THE EFFECT OF SPACE CHARGE AND INITIAL VELOCITIES ON THE POTENTIAL DISTRIBUTION AND THERMIONIC CURRENT BETWEEN PARALLEL PLANE ELECTRODES.

By Irving Langmuir.

Abstract.

Effect of space charge and cathode temperature on thermionic current and potential distribution.—I. *Case of parallel plane electrodes.* (*a*) *Current limited by space charge.* The results obtained by E. Q. Adams (unpublished), Epstein, Fry and Laue are discussed and summarized, certain errors are pointed out, and the equations are put in a form adapted to easy numerical calculation. Assuming the normal components of the velocities of the emitted electrons have the Maxwell distribution, the integration of Poisson's equation between proper limits leads to a numerical relation between the new variables $\xi = 2(x - x_m)[2\pi^3 e^2 i^2 m/k^3 T^3]^{1/4}$ and $\eta = e(V - V_m)/kT$, where x_m and V_m give the position and voltage of the plane of minimum potential, and k is the Boltzmann gas constant. Denoting values at the cathode by the subscript 1, and inserting values of constants: $\eta_1 = \log (i_0/i)$, where i_0 is the saturation current; $V - V_1 = T(\eta - \eta_1)/11{,}600$; $\xi - \xi_1 = 9.180 \times 10^5 T^{-3/4} i^{1/2}(x - x_1)$. These equations and *the tables of* $\xi(\eta)$ for various values of η enable, for a given cathode temperature T, the potential distribution for a given current i, or vice versa, to be computed. An approximate solution for the current is: $i = [(2^{1/2}/9\pi)(e/m)^{1/2}(V - V_m)^{3/2}/(x - x_m)^2](1 + 2.66\eta^{-1/2})$, which reduces to the usual three halves power law equation if we neglect V_m and x_m and the correction factor in η. (*b*) *Equilibrium condition with anode at great distance, current zero.* If the only retarding field is that of the space charge, the density of charge is: $\rho = kT\rho_1/[(kT)^{1/2} + x(2\pi e\rho_1)^{1/2}]^2$ where $\rho_1 = i_0[2\pi m/kT]^{1/2}$. Except near the cathode this is approximately equal to $kT/2\pi ex^2$; hence ρ is proportional to the absolute temperature of the cathode and inversely proportional to the square of the distance away. The potential gradient at the cathode is $X_1 = [8\pi\rho_1 kT/e]^{1/2}$. Equations are also given for the case where an external retarding field X_∞ is applied. II. *In the case of concentric cylindrical electrodes*, the current is: $i = (8^{1/2}/9)(e/m)^{1/2}[V - V_m + \frac{1}{4}V_0\{\log (V/\lambda V_0)\}^2]^{3/2}/r$, where V_0 is the initial energy of the electrons expressed in volts $(3kT/2e)$, r is the radius of the anode, and λ is a constant between 1 and 2, not yet experimentally determined. The deviations from the three halves power law are not more than one quarter as much as for parallel planes and amount to only about 3 per cent at 130 volts.

THE effect of space charge on the potential distribution near an electron-emitting plane cathode was calculated by Richardson [1] for the case where the opposing electrode is at infinite distance and the

[1] Phil. Trans., A, *201*, 516 (1903).

potential gradient at this second electrode is zero. The electrons were assumed to be emitted with velocities distributed in accordance with Maxwell's Law. Under these conditions no current flows between the electrodes. Child [1] and Langmuir [2] treated the case where current flows under the influence of an accelerating field insufficient to cause saturation. Neglecting the effect of the initial velocities of the electrons they found that the current varied with the three halves power of the potential difference between anode and cathode and that the potential varied with the four thirds power of the distance from the cathode.[3]

Schottky [4] made approximate calculations for the case of small currents, taking into account the effects of initial velocities. Laue [5] gives an exhaustive treatment of the case where one or more heated electrodes (parallel planes, concentric cylinders, spheres, etc.) are in equilibrium with an electron atmosphere, *i.e.*, when no current flows.

Epstein [6] gives a clear and complete treatment of the effect of initial velocities for parallel plane electrodes when the current is limited by space charge and tabulates a function by which these calculations are facilitated. Unfortunately Epstein has committed either an error or an oversight in the use of the Boltzmann constant k so that it becomes necessary to substitute $2k$ in place of each k that occurs in his equations in order to be able to use correctly the customary value $k = 1.37 \times 10^{-16}$ erg per degree. Since Epstein gives no numerical values of the constants

[1] PHYS. REV., *32*, 492 (1911).

[2] PHYS. REV., *2*, 450 (1913); Phys. Zeitschr., *15*, 348 (1914).

[3] Lilienfeld (PHYS. REV., *3*, 364, 1914; and many subsequent papers) claims to have discovered the three halves power law in some of his work in 1910. A very careful study of Lilienfeld's 1910 paper (Ann. d. Physik, *32*, 674, 1910), made in connection with an Interference before the U. S. Patent Office (Interference No. 40380, Arnold *vs.* Langmuir, Langmuir Record, pages 352 to 395) shows that in Lilienfeld's experiments the current did not even approximately vary with the three halves power of the voltage. The original data upon which Lilienfeld bases his claim are those given on page 698 of his 1910 paper. It there appears that no current flowed until the difference of potential between the sounding electrodes was 102 volts. When this voltage was raised to 116 volts the current increased 17-fold or with the 22d power of the voltage instead of the three halves power.

Dr. A. W. Hull and the writer have constructed and studied a tube as nearly as possible identical with Lilienfeld's tube of 1910, with the result that we have found that the type of discharge observed by Lilienfeld depends upon secondary electron emission from the walls of the narrow glass tubes under the influence of electron bombardment. Undoubtedly traces of residual gas are essential in starting the discharge, but it has not proved possible to stop the discharge by improvement of the vacuum. These experimental results are described in the Langmuir Record, p. 382. A discussion and summary have been published by the writer elsewhere (General Electric Rev., *23*, 513, 1920).

[4] Phys. Zeitschr., *15*, 526 (1914); Ann. der Physik, *44*, 1011 (1914).

[5] Jahrb. d. Radioakt. u. Elektronik, *15*, 205 (1918).

[6] Ber. d. Deut. phys. Ges., *21*, 85 (1919).

in his equations, his unusual value for k must lead to error in all numerical calculations unless the whole laborious derivation is gone through.

Fry [1] treats exactly the same problem as Epstein apparently without knowledge of this earlier work and obtains essentially similar results. Fry's equations, however, are expressed in very unusual nomenclature and this makes it not only difficult for others to apply his results but has caused Fry himself to commit serious errors in all his numerical calculations. Thus $\bar{v}_0$ is defined (p. 444) as the *average* velocity component of the *emitted* electrons normal to the surface. Nowhere does he state how this velocity may be calculated from the temperature. As a matter of fact this kind of an average is totally different from those customarily used in the kinetic theory. Before coming to his final equations Fry replaces $\bar{v}_0$ by another quantity which is defined as the "potential change, $\overline{V}_0$, which would give to an electron an energy equal to the average energy of those shot out from the cathode." Now the "*average energy*" of the electrons is not equal to the energy of an electron moving with the average velocity $\bar{v}_0$ nor is it four times this energy as is implied by the second of Fry's equations on p. 449. The third equation is also in error probably due to confusion between the many possible kinds of averages. These errors are all easily avoided if the temperature is brought into the equations at an early stage.

In the summer of 1913 at the request of the writer, Dr. E. Q. Adams undertook an analysis of the space charge problem, taking into account the initial velocities in accordance with Maxwell's Law. He arrived at the complete and correct solution but unfortunately failed to publish his results. In view of the discrepancies which have occurred in the publications of Epstein and Fry, it seems desirable to summarize and compare the results of these three sets of calculations and to present the equations in a form adapted to numerical computation. At the same time it is worth while to correlate these with the equations developed by Richardson and Laue. No attempt will be made to give complete derivations of the equations for these are very satisfactorily given in the publications referred to.

Case I. Current Limited by Space Charge.[2]

We will consider here the case treated by Epstein, Fry and Adams. Electrons are emitted in accordance with Maxwell's Law from a plane cathode of infinite extent. The anode is a parallel plane surface at a positive potential (with respect to the cathode) such that the current is less than the saturation current.

[1] Phys. Rev., *17*, 441 (1921).

[2] The nomenclature adopted is essentially that of Epstein except where confusion might arise because of a different meaning given to the same symbol by Fry.

In accordance with the usual derivation of the Richardson equation, let us assume that each unit of volume of the metal contains N_v electrons. Then Maxwell's Law states that the number dN_v of electrons in this volume having velocity components (in the direction of the x-axis) lying between v and $v + dv$ is

$$dN_v = N_v \sqrt{\frac{m}{2\pi kT}} \epsilon^{-\frac{mv^2}{2kT}} dv. \tag{1}$$

Here m is the mass of the electron (9.01×10^{-28} g.); k is the Boltzmann constant (1.372×10^{-16} erg per degree), and T is the absolute temperature of the cathode.

The form of assumption above, which is that made by Epstein, is open to the objection that it postulates knowledge that we do not possess of the internal structure of the metal of the cathode. The assumption however is mathematically equivalent to the following which is free from this objection and which has received a certain amount of experimental verification on the part of Richardson and others. Let N_s be the number of electrons emitted per unit time per unit area from a plane surface. Then according to Maxwell's Law [1] the number dN_s of electrons emitted per unit time per unit area which have velocity components normal to the surface lying between v and $v + dv$ is

$$dN_s = N_s \frac{mv}{kT} \epsilon^{-\frac{mv^2}{2kT}} dv. \tag{2}$$

If the current flowing to the anode is less than the saturation current (corresponding to N_s) it is evident that this must be due to a retarding potential gradient close to the surface of the cathode by which the more slowly moving electrons are forced back to the cathode. If the potential of the anode is positive, there must then be a surface between the cathode and anode at which the potential is a minimum. Using x as abscissa to measure distances in a direction normal to the cathode surface, we let x_1, x_2 and x_m be the abscissas of the cathode, anode and the surface of minimum potential respectively. Similarly, if V is the potential at any surface represented by the abscissa x, then V_1, V_2 and V_m are respectively the potentials of the cathode, anode and surface of minimum potential.

Let i_0 be the saturation current from the cathode obtainable by use of higher anode potentials. Electrons corresponding in number to i_0 are being *emitted* continuously from the cathode even when the current is not saturated, but a certain fraction of them are then made to return by the retarding field. The actual current i which flows between cathode and anode consists of those electrons which are emitted with sufficient

[1] See Richardson, Emission of Electricity from Hot Bodies, 1916, page 141.

velocity components to enable them to move against the potential difference $V_1 - V_m$.

By means of Equations (1) or (2) the following relation is obtained between i, i_0 and $V_1 - V_m$:

$$i = i_0 \epsilon^{-\frac{e(V_1 - V_m)}{kT}}, \tag{3}$$

where e is the negative charge on the electron (*i.e.*, 4.774×10^{-10} e.s.u.).

The treatment of the problem by Adams, Epstein and Fry is based on the following facts. Between x_1 and x_m there are two groups of electrons, those moving away from and those moving back towards the cathode. Among the former all velocities from zero to ∞ are present in accordance with Maxwell's Law. Although each electron loses velocity as it moves through the retarding field, the average velocity of all the electrons at any place remains constant because the more slowly moving electrons are continually being sorted out and sent back to the cathode. Considering only velocity components normal to the surface, among the electrons returning to the cathode, we see that all velocities are not present, for the velocity acquired by the electrons at any place of potential V cannot exceed that corresponding to a fall through the potential difference $V - V_m$. Between x_m and x_2 electrons are moving only away from the cathode. At any point of potential V the normal velocities may have any value above a certain minimum corresponding to a potential difference $V - V_m$.

The above conditions regarding the distribution of velocities are taken into account by means of proper choice of the limits in the integration of Eqs. (1) or (2).

The first step in the mathematical treatment is to calculate the space charge or the electron density at any point by means of the integrations just referred to. Then by Poisson's equation

$$d^2V/dx^2 = -4\pi\rho. \tag{4}$$

To determine the potential distribution, two more integrations must be carried out. The first integral can be expressed in terms of the Probability Integral, but the second requires the numerical calculation of a new function. In connection with these integrations it is necessary to choose values for the integration constants. For the first integration the conditions imposed are:

I. $dV/dx = 0$ when $V = V_m$;

and for the second integration:

II. $V = V_m$ when $x = x_m$.

The first of these conditions makes it impossible to apply the resulting equations to cases in which there is no potential minimum between the cathode and anode. For example the potential distribution between two electrodes with a potential slightly *greater* than that needed to give saturation cannot be calculated by this method, for the condition I. is not fulfilled. If in this case we impose the proper conditions for fixing the integration constants even the first integration cannot be performed. Similarly the equations resulting from these calculations must not be used where the retarding potential gradient extends up to the surface of the anode.

Before carrying out the numerical calculations required in the second integration it is desirable to reduce the equations to a form in which only pure numbers occur. This may be accomplished by introducing the new variables:

$$\eta = e(V - V_m)/kT, \tag{5}$$

$$\xi = 4(\pi/2kT)^{3/4}m^{1/4}(ei)^{1/2}(x - x_m). \tag{6}$$

It is seen that η is a measure of the potential of any point with respect to that of the minimum potential surface, this potential difference being measured in a new kind of unit. Similarly ξ measures the distance of any point from the minimum potential surface in units whose magnitude depends upon the current and the temperature.

The quantities η and ξ are the same as those represented by the same symbols by Fry although he expresses them in equations quite different from (5) and (6). Adams uses exactly the same variables in his calculations. Epstein puts his equations in slightly different form, using the variables τ and G which are related to those used above as follows:

$$\tau = \pm \sqrt{\eta}, \tag{7}$$

$$G = \pm \tfrac{1}{2}\xi. \tag{8}$$

Epstein adopts a different convention in regard to signs from that here used. He takes G always positive while we shall find ξ to be of the same sign as $x - x_m$. Epstein however takes τ to be of the same sign as $x - x_m$.

Epstein simplifies Eq. 6 by grouping together factors under the symbol L thus:

$$L = 2(\pi/2kT)^{3/4}m^{1/4}(ei)^{1/2}. \tag{9}$$

Equation 6 thus becomes:

$$\xi = 2L(x - x_m). \tag{10}$$

By the introduction of the new variables η and ξ the second integration referred to previously takes the form

$$\xi = \int_0^\eta \frac{d\eta}{\left[\epsilon^\eta - 1 \pm \epsilon^\eta P(\sqrt{\eta}) \mp \frac{2}{\sqrt{\pi}}\sqrt{\eta}\right]^{1/2}}. \tag{11}$$

Here P represents the probability function so that

$$P(\sqrt{\eta}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\eta}} \epsilon^{-\eta^2} d\eta. \tag{12}$$

In Eq. (11) the upper or the lower signs are to be taken according as $x - x_m$ is negative or positive, respectively.

When the relation between ξ and η has been found by Eq. (11) then Eqs. (5) and (6) contain the complete solution of the problem of the potential distribution between the electrodes in terms of the current and the temperature, etc. To make practical use of these results, however, it is necessary to prepare a table of the function $\xi(\eta)$. Adams in 1913 calculated this function to about four figures while Epstein and Fry have given only rough values sometimes inaccurate to several per cent. Adams's calculations have recently been repeated by improved methods and to a higher degree of accuracy by Miss Katharine B. Blodgett. For small values of η the method given by Epstein (in which τ is used as a parameter instead of η) is much more convenient for calculating ξ than that used by Adams or Fry. For this reason values of ξ corresponding to values of τ from -0.8 to $+0.8$ have been calculated by means of Epstein's integral for intervals of 0.1 unit. These results are recorded in Table I. From these by interpolation, using Newton's method, the values of ξ corresponding to integral decimals of η have been calculated. For ordinary use it is more convenient to use a table containing ξ as a function of η rather than of τ, since η is proportional to the potentials measured from the minimum potential surface. For purposes of interpolation for small values of η or τ, however, the table having τ as a parameter is more accurate.

TABLE I.

ξ in Terms of τ.

τ.	$-\xi$.	$+\xi$.	τ.	$-\xi$.	$+\xi$.
0.0	0.00000	0.00000	0.5	0.90283	1.09092
0.1	0.19621	0.20372	0.6	1.05917	1.33009
0.2	0.38476	0.41482	0.7	1.20714	1.57591
0.3	0.56547	0.63315	0.8	1.34659	1.82830
0.4	0.73821	0.85857			

The final values of ξ as a function of η are given in Table II. These data are believed to be accurate to the number of figures given, as they have been calculated to two extra decimal places and have been checked by using both Simpson's and Weddle's rules in evaluating the integral of Eq. (11).

TABLE II.

ξ in Terms of η.

ξ is always of the same sign as $x - x_m$.

η.	$-\xi$.	$+\xi$.	η.	$-\xi$.	$+\xi$.
0.00	0.0000	0.0000	6.5	2.4990	7.1924
.05	.4281	.4657	7.0	2.5112	7.5345
.10	.5941	.6693	7.5	2.5206	7.8690
.15	.7167	.8296	8.0	2.5280	8.1963
.20	.8170	.9674	9.0	2.5382	8.8323
.25	.9028	1.0909	10	2.5444	9.4465
.30	.9785	1.2042	11	2.5481	10.0417
.35	1.0464	1.3098	12	2.5504	10.6204
.40	1.1081	1.4092	13	2.5518	11.1845
.45	1.1648	1.5035	14	2.5526	11.7355
.50	1.2173	1.5936	15	2.5531	12.2747
.6	1.3120	1.7636	16	2.5534	12.8032
.7	1.3956	1.9224	18	2.5537	13.8313
.8	1.4704	2.0725	20	2.5538	14.8260
.9	1.5380	2.2154	25	2.5539	17.1931
1.0	1.5996	2.3522	30		19.4253
1.1	1.6561	2.4839	35		21.5522
1.2	1.7081	2.6110	40		23.5939
1.4	1.8009	2.8539	45		25.5643
1.6	1.8813	3.0842	50		27.4740
1.8	1.9515	3.3040	60		31.141
2.0	2.0134	3.5151	70		34.642
2.2	2.0681	3.7187	80		38.007
2.4	2.1168	3.9158	90		41.258
2.6	2.1602	4.1071	100		44.412
2.8	2.1990	4.2934	150		59.086
3.0	2.2338	4.4750	200		72.479
3.2	2.2650	4.6524	300		96.877
3.4	2.2930	4.8261	400		119.185
3.6	2.3183	4.9963	500		140.068
3.8	2.3410	5.1634	600		159.885
4.0	2.3615	5.3274	700		178.861
4.5	2.4044	5.7259	800		197.146
5.0	2.4376	6.1098	900		214.850
5.5	2.4634	6.4811	1000	2.5539	232.054
6.0	2.4834	6.8416			

For large values of η the function ξ has been calculated by a series expansion.

The following expression for negative values of ξ is accurate to one unit in the fifth decimal place for all values of η greater than 3.0:

$$\xi = -2.55389 + \sqrt{2}\,\epsilon^{-\eta/2} - 0.0123\epsilon^{-\eta} + \frac{1}{3\sqrt{2}}\left(\sqrt{\frac{\eta}{\pi}} + 1\right)\epsilon^{-3\eta/2}. \quad (13)$$

For positive values of ξ Adams obtained the following series which is accurate to one unit in the fourth decimal place when η is greater than 8:

$$\begin{aligned}\xi = 1.25520\eta^{3/4} + 1.66854\eta^{1/4} - 0.50880 - 0.1677\eta^{-1/4} + 0.1441\eta^{-3/4} \\ - 0.0145\eta^{-5/4} - 0.069\eta^{-7/4} + 0.036\eta^{-9/4} + 0.083\eta^{-11/4}.\end{aligned} \tag{14}$$

The coefficient of the first term is equal to $(2/3)\sqrt{2}\pi^{1/4}$ so that as a first approximation for large values of η we have

$$\xi = (2/3)\sqrt{2}\pi^{1/4}\eta^{3/4}. \tag{15}$$

By substituting this in Eq. (6), squaring, combining with Eq. (5) and neglecting V_m and x_m compared with V and x, we obtain

$$i = \frac{\sqrt{2}}{9\pi}\sqrt{\frac{e}{m}}\frac{V^{3/2}}{x^2}, \tag{16}$$

which is the usual three halves power law equation as derived by Child and Langmuir.

If we take into account the second term in the expansion of (14) and consider that V_m and x_m are not negligible, we obtain

$$i = \frac{\sqrt{2}}{9\pi}\sqrt{\frac{e}{m}}\frac{(V - V_m)^{3/2}}{(x - x_m)^2}\left(1 + \frac{2.66}{\sqrt{\eta}}\right). \tag{17}$$

Numerical Calculations.—In applying the foregoing results to the numerical solution of problems we proceed as follows:

By combining Eqs. (3) and (5) we obtain

$$\eta_1 = \log\frac{i_0}{i}, \tag{18}$$

where i_0 is the saturation current and i is the actual current between anode and cathode. This gives the value of η corresponding to the surface of the cathode.

Let us now place in Eqs. (5) and (9) the values of the constants $e = 4.774 \times 10^{-10}$ c.g.s. units; $k = 1.372 \times 10^{-16}$ erg/degree; $m = 9.01 \times 10^{-28}$ g.

Then if we express V in volts, we find $e/k = 11{,}600$ degrees per volt. By applying Eq. (5) first to the surface of the cathode and then to any other point we find

$$V - V_1 = T(\eta - \eta_1)/11600. \tag{19}$$

If we express i in amperes, then Eq. (9) becomes

$$L = 4.590 \times 10^5 T^{-3/4}\sqrt{i}\ \text{cm}^{-1}. \tag{20}$$

By applying Eq. (10) first to the surface of the cathode and then to another point we obtain, by subtraction,

$$2L(x - x_1) = \xi - \xi_1. \tag{21}$$

These last four equations together with the relation between ξ and η given by Table II. furnish the complete solution of the problem of the potential distribution and the magnitude of the current between parallel planes. Knowing the saturation current and choosing a value of the current i we find η_1 from Eq. (18) and L from Eq. (20). From the table we now look up the value of $-\xi$ corresponding to η_1 and call this value $-\xi_1$. If we wish to measure distances and voltages from the surface of the cathode, then V_1 and x_1 become zero. By choosing any value of x we then find ξ from Eq. (21) and from the table obtain the corresponding value of η. Eq. 19 then gives the voltage at this point.

If it is desired to use the approximation formula (17), the value of V_m is found from Eq. (19) by placing $V_1 = 0$ and $\eta_m = 0$ thus

$$V_m = -T\eta_1/11600 \tag{22}$$

in which η_1 may be found from Eq. (18).

Similarly from Eq. (21) we find

$$x_m = -\xi_1/2L, \tag{23}$$

in which $-\xi_1$ is a number (always less than 2.56) found by the table from the corresponding value of η_1, and L is obtained from Eq. (20). The quantity η according to Eq. (19) is then

$$\eta = \eta_1 + (11600V/T). \tag{24}$$

As an illustration let us take the example considered by Fry, of a surface of tungsten at 2400° K capable of giving a saturation current of 0.16 amp. per cm². The anode is a parallel surface at a distance of 0.5 cm. From Eq. (20) we find $L = 1339\sqrt{i}$. Taking the cathode as origin we then obtain [1] from Eqs. (19) and (21) for the surface of the anode at $x_2 = 0.5$,

$$V = 0.207(\eta - \eta_1), \tag{25}$$

$$\xi_2 = \xi_1 + 1339\sqrt{i}. \tag{26}$$

[1] Fry obtains equations of this form but gets the coefficients 0.3 and 1010 instead of 0.207 and 1339. These differences are due to two errors arising from confusion of different kinds of averages. The *average velocity of the emitted electrons* which Fry denotes by $\bar{v}_0$ is actually equal to $(\pi kT/2m)^{1/2}$ but if we substitute this in Fry's equations we get different equations from those obtained here. The potential change $\overline{V}_0$ is defined as that "which would give to an electron an energy equal to the *average energy* of those shot out from the cathode." This average energy as Richardson has shown is $2kT$ whereas if we substitute the value of $\bar{v}_0$ in the second equation on page 449 of Fry's paper we find that his value for the average energy is πkT. The third equation on the same page is also wrong for the exponent has the value $V'e/\pi kT$ whereas it should be $V'e/kT$. On page 450 Fry states that the value of $\overline{V}_0$ for tungsten at 2400° K is 0.3 volt. This corresponds to the value $(3/2)kT$ which is the average energy of the electrons in a given volume and not the energy of the electrons passing through a given surface ($2kT$).

TABLE III.

Current between Parallel Plane Electrodes 0.5 cm Apart.

1	2	3	4	5	6	7	8	9	10	11
i/i_0.	i amp.	η_1.	ξ_1.	ξ_2.	η_2.	V_2 volts.	V_m volts.	x_m cm.	i/i_3.	i_4/i_3
0.001	0.00016	6.908	−2.509	14.4	19.2	2.5	−1.43	0.074	4.22	4.30
0.01	0.0016	4.605	−2.411	51.2	122.4	24.4	−0.95	0.0224	1.424	1.44
0.1	0.016	2.303	−2.094	167.3	638.6	131.6	−0.48	0.0062	1.134	1.138
1.0	0.16	0.000	−0.000	535.6	3,117.4	645.0	0.00	0.0000	1.045	1.048

Taking the current i to be a decimal fraction of the saturation current i_0 as indicated in the first two columns, η_1 is calculated by Eq. (18) (Col. 3). From this, ξ_1 (Col. 4) is found from Table II., keeping in mind that ξ_1 must be negative since at the cathode x is zero and therefore less than x_m. From Eq. (26) the value of ξ_2 is found (Col. 5) and then η_2 (Col. 6) by Table II. The voltage given in Col. 7 is that of the anode with respect to the cathode calculated by Eq. (25) from η_2 and η_1. Col. 8 gives the potential of the minimum potential point with respect to the cathode, calculated by Eq. (22), while x_m (Col. 9) is the distance of this point from the cathode calculated by Eq. (23).

Col. 10 gives data for comparing the results of these calculations with those obtainable from the ordinary three halves power law. When i and V in Eq. (16) are expressed in amperes and volts and the values of e and m are introduced, the equation takes the form

$$i_3 = 2.336 \times 10^{-6}(V_2^{3/2}/x^2) \text{ amp./cm}^2. \tag{27}$$

We thus denote by i_3 the current *calculated* in this way from the corresponding value of voltage given in Col. 7 (V_2). The ratio i/i_3 of the actual current i (Col. 2) to that given by the three halves power law, Eq. (27), is tabulated in Col. 10.

Col. 11 illustrates the degree of accuracy of the second approximation equation (17). Let us denote by i_4 the currents calculated by Eq. (17) from the corresponding values of V_2 (Col. 7). Col. 11 contains the ratios of i_4 to i_3. The close agreement between Cols. 10 and 11 shows that the currents calculated by Eq. (17) are very nearly equal to the actual currents i. The figures in Col. 11 give directly the factor by which the currents calculated by the ordinary three halves power law should be multiplied in order to obtain the results that could be got from the better approximation of Eq. (17).

The deviations from the three halves power law indicated in Col. 10 are very much less than those calculated by Fry from the same data.

Fry states in his conclusion that the errors in current involved in the use of the three halves power law (in the example here considered) "may be close to 50 per cent for voltages as high as 40 or 50 volts." Performing calculations like those involved in Table III. shows, however, that the actual current at an anode voltage of 50 is only 27 per cent larger than calculated by the three halves power law.

The deviations are however rather large and would be very important if space charge phenomena are to be used for the determination of e/m. It must not be thought that the deviations are as large as this in the case of cylindrical electrodes. The measurements made by Dushman [1] of electron currents, limited by space charge, from a wire to a concentric cylindrical anode, are quite incompatible with deviations as great as those indicated by Table III.

Schottky [2] pointed out that the correction of the three halves power law for cylinders must be much smaller than for parallel planes. The term corresponding to x_m should vanish since it could only change the effective diameter of the cathode and this diameter does not enter the equation for cylinders. The term containing V_m should remain subtracted from the anode voltage as in the case of parallel planes.

The effect of initial velocities on the current between cylinders will be most marked near the anode where the field is weakest. The space charge at any point will be reduced approximately in the ratio $[(V/(V + V_0)]^{1/2}$, where V is the voltage at the point and V_0 is the voltage corresponding to the average initial energy of the electrons in a radial direction. Substituting this corrected space charge in the differential equation and considering that the correction is small we are enabled to obtain an approximate equation for the effect of initial velocities on the current between concentric cylinders. This equation is

$$i = \frac{2\sqrt{2}}{9}\sqrt{\frac{e}{m}}\left[V - V_m + \frac{V_0}{4}\left(\log\frac{V}{\lambda V_0}\right)^2\right]^{3/2} \Big/ \beta^2 r, \qquad (28)$$

where i is the current per unit of length, V is the potential of the anode, V_m is the potential at the minimum potential surface (given as before by Eqs. (22) and (18)), r is the radius of the anode, β is a constant nearly equal to unity and λ is a numerical constant whose value probably lies between 1 and 2 and must be determined by experiment. Even without knowing the exact value of λ the magnitude of the corrections may be estimated from this equation. The average kinetic energy component normal to the surface among the electrons leaving a surface is kT while each component parallel to the surface is $\frac{1}{2}kT$. In the case of a small

[1] Physical Review, *4*, 121 (1914).

[2] Physik. Zeitschr., *15*, 624 (1914).

wire in a large cylinder not only the normal component but the component tangent to a cross section of the cathode wire will be effective in producing radial velocity components. Therefore the average radial energy component is $(3/2)kT$,

$$V_0 = (3/2)kT/e = T/7{,}733 \text{ volts}. \tag{29}$$

If V and V_m in Eq. (28) are expressed in volts, r in cm, and i in amp. per cm, then the coefficient in Eq. (28) becomes 14.68×10^{-6}.

As an example comparable with that illustrated in Table III., let us consider a tungsten filament 0.25 mm in diameter, at 2,400° K, in the axis of a cylindrical anode, 1 cm in diameter. If the electron emission at this temperature is 0.16 amp. per cm², the saturation current i between cathode and anode is 0.0126 amp. per cm of length. According to Eq. (29) we find that $V_0 = 0.31$ volt. To find the anode voltage at which the current becomes saturated, we place $V_m = 0$ in Eq. (28) and solve for V. If we assume that $\beta = 1$ and $\lambda = 1$, we find $V = 55$ volts. The current calculated from Eq. (28) is 5 per cent greater than if calculated from the ordinary three halves power law neglecting initial velocities. This correction is several times smaller than that applying to parallel planes as given in Table III.

In order to compare the corrections at voltages of 24.4 and 131, which were used in Table III., we must carry out the calculation at higher currents so that saturation does not occur. If the filament temperature is raised to 2,540° K, the saturation current will be 0.045 amp. per cm and the saturation voltage will be 131. The current calculated by Eq. (28), taking $\lambda = 1$ and $\beta = 1$, is then 1.033 times greater than by the ordinary three halves power law whereas by Table III. the ratio is 1.134 for parallel planes. Thus the correction for initial velocities for cylinders is about one fourth as great as for planes. A similar result is obtained for an anode voltage of 24.4. We then find $V_m = -0.53$ volt, and by Eq. (28), $i = 0.004$ amp. per cm, this calculated current being 1.12 times as great as by the three halves power law, while Table III. gives 1.424 for planes. If we had taken $\lambda = 2$ instead of 1 the corrections for cylinders would have been about one fifth those for planes instead of one fourth.

Experiments are now in progress to measure accurately the relation between the voltage and the current in cylindrical anode tubes so as to test Eq. (29) and determine the number represented by λ.[1]

Case II. Condition of Equilibrium (Zero Current) without External Field (Richardson-Laue).

Consider a plane cathode with the opposing plane anode parallel to it but at an infinite distance from it, and let the potential of the anode

[1] See Note at end of paper.

be such that there is no potential gradient near the anode. Under these conditions we shall see that the anode is at an infinite negative potential so that no current flows between the electrodes. This condition may be approached in Case I. if we increase the distance between the electrodes and if we make the anode potential such that only a very small current flows. The potential distribution near the cathode then becomes close to that calculated by the method we shall now consider. However the method of Case I. involves difficulties when the current is a very minute fraction of the saturation current, for η_1 by Eq. (18) becomes very large, and thus ξ_1 (by Table II.) becomes almost exactly equal to -2.5539, while L, by Eq. (20), is close to zero. When it is attempted to calculate the potential gradient near the cathode by Eqs. (19) and (21), the Eq. (21) is found to become nearly indeterminate and therefore unsuitable for this calculation. The equations given below are then more convenient.

Since we are dealing with equilibrium conditions, the distribution of electrons is given by Boltzmann's equation

$$\rho = \rho_1 \epsilon^{\frac{Ve}{kT}}, \tag{30}$$

where ρ_1 is the electron space charge at the surface of the cathode, and ρ is the space charge at any point whose potential with respect to the cathode is V. Substituting this value of ρ in (4) (Poisson's equation), integrating twice, and imposing the condition that $dV/dx = 0$ when $x = \infty$, we find

$$V = -\frac{kT}{e} \log \left[\frac{2\pi e \rho_1}{kT} (x - x_0)^2 \right], \tag{31}$$

where x_0 is an integration constant. This is essentially the equation obtained by Laue.

The space charge ρ_1 may be calculated from Eq. (2) by integration by considering that dN_s/v is the number of electrons per unit volume having velocity components between v and $(v + dv)$ and by noting that the saturation current i_0 is equal to eN_s. Thus we find

$$\rho_1 = i_0 (2\pi m/kT)^{1/2}. \tag{32}$$

We shall find it more convenient to use the same nomenclature as in Case I. Let us denote by L_0 the value of L calculated from Eqs. (9) or (20) by placing $i = i_0$. Using this we can eliminate i_0 from (32) and obtain

$$\rho_1 = (kT/\pi e) L_0^2. \tag{33}$$

The integration constant x_0 in Eq. (31) can be eliminated if we measure x and V from the surface of the cathode, for then $V = 0$ when $x = 0$.

Substitution of the value of ρ_1 from Eq. (33) into Eq. (31) gives

$$\sqrt{2}L_0x = \epsilon^{-\frac{Ve}{2kT}} - 1. \quad (34)$$

From Eqs. (30) and (34) we may eliminate the exponential term and obtain a convenient expression for the space charge at any point:

$$\rho = \rho_1/(\sqrt{2}\, L_0x + 1)^2. \quad (35)$$

When x is very large compared to $1/L_0$ so that the term 1 in the denominator may be neglected, Eqs. (33) and (35) can be combined to give the simple relation

$$\rho = kT/2\pi ex^2\,. \quad (36)$$

Thus, except very close to the cathode, *the electron density is independent of the material of the cathode, is proportional to the temperature and is inversely proportional to the square of the distance from the cathode.*

Eq. (34) can also be obtained as a limit for Case I. for large negative values of ξ. Taking only the first variable term in the expansion (13) we find

$$\xi - \xi_1 = \sqrt{2}(\epsilon^{-\eta/2} - \epsilon^{-\eta_1/2}). \quad (38)$$

From Eqs. (9) and (18)

$$L^2 = L_0^2\epsilon^{-\eta_1}. \quad (39)$$

Placing $x_1 = 0$ in Eq. (21) and combining with Eqs. (38) and (39) we find

$$\sqrt{2}\, L_0x = \epsilon^{-(\eta-\eta_1)/2} - 1,$$

which becomes identical with (34) when we substitute the value of $\eta - \eta_1$ from Eq. (5).

The potential gradient dV/dx at the surface of the cathode, which we may call X_1, is found by differentiating (34) to be

$$X_1 = -2\sqrt{2}\frac{kT}{e}L_0. \quad (40)$$

As an illustration of the use of these equations, let us consider a large flat tungsten surface at 2,400° K in a large glass bulb. The conditions then correspond closely to those assumed in Case II. With Fry we may assume the saturation current obtainable from the tungsten to be $i_0 = 0.16$ ampere per cm². Substituting the values of e, m, and L_0 in Eq. (40) as in Eqs. (19) and (20) we find

$$X_1 = -111.92T^{1/4}\sqrt{i_0} \text{ volts per cm}, \quad (41)$$

which in the example considered gives $X_1 = -314$ volts per cm. This is the electric field at the surface of the cathode due to the space charge of the electron atmosphere near the cathode.

Placing $T = 2{,}400$ and $i_0 = 0.16$ in Eq. (20) we find $L_0 = 535$ per cm, which enables us to solve Eq. (34) for V, giving

$$V = -0.95 \log_{10} (758x + 1) \text{ volts.}$$

The voltages calculated from this for several values of x are given in Table IV.

The space charge ρ_1 at the surface of the cathode is found from Eq. (32) to be

$$\rho_1 = 19260 i_0 / \sqrt{T} \text{ e.s.u. per cm}^3,$$

TABLE IV.

Potential Distribution and Electron Density near a Tungsten Cathode at 2,400° K, without External Field.

Distance from Cathode, Cm.	Potential Volts.	Electrons per Cm³.
0	0.00	1.32×10^{11}
0.0001	−0.03	1.14×10^{11}
0.001	−0.23	4.26×10^{10}
0.01	−0.88	1.79×10^{9}
0.1	−1.79	2.23×10^{7}
1	−2.74	2.28×10^{5}
10	−3.69	2.29×10^{3}

when i_0 is expressed in amperes per cm². Inserting the values of i_0 and T gives $\rho_1 = 62.8$ e.s.u. per cm³, or 1.315×10^{11} electrons per cm³. The electron densities for other distances from the cathode calculated by Eq. (35) are given in the last column of Table IV.

CASE III. CONDITION OF EQUILIBRIUM (ZERO CURRENT) WITH RETARDING FIELDS.

Consider an electron-emitting plane surface (cathode) and a second parallel plane electrode (anode) at a great distance. Let the anode be at such a large negative potential that the electrons are pressed back against the cathode so that only a negligible number pass beyond a certain distance from the cathode. Beyond this distance there is a uniform potential gradient which we may represent by X_∞. Laue has given the solution for the potential distribution in the form

$$-\frac{Ve}{kT} = \log\left[\sinh\frac{X_\infty e(x - x_0)}{2kT}\right]^2 - \log\frac{X_\infty^2 e}{8\pi k T \rho_1}. \quad (42)$$

If we put

$$X_1 = -(8\pi k T \rho_1 / e)^{1/2}, \quad (43)$$

Eq. (42) can be transformed into the more convenient form

$$\sinh^{-1}\left(\frac{X_\infty}{X_1}\epsilon^{-\frac{Ve}{2kT}}\right) - \sinh^{-1}\left(\frac{X_\infty}{X_1}\right) = -\frac{X_\infty ex}{2kT}. \qquad (44)$$

It should be noted that the X_1 defined by Eq. (43) is the same as that given by Eq. (40), as can be readily proved by substituting the value of L_0 from Eq. (33) into Eq. (40). For values of X_∞ so small that only the first term in the series expansion of $\sinh^{-1}$ is needed, the above equation reduces to Eq. (34) of Case II. after the value of X_1 from Eq. (40) is introduced.

By differentiation of (44), the potential gradient dV/dx at the surface of the cathode, which we may call X_s, is found to be

$$X_s = -\sqrt{X_1^2 + X_\infty^2}.$$

These equations lend themselves readily to numerical calculations. The value of X_1 is found from Eq. (41) and it is then only necessary to place $e/k = 11{,}600$ degrees per volt in Eq. (44), in order to calculate the potential distribution.

Research Laboratory, General Electric Company,
Schenectady, N. Y.,
November 14, 1922.

[1] *Note* added Jan. 8, 1923. Recent mathematical analysis shows that an error was made in the calculation of β in the writer's paper on space charge (Physical Rev., *2*, 450, 1913, and Physik. Zeitschr., *15*, 348, 1914), in which β was found almost exactly equal to unity for all cases where the radius of the anode (r) is more than twenty times that of the cathode (a). The following table gives the new values of β^2 for several values of r/a.

r/a.	β^2.	r/a.	β^2.	r/a.	β^2.
7.0.......	0.8870	16.0	1.0513	121.5	1.0722
8.0.......	0.9252	20.09	1.0718	221.4	1.0531
9.0.......	0.9547	29.96	1.0908	735.1	1.0225
10.0.......	0.9782	44.70	1.0945	2,440	1.0064
12.0.......	1.0122	66.69	1.0889	22,026	0.9990

The function β approaches unity in a series of oscillations of decreasing amplitude. The agreement observed by Dushman between experiment and the ordinary space charge equation for cylinders (taking β equal to unity) was due to two compensating errors. The corrections corresponding to the terms containing V_m and V_0 in Eq. (28) amount to 8.1 per cent at 35 volts on the anode, 5.0 per cent at 75 volts and 3.5 per cent at 130 volts (assuming $\lambda = 1$). The correction due to β^2, which should have been put equal to 1.079 instead of unity, is 7.3 per cent in the opposite direction. In the range from 35 to 90 volts the currents calculated with these two corrections agree with Dushman's observations within the experimental error. At 130 volts the observed current was about 4 per cent higher than that calculated. This may have been due to a trace of ionization of residual gas.