AN APPLICATION OF THE ELECTRON THEORY TO THE HALL EFFECT.

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Abstract.

Modified electron theory of the hall effect.—It is generally supposed that the simple electron theory of conduction leads to a negative coefficient for this effect and that the positive coefficients found for many metals are anomalous. But if we suppose the free path instead of the free time of motion of the electrons to be independent of the velocity, then in the *case of isotropic conductors*, the effect comes out zero. Considering the group of electrons having a given speed immediately after collision with molecules, those moving with the electric field E will be retarded and will be more deflected by a magnetic field H than those moving against E, giving a positive coefficient, but electrons moving at right angles to both E and H will give a negative coefficient, and assuming all directions equally probable, the resultant effect for each group and therefore for all is found to be zero. By slight modification of the assumptions of the simple theory, either positive or negative coefficients may be obtained. *In the case of a crystal*, on account of the asymmetry of free paths, a positive effect might be expected for one orientation and a negative for one at right angles. Though the experimental results are not as simple as this, the fact that the Hall effect has never been detected for mercury and that nearly all metals with negative coefficients have face-centered crystal lattices, while those with positive coefficients have body-centered or hexagonal lattices, indicates that the distribution of free paths with reference to E and H is of importance.

Modified electron theory of conductivity.—The same assumptions as to free path lead to the expression for specific conductivity which has been obtained by Lorentz and others. According to this theory, the electrons with velocities parallel to the field take no part in conduction, since the retarding effect on half of them balances the accelerating effect on the other half. In the case of a crystal, the conductivity should depend on the direction of the field with respect to the crystal axis, as in fact it does.

In 1879 Hall discovered that when an electric current flows in a conductor in a transverse magnetic field, an electromotive force is produced which is at right angles both to the primary current and to the magnetic field. This is as we should expect, but the variation in sign which occurs in different metals has never been satisfactorily explained. Supposing the current to be due to the transportation of negative charge, we anticipate a negative value for the "Hall coefficient" and no excuse is apparent for the positive values which are found in many cases; it has been one of the outstanding failures of the electron theory that it has in no way cleared up the matter but has only made the earlier view more precise.

In the figure the electric field is supposed acting in the direction of

the x axis (*i.e.*, the force on the electron is in the negative x direction); the electronic current is flowing from right to left and the magnetic field is supposed directed outward from the paper. A fundamental law of electromagnetism tells us that upon a negative charge so moving in such a field a force is exerted tending to deflect the charge downward. This is all so simple that we are indeed shocked when we find experiment showing in many cases a deflection in the opposite direction. In the cases of bismuth, nickel, silver and other substances the electronic current is found deflected as it should be by the simple theory (in these cases the Hall coefficient is given a negative sign); but in an equally large number of cases (tellurium, antimony, iron, etc.) the deflection is found to be in the opposite direction. The enormous range which has been found in the absolute values of the coefficients also seems quite out of keeping with the theory. Tellurium and bismuth have by all odds the largest values for their Hall coefficients and in one case the value is positive and in the other it is negative. In most cases the effect is extremely small and there is no indication whatever of a preponderance of negative values among the elements.

The discord between the theory and the observed phenomena is made more striking when we remember how satisfactory the electromagnetic theory is when applied to other similar phenomena. In the case of conduction through gases which has furnished us with the model for our theory of metallic conduction, the deflection of the electron moving through a magnetic field is found to be quite in accord with the theory. And again in the cases of metallic conductors, it is of course true that whatever the effect on the electrons, the conductor as a whole when carrying a current in a magnetic field always is subject to a force which in direction and magnitude is quite in accord with the theory and depends not at all upon the material of the conductor. If this force is to be considered as the aggregate of the forces impressed upon the individual electrons, how is this to be brought into consistency with the Hall phenomenon?

With the aid of hypotheses introduced *ad hoc* the contradictions which the Hall effect seems to present may be explained away. Thomson has accounted for the effect by assuming the agency of intra-atomic magnetic fields; Riecke, Drude and others have hypothecated positive as well as negative carriers of electricity in metals;[1] but these hypotheses cannot be said to have met with general acceptance and have certainly been unfruitful in their effects upon the theory of metals. Collateral

[1] Thomson, Corpuscular Theory of Matter, Chap. V., and Drude, Ann. der Phys., 3, p. 369 (1900).

evidence seems significantly lacking for either hypothesis. Lorentz has raised serious objections to the theory of Drude. And yet the positive carrier theory does still exist as a competitor of the simple electron theory and bears evidence of the important place which the Hall effect has had in shaping the theory of conduction.

It is the purpose of this paper to show that the electron theory does not lead inevitably to a negative Hall coefficient as has been generally supposed but in its simplest form leads (to the same degree of approximation as that hitherto employed) to a zero coefficient. This still leaves the observed effect unexplained. It does however, and without introducing assumptions expressly for that purpose, show that our intuitive anticipation of a negative coefficient is not justified by the electron theory and so removes the phenomenon from the rather paradoxical position which it has hitherto occupied. It is not to be expected that this simple view of conduction can lead to a complete explanation of the phenomenon; but by slight modifications of the assumptions made in this paper, either positive or negative coefficients may be obtained and it seems reasonable to hope that with our increasing knowledge of metallic structure we may develop a less artificial theory of conduction which will account for many of the galvanomagnetic and thermomagnetic effects.

In the electron theory as usually developed it has been assumed that the *time* of free flight was not affected by the presence of the electric and magnetic fields. The electric field modifies the velocity of the electron and the assumption that the time of flight is unchanged tacitly involves the assumption that the free path of the electron is proportional to the velocity. No attempt seems to have been made to justify this assumption and indeed it appears to have been made unconsciously. It is surely more reasonable to assume that the free *path* rather than the free time is independent of the velocity. This is the assumption of this paper.

With this exception the assumptions made are quite conventional. Electrons are supposed moving with thermal velocities among fixed atomic centers and all directions are equally probable after reflection from an atomic center. The ratio of the electrical drift velocity to the thermal velocity is considered small.

During any interval of time there will occur in the conductor a large number of electronic reflections, in all directions and with thermal velocities ranging about a mean thermal velocity $\bar{v}_0$ and free paths ranging about a mean free path $\bar{\lambda}$. In most cases the conductor may be considered isotropic. Even though the metal consists of crystals, these crystals are arranged in random fashion among themselves so that in any reasonably large volume of the metal the mean free path (as well as the thermal velocity) will be the same in one direction as in another.

Of all the electrons in the conductor under consideration which have been reflected during an element of time, we first consider the contribution to the Hall effect only of those with some definite thermal velocity v_0 and free path λ. Representatives of this group are found in all parts of the conductor, going in all possible directions but all with the same thermal velocity and free path.

The free path here considered is the one which actually exists in the presence of the electric and magnetic fields and it is supposed that for an isotropic substance, even under these conditions, the number of electrons with free path λ and thermal velocity[1] v_0 is the same in all directions.

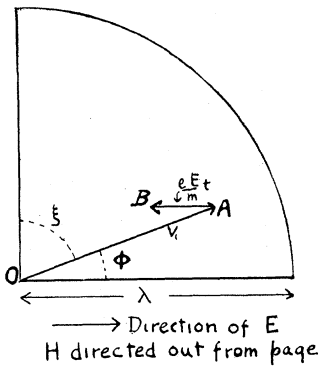

Fig. 1.

We first consider the deflection produced by a magnetic field upon a single representative of this group. The following notation applies:

n = number of electrons of this group reflected per cubic centimeter per unit solid angle per second.
T = time of free flight.
T_0 = time of free flight without the electric field.
v_0 = thermal velocity.
v_{0y} = y component of thermal velocity.
λ = free path of electron.
Y = total displacement in y direction during a free path.
e, m = the absolute value of the charge and mass of the electron.
E = electric field (directed to the right along the x axis of the figure).
H = magnetic field (directed out from the page in the figure).
ϕ = the angle between the path of the electron and x axis.
ξ = the angle between the path of the electron and y axis.
ψ = the angle between the path of the electron and the principal axis of the crystal.

Taking one electron of the v_0 group and referring to the figure, the

[1] That is, the velocity at the instant of reflection.

radius represents the free path λ, the constant vector OA and the variable vector AB represent, respectively, the thermal velocity v_0 and the velocity due to the electric field, that is $(eE/m)t$ in the negative x direction. The resultant velocity of the electron is the vector sum of v_0 and $(eE/m)t$.

The magnetic force on electrons in the y direction is

$$\frac{He}{c}\left(v_0 \cos\phi - \frac{eE}{m}t\right).$$

Hence

$$\frac{d^2y}{dt^2} = \frac{He}{cm}\left(v_0 \cos\phi - \frac{eE}{m}t\right)$$

and therefore the displacement of the electron in the y direction is

$$y = v_{0y}t + \frac{He}{2cm}\left(v_0 \cos\phi \cdot t^2 - \frac{1}{3}\frac{eE}{m}t^3\right). \tag{1}$$

The average velocity during the free path is

$$v_0 - \frac{eE}{2m}T_0 \cos\phi$$

and the free time therefore equals

$$T = \frac{\lambda}{v_0 - \frac{eE}{2m}T_0 \cos\phi} = T_0\left(1 + \frac{eE}{2mv_0}T_0 \cos\phi\right). \tag{2}$$

Substituting in (1)

$$Y = v_{0y}T_0\left(1 + \frac{eE}{2mv_0}T_0 \cos\phi\right) + \frac{HeT_0^2}{2cm}\left\{v_0 \cos\phi + \frac{eE}{m}T_0(\cos^2\phi - \tfrac{1}{3})\right\}$$

since we may neglect terms of the order of eE/mv_0 in the parenthesis multiplying $HeT_0^2/2cm$.

Considering now all electrons which belong to our group, which are of course travelling in all directions, we find that their contribution to the Hall current is

$$I_y = 2\pi \int_0^\pi neY \sin\phi \, d\phi.$$

Terms in v_{0y} drop out by symmetry with respect to the xz plane and so

$$I_y = \frac{\pi ne^2HT_0^2}{cm}\int_0^\pi \left\{v_0 \cos\phi + \frac{eE}{m}T_0(\cos^2\phi - \tfrac{1}{3})\right\} \sin\phi \, d\phi \tag{3}$$
$$= 0.$$

As this particular group gives a Hall current of zero, so will any other

group with velocity v_0' and free path λ' and the total Hall current from the theory is zero.

It will be instructive to point out qualitatively how the substitution of the constant path for the constant time hypothesis has led to this result. If we fix the attention on two electrons, moving respectively to the right and left along the x axis in the figure, the former (going against the current) will on the average be going slower than the one which is moving with the current, due to the retardation of the field during the time of free flight. The magnetic force is of course greater upon the more rapidly moving electron and in equal times this one would be deflected the more but *in the same path* the slower of two electrons is deflected the more by a magnetic field. Therefore if all electrons in the conductor were like these two, it would be those going *against* the field which would have the larger deflection and the substance would have a *positive* Hall coefficient. However when electrons with thermal velocities nearly normal to the direction of the applied electric field are considered the reverse is the case and these alone lead to a negative coefficient. In the case of isotropic substances where equal numbers travel in each direction and with similar distributions of free paths in each direction, the total effect has been shown above to be zero. If however in any particular case the free paths in the direction of the primary current were on the average longer than those at right angles thereto, the electrons going in this direction would be most deflected and such a conductor would under these circumstances have a positive coefficient. While it is difficult to see how this assymmetry of path can exist in the case of most conductors, it should be a factor in the case of a crystal and this case is examined more fully in the last part of this paper.

II.

The expression for the electrical conductivity of a metal as given by Drude is

$$\sigma = \frac{Ne^2\bar{\lambda}\bar{v}_0}{4\alpha\Theta}.$$

More recently errors in his derivation have been pointed out and the expression

$$\sigma = \frac{Ne^2\bar{\lambda}\bar{v}_0}{3\alpha\Theta}$$

is obtained [1] which goes over into the expression obtained by Lorentz

[1] Bohr "Dissertation," page 54; Swann, Phil. Mag., 27, p. 441 (1914); Livens, Phil. Mag., 29, p. 173 (1915). Mayer has recently summarized the literature on this subject [Jahab. Rad. und Elec. 18, p. 201 (1922)] but makes no reference to these papers.

if account is taken of the Maxwellian distribution of velocity among the electrons. This expression does not seem as yet to have displaced in the literature the one obtained by Drude. By a method quite similar to that used in the first part of this paper the expression for conductivity can be obtained. The method used seems to the writer somewhat simpler than those hitherto used for the derivation, and is interesting as showing that the correction which has been introduced above in the classical treatment of the Hall effect is essentially the same one as that which has already been applied to the treatment of conduction.

At first it will be convenient to neglect the distribution in the thermal velocities of the electrons and it will be supposed that they all have the mean velocity $\bar{v}_0$. As before all the electrons will be divided into groups having free paths λ_1, λ_2, λ_3, etc., and corresponding free times $T_{0,1}$, $T_{0,2}$, $T_{0,3}$, etc., in the absence of the field, which become altered to T_1, T_2, T_3, etc., by the influence of the field. The electric field is supposed directed along the x axis as before and X_1 represents the displacement in this direction of some electron of the first group during its free flight.

$$X_1 = -v_0 T \cos\phi + \frac{eET^2}{2m},$$

$$= -v_0 T_0 \cos\phi + \frac{T_0^2 eE}{2m}\sin^2\phi,$$

substituting in the value of T from (2) and dropping higher order terms as before. To get the contribution to the current from all electrons of the first group we must integrate for all directions

$$I_1 = 2\pi e \int_0^\pi X_1 n_1 \sin\phi \, d\phi, \tag{4}$$

$$= \frac{4\pi n_1 e^2 T_{0,1}^2 E}{3m},$$

where I_1 is the current contribution of all electrons in the first group and n_1 is the number of electronic collisions per second, per cubic centimeter, per unit solid angle producing electrons which belong to this group.

The total current then is equal to

$$\frac{4\pi e^2 E}{3m}\sum_1^\infty n_1 T_1^2 = \frac{4\pi e^2 E \nu \overline{T_0^2}}{3m} = \frac{8\pi e^2 E \nu \overline{T_0}^2}{3m},$$

where $\overline{T_0^2}$ and $\overline{T_0}^2$ represent the mean value of the square and the square of the mean, respectively, and ν is the total number of collisions per second per cubic centimeter per unit solid angle, which occur. It has been assumed here that the probability of a free path of any particular

length is an exponential function of the free path (or free time); *i.e.*, $dn = \nu e^{-T} dT$. It is well known that in such a case the mean value of T^2 is twice the square of the mean value T, which accounts for the factor 2 which has been introduced in the last member of the above equality.

Let N represent the number of free electrons per c.c. of the conductor, and $\alpha\Theta$ the energy of thermal agitation per molecule at the temperature Θ

$$4\pi\nu = \frac{N\bar{v}_0}{\lambda} = \frac{N}{T_0}$$

$$I = \frac{2}{3}\frac{Ne^2\overline{T}_0 E}{m} = \frac{Ne^2\bar{\lambda}\bar{v}_0 E}{3\alpha\Theta},$$

and the expression for the specific conductivity is

$$\sigma = \frac{Ne^2\bar{\lambda}\bar{v}_0}{3\alpha\Theta}, \tag{5}$$

which is identical with the expression obtained by Swann.[1]

Swann has shown that if account be taken of the variation in the thermal velocities among the electrons the above expression reduces to the expression which has been obtained by Lorentz.

The expression (5) differs from that of Drude with which it is comparable by a factor of 4/3. A factor 2 has been introduced by taking account of the difference between the mean of the free path squared and the square of the mean free path, and a factor 2/3 introduced by taking the free path rather than the free time as unchanged by the field. In Drude's theory all electrons contributed equally to the current regardless of the direction in which they were moving; in the present treatment it is seen that those electrons which move in a direction parallel to the field take no part in the conduction. Electrons moving forward are accelerated, it is true, and those moving backward are retarded but as each has the same free path this can produce no displacement of electricity.

The significance of the different numerical factor which is introduced into the classical expression for conduction is probably not great. More significant is the dependence of the conductivity upon the direction of the impressed field in the case of an anisotropic substance (such as a crystal) which the present theory leads us to expect. As has been pointed out, those electrons which move in directions parallel to the impressed field play no part in the conduction. In a crystal the distances between atoms are different in different directions. The deflections which the

[1] Phil. Mag., S. 6, Vol. 27, pp. 441–455, 1914.

electrons suffer in the field are proportional to the squares of the free paths, and so the crystal should show greater conductivity when it is so oriented that the longer free paths lie in directions normal to the field and so become the paths of those electrons which play a part in conduction, than when these longer paths lie parallel to the field. It is well known that in crystals the conductivity is indeed a function of the direction of the impressed field.

III.

In the consideration of the Hall effect it has been assumed that the substance was isotropic. In many cases, however, the mean value of λ (or T_0) must be different in different directions. It may be supposed that although the theory given offers no explanation for the observed effect in isotropic substances, it may throw some light on the phenomena which have been observed in the case of a crystal. The theory is far from agreeing with the experimental results, but because of the prominent position which this model of the conductor has occupied and because the emphasis which the theory places upon distribution of free paths may indeed suggest a more satisfactory explanation, the theory will be extended to cases where λ is not considered an isotropic function.

In the case of a crystal with symmetrically placed atoms, the free path must be fairly constant for a given direction but changing very suddenly with slight changes in direction. A smoothed-over value of λ will here be considered which is thought of as a function of ψ, the angle between the principal crystal axis and the direction vector. n may also be thought to depend upon the relative position of the atoms and so also to be a function of ψ.

Under these conditions T_0 and n must remain under the integration sign and expression (3) is so modified as no longer to equal zero. The expression for the Hall coefficient, R, may be obtained. By definition

$$R = \frac{E_y}{Hi_x}, \tag{6}$$

where E_y is the electrical field intensity in the y direction applied in experiment to neutralize the Hall current, and i_x the primary current density.

Upon equation (3) one integration has already been performed which was possible because of the symmetry of n and T about the x axis. Now that this symmetry no longer exists, since n and T_0 are functions of ψ as well as of ϕ, equation (3) must be changed to the form

$$I_y = \frac{e^2H}{2cm}\int \left\{ nT_0^2v_0 \cos\phi + nT_0^3\frac{eE}{m}(\cos^2\phi - \tfrac{1}{3}) \right\} d\omega,$$

where the integration is to be carried throughout the total solid angle of which $d\omega$ is an element. The first term in the integrand drops out as before. A similar change is made in equation (4) and the expression for the current due to E_y is then equated to the Hall current.

$$-\frac{e^2E_y}{2m}\int nT_0^2 \sin^2\xi\, d\omega = \frac{e^3HE_x}{2cm^2}\int nT_0^3(\cos^2\phi - \tfrac{1}{3})d\omega.$$

Equation (6) then becomes when we substitute the value of i_x

$$R = \frac{E_y}{Hi_x} = \frac{2\int nT_0^3(\cos^2\phi - \frac{1}{3})d\omega}{ce\int nT_0^2 \sin^2\phi\, d\omega \cdot \int nT_0^2 \sin^2\xi\, d\omega}$$

or in terms of the observed conductivities σ_y and σ_x

$$R = \frac{e^3}{2\sigma_y\sigma_x cm^2}\int nT_0^3(\cos^2\phi - \tfrac{1}{3})d\omega. \qquad (7)$$

If the principal axis of the crystal is parallel to the primary electric field

$$R = \frac{\pi e^3}{\sigma_y\sigma_x cm^2}\int_0^\pi nT_0^3(\cos^2\phi - \tfrac{1}{3})\sin\phi\, d\phi.$$

Under these conditions the integration will give R a positive value if nT_0^3 has a larger value in directions near the principal axis than in directions perpendicular thereto, and a negative value if the opposite be the case.

If, on the other hand, the current be flowing in a plane normal to the principal axis, the Hall coefficient should from the nature of equation (7) have an opposite sign to that in the preceding case.[1] The value in this case may vary somewhat, depending on the direction of the magnetic field relative to the principal axis. Considering only the case in which the magnetic field is perpendicular to the crystal axis, it is seen that the value of the product $\sigma_y\sigma_x$ is the same as in the case just discussed (where the electric field was parallel to the crystal axis). If nT_0^3 be supposed to have its largest value along the crystal axis, the integrated quantity in (7) will have a smaller value in the present case, making the Hall coefficient different in absolute value as well as in sign from the case first discussed.

Comparing these results with the observations of Everdingen[2] for

[1] Supposing nT_0^3 to have its larger value in the direction of the principal axis, if this axis is in the plane perpendicular to the x axis in the figure, nT_0^3 will on the average have larger values when $\phi = \pi/2$ than when $\phi = 0$ or π. Since $\cos^2\phi - 1/3$ is negative for these values (and its unweighted mean value is 0) the average value of $nT_0^3(\cos^2\phi - 1/3)$ must be negative.

[2] Comm. Leiden No. 61. See also Lownds, Ann. d. Phys., 9, p. 677 (1902).

bismuth it will be seen that the theory as developed is not competent to account for the observed phenomena. The table gives the values of the coefficients found for electric and magnetic fields parallel to and perpendicular to the crystal axis.

Hall Coefficient for Bismuth with Magnetic Field of 4,600 Gauss.

Electric Field.	Magnetic Field.	
	⊥	∥
∥	− 8.0	
⊥	−10.6	−0.2
⊥	− 8.8	−0.0
⊥	− 8.2	+0.6

Values are given for the electric field in several directions perpendicular to the crystal axis. The resistance was found to be $\sqrt{5/3}$ greater along the principal axis than perpendicular thereto. It is seen that the direction of the magnetic field is very important; the important rôle assigned by the theory to the direction of the electric field is not in agreement with the experiment.

Many data have been obtained for the effect of temperature upon the Hall effect. In the case of bismuth in particular, this effect is very remarkable, but a comparison with the theory in this case will not be made. As is well known, the simple theory gives the wrong temperature coefficient for resistance and until this matter can be cleared up it is hardly profitable to apply it to the varied effects of temperature upon the Hall coefficient.

The increase of the coefficient observed in exceedingly thin plates is directly predicted by the theory. The asymmetry of the free paths on the surface should have an important effect and the relative importance of this surface phenomenon increases as the thickness decreases. The parallelism which is in general to be observed between resistance and the magnitude of the Hall effect is anticipated by the present theory as it was indeed by the older application of the electron theory.

Not only were the theories unable to account for the positive Hall coefficients but until recently it could be said that the variation in sign of these coefficients was paralleled by no other known property of the material. Today we find a very remarkable parallelism between this sign and the crystal structure. With but two or three doubtful exceptions, we find the positive coefficient in metals with body-centered cubic or hexagonal close packed lattices while those metals with face-centered cubic lattices have negative coefficients. A very obvious effect of the

crystal structure will be an alteration of the relative free path lengths in different directions, and it is the importance of this path distribution which the above theory tends to emphasize.

The Hall effect has never been detected in mercury. Experimental difficulties make this result of doubtful value, but if it be accepted, we find here in a liquid conductor where the isotropism is more truly realized than in a crystalline substance a check with the theory.

It has been the main purpose of this paper to show that by a reasonable application of the same electron theory which has been applied to explain conduction the paradox of the Hall effect disappears. Hitherto it has been felt, intuitively, that a current of negative electrons should lead to a negative Hall coefficient and the electron theory has done nothing to weaken this belief. The electron theory is now shown to contradict this belief. The unreality of the theory, with its substitution of conventionalized assumptions for an intimate knowledge of intra-atomic fields, does not at all vitiate its conclusion that any presumption which has been made as to sign of the effect has been unjustified. Since in the above case a null effect was derived, by slight modification of hypotheses either a positive or negative may be obtained.[1] A hitherto unknown emphasis is laid upon the importance of the distribution of paths of the electron in the conductor. The remarkable parallelism which exists between the crystal structure as determined by x-ray analysis and the sign of the Hall effect seems to the author to strengthen the probability that this path distribution holds the key to the effect. The theory given is obviously unsatisfactory, but it at least seems suggestive of a possible explanation of the phenomena when a more exact mechanism of conduction is formulated.

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[1] If we assume, for instance, that the electron retained after collision some of its directed velocity acquired in the electric field, we should obtain a *positive* coefficient.