation of the form

$$\boldsymbol{s} + \Phi\boldsymbol{s} = 0,$$

and admits of a symbolical solution in the form

$$\boldsymbol{s} = [1 - \int\Phi d\tau + \int\Phi d\tau\int\Phi d\tau - \int\Phi d\tau\int\Phi d\tau\int\Phi d\tau \cdots]\boldsymbol{q},$$

where $\boldsymbol{q}$ is any arbitrary vector.

Special cases of motion can be solved by aid of the following relations, which are readily established from (5) by writing $\boldsymbol{c} - \boldsymbol{v} = \boldsymbol{u}$:

$$\boldsymbol{s}\cdot\boldsymbol{v} = \sqrt{c^2 - v^2}\,\sqrt{c^2 - s^2}, \tag{7}$$

$$\boldsymbol{c} - \boldsymbol{v} = \boldsymbol{s}\,\frac{1 - \dfrac{\boldsymbol{v}\cdot\boldsymbol{c}}{c^2}}{\sqrt{1 - \beta^2}} = \boldsymbol{s}\,\frac{c^2 - v^2}{c\sqrt{c^2 - v^2} + \boldsymbol{s}\cdot\boldsymbol{v}}. \tag{8}$$

It may be noted that (7) enables us to transform (6) into the symmetrical form

$$\frac{\boldsymbol{s}}{\sqrt{c^2 - s^2}} = -\frac{\dot{\boldsymbol{v}}}{\sqrt{c^2 - v^2}}, \tag{9}$$

exhibiting the close relationship which exists between the form of the lines of force and the path of the electric charge.

Rectilinear Motion.

Let the charge move with uniform acceleration along the axis of x. Then if s_x, s_y, s_z denote components along the axes and m', n' are constants, we can write from (6)

$$s_y = cm', \qquad s_z = cn';$$

and from (7)

$$v^2 s_x^2 = (c^2 - v^2)(c^2 - c^2m'^2 - c^2n'^2 - s_x^2),$$
$$s_x^2 = (c^2 - v^2)(1 - m'^2 - n'^2) = (c^2 - v^2)l'^2,$$

where $l'^2 + m'^2 + n'^2 = 1$.

$$\therefore\quad \boldsymbol{s}\cdot\boldsymbol{v} = l'v\sqrt{c^2 - v^2},$$

We get then by resolving (8) along the axes and writing c_x, c_y, c_z for components of velocity of emission of the moving elements

$$c_x = cl = \frac{l'(c^2 - v^2)\sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2} + l'v\sqrt{c^2 - v^2}} + v$$
$$= \frac{cl' + v}{1 + \dfrac{vl'}{c}}, \tag{10}$$

$$c_y = cm = \frac{cm'\sqrt{1-\beta^2}}{1+\dfrac{vl'}{c}}.$$

$$c_z = cn = \frac{cn'\sqrt{1-\beta^2}}{1+\dfrac{vl'}{c}}.$$

These equations are the formulæ of transformation of velocities c_x, c_y, c_z to an observer in a system K' moving with the charge with velocity v with respect to the system K to which all our equations have hitherto referred. The moving elements are emitted, therefore, with velocities cl', cm', cn' to an observer in K' and, as l', m', n' are constants of integration, the directions of emission are therefore constant to an observer moving with the charge.

If K' is the system when the charged particle has the velocity $\boldsymbol{v}$ and K'' when it has the velocity $\boldsymbol{v} + d\boldsymbol{v}$ or $d\boldsymbol{v}'$ relative to K', Page[1] assumes that "the angle between any particular tube of strain in the immediate vicinity of the charged particle and the direction of $d\boldsymbol{v}$ must have the same value to an observer in K'' when the charge is in K'' as this angle has to an observer in K' when the charge is in K'."

When the acceleration is constant in direction, this assumption requires a direction of emission which appears constant to an observer moving with the charge as shown above.

The values for l, m, n given by Murnaghan appear in different form. We obtain his values if we write, using α and β as arbitrary constants,

$$l'/(\alpha^2+\beta^2-1) = m'/2\beta = n'/2\alpha = (\alpha^2+\beta^2+1)^{-1},$$

giving

$$Rl = (c+v)(\alpha^2+\beta^2) - (c-v),\ Rm = 2\beta(c^2-v^2)^{1/2},\ Rn = 2\alpha(c^2-v^2)^{1/2},$$

where

$$R = (c+v)(\alpha^2+\beta^2) + (c-v).$$

Motion in a Circle with Uniform Speed.

Let a be the radius of a circle described by a charge e in the plane of (xy) and p be its angular velocity. Using $\boldsymbol{X}_1$, $\boldsymbol{Y}_1$, $\boldsymbol{Z}_1$ to represent unit vectors along the axes and referring to the center of the circle as origin and to axes rotating with the electron, we have $\boldsymbol{v} = ap\boldsymbol{Y}_1$, $\dot{\boldsymbol{v}} = -ap^2\boldsymbol{X}_1$, $\dot{\boldsymbol{s}}_f = \dot{\boldsymbol{s}}_m + p(\boldsymbol{Z}_1 \times \boldsymbol{s})$ in accordance with the usual formula for the rate of change of a vector quantity with respect to moving axes,[1] where the notation $\dot{\boldsymbol{s}}_m$ refers to the moving system.

[1] Amer. Jour. Sci., 38, p. 169, 1914.

[2] Whittaker, Analytical Dynamics.

Equation (6) becomes

$$\frac{\dot{\boldsymbol{s}}_m + p(\boldsymbol{Z}_1 \times \boldsymbol{s})}{ap\cdot(\boldsymbol{s}\cdot \boldsymbol{Y}_1)} = \frac{ap^2 \boldsymbol{X}_1}{c^2 - p^2a^2}.$$

We derive the scalar equations

$$\dot{x} - py = a^2p^3y/(c^2 - p^2a^2), \qquad \dot{y} + px = 0, \qquad \dot{z} = 0.$$

Solving, we find

$$x = A \sin (q\tau - \epsilon), \qquad y = A\omega \cos (q\tau - \epsilon), \qquad z = B, \tag{11}$$

where

$$q = p/\omega, \qquad \omega^2 = 1 - a^2p^2/c^2 = 1 - v^2/c^2.$$

Using

$$(\boldsymbol{s}\cdot\boldsymbol{v})^2 = (c^2 - s^2)(c^2 - v^2),$$

we get

$$y^2v^2 = (c^2 - v^2)(c^2 - x^2 - y^2 - z^2),$$

giving

$$B = \sqrt{c^2 - A^2}.$$

Referring to fixed axes X, Y, Z, we get for the components (X, Y, Z) of $\boldsymbol{s}$

$$X = x \cos p\tau - y \sin p\tau, \quad Y = x \sin p\tau + y \cos p\tau, \quad Z = \sqrt{c^2 - A^2},$$

$$\boldsymbol{s}\cdot\boldsymbol{v} = A\omega ap \cos (q\tau - \epsilon). \tag{12}$$

We have from (8)

$$\boldsymbol{c} = \boldsymbol{s}/K + \boldsymbol{v},$$

where

$$K = \frac{c}{\sqrt{c^2 - v^2}} + \frac{\omega Aap \cos (q\tau - \epsilon)}{c^2 - v^2}$$

$$= \frac{1}{\omega} + \frac{\omega Aap \cos (q\tau - \epsilon)}{c^2\omega^2} \tag{13}$$

Hence

$$cl = X/K - ap \sin p\tau; \quad cm = Y/K + ap \cos p\tau; \quad cn = \sqrt{c^2 - A^2}/K. \tag{14}$$

This solution can be thrown into the form given by Murnaghan[1] by the substitutions, using his notation

$$g = (1 - \omega)/2\omega, \qquad \tan \epsilon = \beta/\alpha,$$

$$S = \frac{\alpha^2 + \beta^2}{g\omega} + \frac{g}{\omega} - \frac{2ap}{c\omega}(\alpha \cos q\tau + \beta \sin q\tau),$$

$$K = \frac{gS}{\alpha^2 + \beta^2 + g^2},$$

$$A = -\frac{2cg\sqrt{\alpha^2 + \beta^2}}{\alpha^2 + \beta^2 + g^2},$$

[1] Loc. cit.

giving

$$\frac{\sqrt{c^2 - A^2}}{cK} = \frac{g^2 - (\alpha^2 + \beta^2)}{gS} = \frac{1}{S}\left[g - \frac{\alpha^2 + \beta^2}{g}\right] = \frac{T}{S}.$$

Lines of Force Due to Uniform Circular Motion.

Varying the notation used in equation (1) we can write for the equations of the lines of force at the time $t = 0$,

$$\xi = a \cos p\tau - cl\tau; \qquad \eta = a \sin p\tau - cm\tau; \qquad \zeta = - cn\tau.$$

Substituting for l, m, we get from (13), (14)

$$\xi = a \cos p\tau - \tau\left[\frac{A \cos p\tau \sin (q\tau - \epsilon) - A\omega \sin p\tau \cos (q\tau - \epsilon)}{1/\omega + Aap \cos (q\tau - \epsilon)/c^2\omega} - ap \sin p\tau\right],$$

$$\eta = a \sin p\tau - \tau\left[\frac{A \sin p\tau \sin (q\tau - \epsilon) + A\omega \cos p\tau \cos (q\tau - \epsilon)}{1/\omega + Aap \cos (q\tau - \epsilon)/c^2\omega} + ap \cos p\tau\right].$$

For purposes of calculation, it is convenient, since only negative values of τ are permissible, to change the sign of τ and write $\tau = - t$. It will also be found convenient to write $\epsilon = \delta - \pi/2$ and to consider only clockwise rotation. In calculating, we deal only with the past history of the charge and, as t increases, we describe the path anticlockwise; we shall therefore use negative values for p and q. In order to avoid cumbering the notation, we shall use the same symbols and merely change their signs, substituting $- p$ for p and $- q$ for q. We get, then,

$$\xi = a \cos pt + t\left[\frac{A \cos pt \cos (qt - \delta) + A\omega \sin pt \sin (qt - \delta)}{1/\omega + Aap \sin (qt - \delta)/c^2\omega} + ap \sin pt\right],$$

$$\eta = a \sin pt + t\left[\frac{A \sin pt \cos (qt - \delta) - A\omega \cos pt \sin (qt - \delta)}{1/\omega + Aap \sin (qt - \delta)/c^2\omega} - ap \cos pt\right].$$

Writing

$$qt = pt/\omega = \theta/\omega = \varphi + \delta,$$

$$\xi p/c = \xi', \qquad \eta p/c = \eta',$$

$$\omega^2 = 1 - a^2p^2/c^2 = 1 - \beta^2,$$

$$c\sqrt{1 - n^2} \cos \alpha = \frac{A\omega \cos (qt - \delta)}{1 + Aap \cdot \sin (qt - \delta)/c^2} = \frac{A\sqrt{1 - \beta^2} \cos \varphi}{1 + \beta A \sin \varphi/c}, \qquad (15)$$

$$c\sqrt{1 - n^2} \sin \alpha = \frac{A\omega^2 \sin (qt - \delta)}{1 + Aap \cdot \sin (qt - \delta)/c^2} + ap = \frac{A \sin \varphi + ap}{1 + \beta A \sin \varphi/c},$$

we get finally

$$\xi' = \beta \cos \theta + \theta\sqrt{1 - n^2}(\cos \theta \cos \alpha + \sin \theta \sin \alpha),$$

$$\eta' = \beta \sin \theta + \theta\sqrt{1 - n^2}(\sin \theta \cos \alpha - \cos \theta \sin \alpha).$$

It will be observed that this alteration in notation is equivalent to taking as the unit of time—the time required to describe one radian—

$1/p$; the distance traversed by light in this time is the new unit of distance c/p so that the velocity of light is unity. The radius of the circle described has the same numerical value (β) as the velocity of the charge.

When β is small $\alpha = \theta - \delta$ and the equations of the lines of force in the plane of the circle for which $n = 0$ may be written in the simple form

$$\xi' = \beta \cos \theta + \theta \cos \delta,$$
$$\eta' = \beta \sin \theta + \theta \sin \delta.$$

The direction of emission is constant and the charge may be said to carry with it its lines of force which may be regarded as straight.

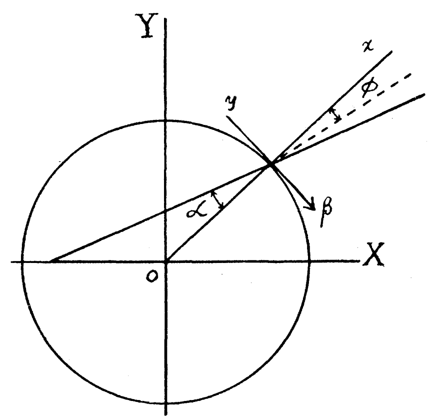

Fig. 1.

In general we have for lines of force in the plane of the circle for which $n = 0$, $A = c$ referred to fixed axes X and Y,

$$\xi' = \beta \cos \theta + \theta \cos (\theta - \alpha),$$
$$\eta' = \beta \sin \theta + \theta \sin (\theta - \alpha),$$

where θ is measured in the usual positive direction,

$\theta - \alpha$ = angle of emission at P as viewed by an observer (K) at the center of the circle,

α = angle of emission with the axis of x where x and y are a set of fixed axes of the system (K) instantaneously coinciding with moving axes at P belonging to a system K' whose direction of motion is $-\beta$ along the y axis.

If we refer the velocity of emission ($c \cos \alpha$, $-c \sin \alpha$, 0) to the system K' by means of the usual relativity formulæ and use (15) in which we put $A = c$, and $n = 0$, we have

$$c \cos \alpha = v_x = \frac{v_x' \sqrt{1 - \beta^2}}{1 - \beta v_y'/c} = \frac{c \cos \varphi \sqrt{1 - \beta^2}}{1 + \beta \sin \varphi},$$

$$- c \sin \alpha = v_y = \frac{v_y' - c\beta}{1 - \beta v_y'/c} = \frac{- c \sin \varphi - c\beta}{1 + \beta \sin \varphi},$$

from which we may deduce, subject to the validity of the use of these transformations for circular motion, that

$$v_x' = c \cos \varphi, \qquad v_y' = - c \sin \varphi.$$

φ is therefore the angle of emission of the "moving element" of a line of force in the plane of (xy) to an observer travelling with the charge. Since $\varphi = pt/\sqrt{1 - \beta^2} - \delta$, the emission-direction of a moving element will rotate with respect to the radius vector around the axis of z with an angular velocity $p/\sqrt{1 - \beta^2}$ in K units. In consequence the emission-direction for a given line of force may never repeat itself for a given position of the orbit.

In the more general case, using the formulæ of (15) we find that φ is the azimuth of the emission in the plane of (xy) with respect to the moving observer. If we use K' units of time, $\varphi = pt' - \delta$, so that the angular velocity of the emission-directions around the axis of z is p. It is possible therefore for the moving observer to regard the emission-directions as fixed, and to consider that the center of his orbit is rotating with respect to them with angular velocity p.

It can be readily shown from the above results that the condition laid down by Page quoted under rectilinear motion is also satisfied for circular motion.

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