RECIPROCAL DIFFRACTION RELATIONS BETWEEN CIRCULAR AND ELLIPTICAL PLATES.

BY JOHN COULSON AND G. G. BECKNELL.

SYNOPSIS.

Diffraction Patterns inside Elliptical Shadows Due to a Point Source of Light.—When a circular disc with its plane originally tangent to the light wave is rotated about an axis in its plane, the Arago spot changes to a figure with four cusps which move out along the axes of the elliptical shadow, two approaching the foci as limits and the other two going outside the shadow. These changes are shown by a set of photographs obtained with a disc 1.6 cm. diameter placed 2 meters from a pin-hole 0.3 mm. in diameter and 5 meters from the photographic plate. The diffraction pattern was found to depend only on the ellipticity of the shadow whether produced by an ellipse or by an inclined disc. Each quadrant of the diffraction pattern was found to be associated with the quadrant of the shadow adjacent to it but on the opposite side of the major axis. Careful measurements of the photographs proves that in each case the diffraction pattern is the *evolute of the geometrical shadow*. The effect is as though each element of the edge of the shadow contributed a spot along its normal, the result being a *caustic curve of diffraction*.

INTRODUCTION.

EARLY in the nineteenth century Poisson[1] predicted that if the Huyghens-Fresnel theory was correct the light intensity behind the center of a small opaque disc illuminated by a point source should be about the same as though the disc were removed. Arago verified this experimentally using a disc about 2 mm. in diameter. In a later paper Arago states that the same phenomenon was discovered by Delisle in 1715.

Some of the more recent investigators, Croft, Arkadiew, and Hufford [2] have repeated and extended the work of Arago. However, none of the writers has drawn attention to the fact that a disc illuminated by a point source of light and placed perpendicular to the wave-normal, when gradually rotated about an axis in its own plane produces a succession of broadening diamond-like figures with concave sides; nor, have they drawn attention to the reciprocal figures produced by ellipses.

[1] Poisson, Verdet, Leçons d'Opt. Phys., Vol. 1, Sect. 66.
Arago, Oeuvres Complètes, Vol. VII., p. 1.
Delisle, Mém. de l'anc. Acad. des Sciences, 1715, p. 166.

[2] Croft, W. B., Phil. Mag., Vol. 38, p. 70 (1894); Nature, Vol. 66, p. 354 (1902).
Arkadiew, W., Phys. Zeit., Vol. 14, p. 832 (1913).
Hufford, PHYS. REV., Vol. III., p. 241 (1914); Vol. VII., p. 545 (1916).

Apparatus and Method.

The discs were turned from brass, the edges beveled and then polished while on the lathe with a fine-grained oil-stone. The sizes ranged from .5 cm. in diameter to over 5 cm.

Very accurate ellipses were produced by a lathe method as follows: A circular cylinder of brass was sawed in two along a plane making the desired angle with its axis. A thin plate of hard-rolled copper was fastened between the sections of the cylinder by means of screws inserted from one end of the cylinder. The cylinder was then chucked in a lathe and turned down so that the minor axis of the elliptical section was equal to the diameter of the circle mentioned above. This elliptical section having been replaced by another sheet of the same material the cylinder was further turned down to such size that the major axis of the new ellipse was equal to the diameter of the disc.

Each of these plates was held at the center of a circular ring (about 20 cm. in diameter) by three No. 44 steel wires. This ring was fitted inside a second ring of slightly larger diameter, so as to slide into any position in its plane. The outer ring was mounted in such a manner as to permit of rotation about a horizontal axis as well as about a vertical axis. By means of clamping screws rotation about any one of the three axes could be made without disturbing the other two adjustments.

The mounted disc was placed in the path of a cone of light emanating from a pinhole source illuminated from behind by a self-regulating carbon arc and a system of condensing lenses. The shadow of the disc was received through a long tube, blackened inside, upon a photographic plate.

The distance between the photographic plate and pinhole was 7 meters, and that between pinhole and disc was 2 meters. The photographic reproductions shown here were produced by a pinhole about .3 mm. in diameter, and were made with a disc 1.6 cm. in diameter and with two ellipses, one having its major axis equal to the minor axis of the other and equal to the diameter of the disc.

Procedure.

The disc was adjusted until the Arago spot was sharply defined at the center of the circular shadow received at the position of the photographic plate. In this position the disc is perpendicular to the wave-normal. The sizes of the Arago spot and the circular shadow are indicated by No. 1, Plate I.

When the disc is rotated about an axis in its own plane, in this case a vertical axis, the Arago spot changes gradually into the series of expanding patterns mentioned before. These are illustrated by Plate I, Nos. 2 to 8, corresponding to angles of rotation of 10°, 15°, 22.5°, 30°, 37.5°, 45°, and 60° respectively. Any one of these figures may be duplicated by an ellipse, whose minor axis equals the diameter of the disc, cut from a circular cylinder at the corresponding angle and placed perpendicular to the wave-normal.

Plate I., No. 9 was produced by an elliptical plate cut at an angle of 37.5°, and placed in the normal position just described. It corresponds to No. 6 for the disc when turned through an angle of 37.5°. In this case the major axis of the ellipse is equal to the diameter of the disc. An ellipse similarly cut, but with its minor axis equal to the diameter of the disc, when placed perpendicular to the wave-normal produces Plate II., No. 10. It will be seen that this diffraction pattern corresponds in shape to that of Plate I., No. 9, and is proportional in size to the dimensions of the ellipse.

When this ellipse is rotated about an axis in its own plane until the shadow produced is a circle (in this case 37.5°) then the Arago spot appears at the center of the shadow, as shown in Plate II., No. 11.

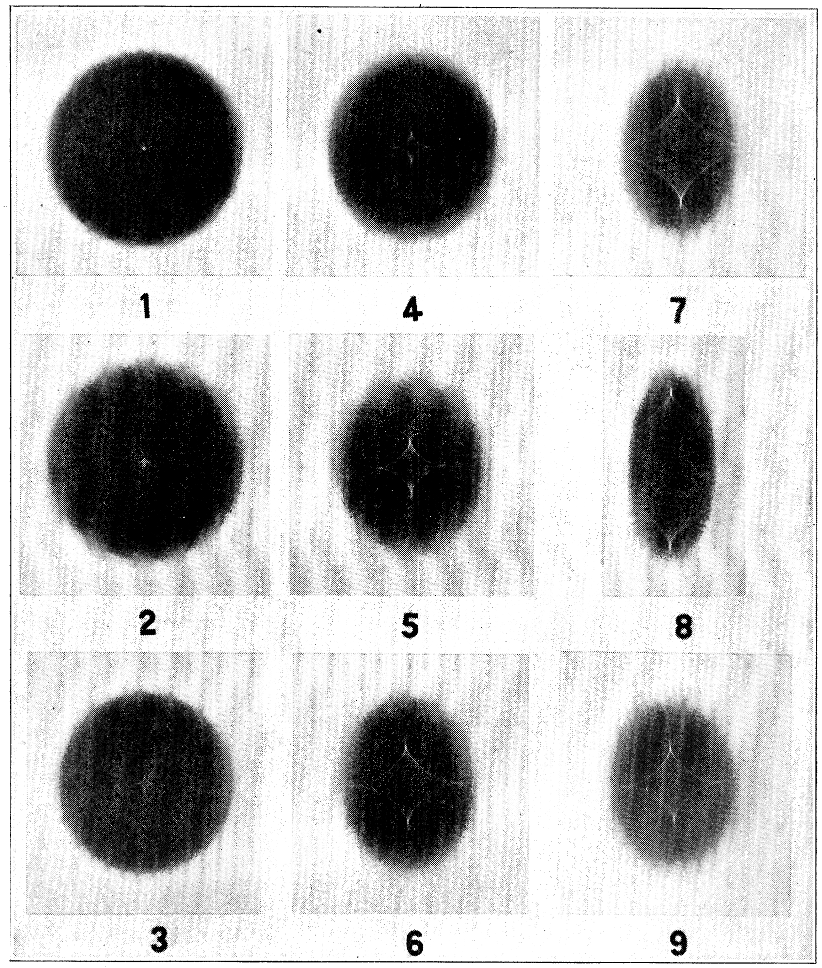
Examination will show that in all cases where the four cusps of the diffraction figure appear within the shadow (Nos. 3, 4, 5, 6, and Nos. 9 and 10) the intensity of light is greatest at the cusps lying on the major axis of the elliptical shadow. This fact is important in the experimental analysis, an account of which is now to follow.

Plate II., No. 16, shows the diffraction pattern produced by an ellipse when a straight-edge in the same plane is made tangent to the ellipse at the end of the major axis, the plane of the ellipse being perpendicular to the wave-normal. The spear-like shadow cast from the point of tangency is directed to the *adjacent* cusp. But, if the straight-edge is made tangent at the end of the minor axis the pointed shadow is directed to the cusp lying on the *opposite* side of the major axis. This is shown by Plate II., No. 17.

From the last two illustrations it is evident that each cusp lying on the major axis is produced by light emanating from the neighborhood of the *nearer* end of the same axis of the elliptical plate; while each of the cusps on the minor axis is due to light coming from the *farther* end of the latter axis. Furthermore, it has been observed that if the straight-edge is made tangent at successive points of the ellipse lying between the ends of the major and minor axes the extremity of the pointed shadow moves along the four-cusped diffraction figure. At any definite position of the straight-edge the bisector of the angle at the tip of the shadow is tangent to the diffraction figure and normal to the geometrical shadow.

If a circle replaces the ellipse the shadow from the point of tangency is directed toward the Arago spot at the center of the circular shadow.

PHYSICAL REVIEW, SECOND SERIES, VOL. XX.
December, 1922



JOHN COULSON AND G. G. BECKNELL.

(See No. 18.) If the disc and straight-edge be now rotated about an axis in their plane figures like Plate II., No. 16 and No. 17 are produced.

Attention may be drawn to the fact that the diffraction bands resulting from any combination of ellipse and tangent straight-edge differ in nature from the quadrantal diffraction figure produced by the ellipse alone. In the case of the former the bands are fringed with the prismatic colors, while in the latter case this color separation does not take place. It appears, also, upon photographing these figures with light of different colors that the dimensions of the main figures are independent of the wave-length.

Photographs 12, 13, 14, and 15 were taken while light was excluded from certain quadrants of an ellipse placed perpendicular to the wave-normal. In No. 13, an opaque screen shuts off all of the ellipse on one side of the major axis. Here the diffraction pattern lies *inside* the shadow of the screen. Hence, it is produced by the light coming from the *opposite* side of the major axis. If, now, the screen is so placed as to exclude light from all of the ellipse on one side of the minor axis the resulting pattern falls *outside* the shadow of the screen. Here, on the contrary, the light producing the diffraction comes from the *same* half of the ellipse into which the pattern falls. (See No. 14.)

Shutting off the illumination from two diagonally opposite quadrants of the ellipse produces the effect shown in No. 12. It will be observed that half of the complete diffraction figure is missing, and that the cusps do not appear. Furthermore, the portion of the figure lying in either quadrant is due to light from an adjacent quadrant, and not from the one diagonally opposite.

No. 15 completes the analysis concerning each quarter of the diffraction figure and the illuminated quadrant which produces it. Here the major axis of the ellipse stands lengthwise of the page, as it does in the other illustrations. In terms of the usual trigonometric convention light is excluded from all but the first quadrant of the elliptical plate, and the resulting diffraction is seen to lie only in the second quadrant of the geometrical shadow. This shows that each illuminated quadrant produces independently its own quarter of the diffraction figure and, therefore, that each of these quarters is due to light from its corresponding *adjacent* quadrant with respect to the major axis. Exactly similar properties are found in the case of the circle rotated through an angle.

The foregoing results may be summarized as follows:

1. The diffraction figures are quadrantal and symmetrical. As the eccentricity of the elliptical shadow gradually increases the cusps which lie upon the minor axis recede indefinitely, while those on the major axis

always lie within the shadow. As the eccentricity approaches zero the figure degenerates into a circular spot which becomes a geometrical point for a point source of light.

2. Each quarter of the diffraction figure is produced by light from the adjacent quadrant with respect to the *major* axis.

3. The intensity of illumination of the cusps lying on the major axis is always greater than that of those lying on the minor axis.

The facts here enumerated have suggested that the diffraction figures are the *evolutes* of an ellipse.

The evolute of a given curve is defined as the locus of its center of curvature. Every normal to the given curve is tangent to its evolute; hence, the evolute is the envelope of these normals.

In the case of an ellipse the cusps of the evolute which lie on the minor axis recede indefinitely with increasing eccentricity of the ellipse, while those on the major axis at the same time approach the foci. With decreasing eccentricity the evolute contracts and, in the limit, becomes a geometrical point. Furthermore, the envelope of the normals from any quadrant of the ellipse is the adjacent quarter of its evolute with respect to the *major* axis. Such an envelope corresponds to the curve formed by the rays as mentioned above.

The difference in the brightness of pairs of opposite cusps is accounted for if we consider the distance from the end of the major axis to the nearest cusp of the evolute, and the distance from the end of the minor axis to the further cusp on that axis. These distances are b^2/a and a^2/b respectively, where a and b are the major and minor semi-axes of the ellipse. Here b^2/a is always less than a^2/b. The distances just mentioned are, also, the radii of curvature of the ellipse at the ends of the major and minor axes.

The equation of the evolute corresponding to the ellipse

$$(x/a)^2 + (y/b)^2 = 1,$$

is

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

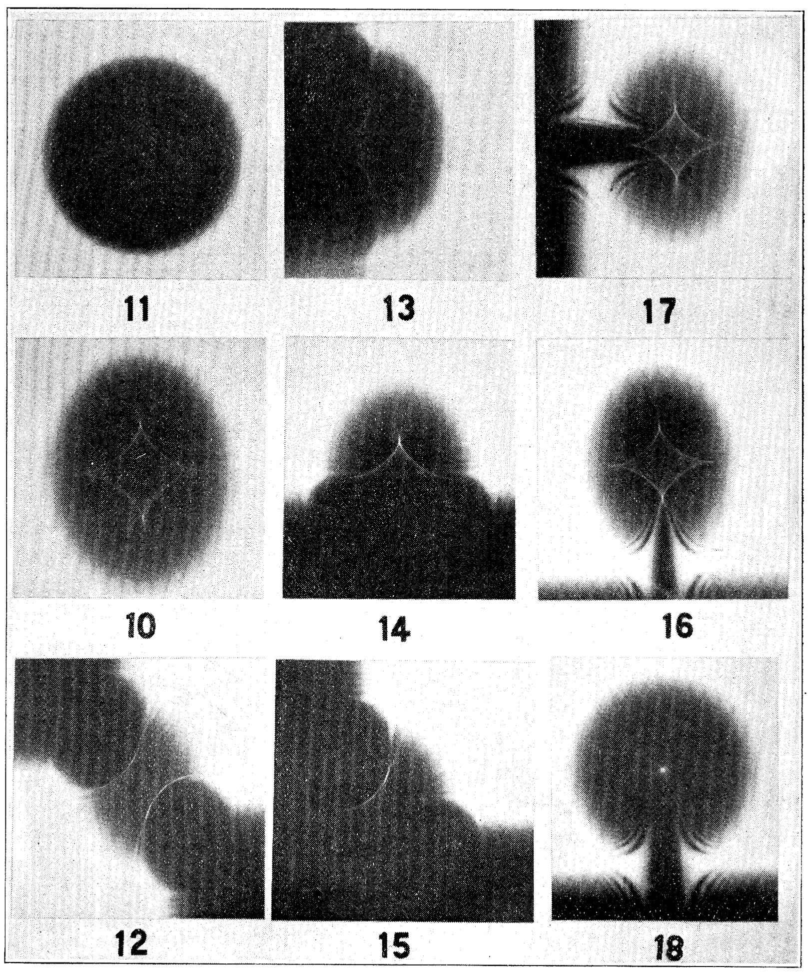
The respective distances between the cusps lying on the major axis and those lying on the minor axis are:

$$2x_0 = 2(a^2 - b^2)/a,$$

and

$$2y_0 = 2(a^2 - b^2)/b.$$

These distances when taken from the photographs permitted the calculation of a and b, which in every case were found to be the dimensions of the corresponding geometrical shadow.



JOHN COULSON AND G. G. BECKNELL.

In order to show the complete coincidence between the diffraction figure and the evolute of the geometrical shadow it was necessary to produce photographically a large sharp-lined figure. This was done by using a disc of large diameter (5.11 cm.) and a very small pinhole source (.12 mm.). The disc was rotated 37.5° with respect to the wave-normal. The figure thus produced, which was about 7 by 8 cm., was projected optically, with a magnification of about 8 diameters, upon a sheet of coördinate paper. The values of x and y for one quadrant of this sharply outlined figure are the points represented by circles in Fig. 19.

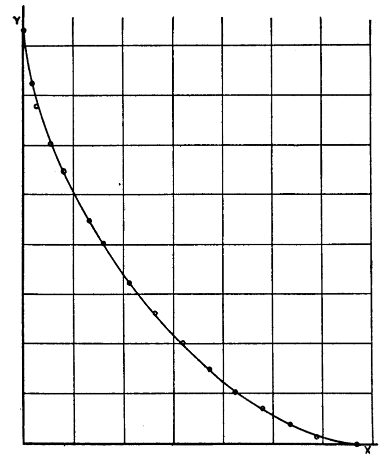


Fig. 19.

The curve itself is the computed evolute of the elliptical shadow. The agreement is well within the limits of observation.

The identity of the diffraction figure and the evolute of the geometrical shadow may be shown more readily from the following property of the evolute. The difference between the radii of curvature at any two points in the same quadrant of an ellipse is equal to the arc cut off of its evolute by the normals to the ellipse at the given points. Hence, it may be proved that the length of the quadrant of the evolute is equal to $(a^3 - b^3)/ab$.

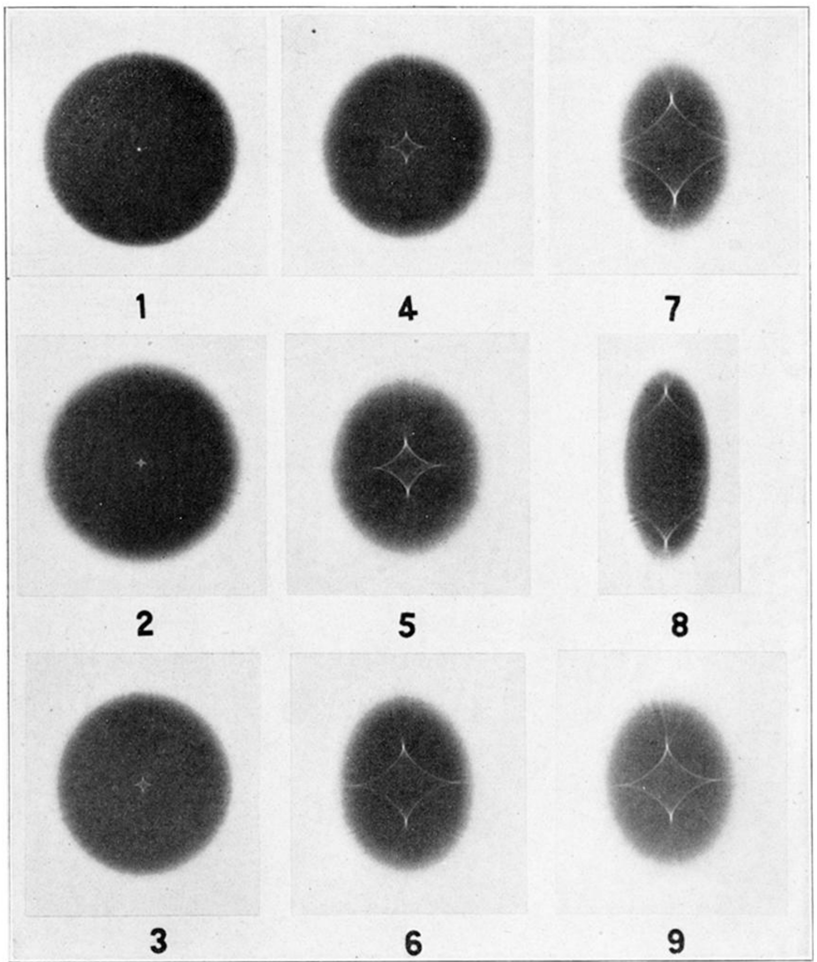
The length of the quadrant of the evolute was measured for several figures (after enlargement) by rolling a self-recording wheel along the periphery of the curve. A planimeter-wheel was used for this purpose. The agreement between the length of the quadrant of the evolute thus measured and the quantity $(a^3 - b^3)/ab$ for the corresponding geometrical shadow was in each case less than one per cent.

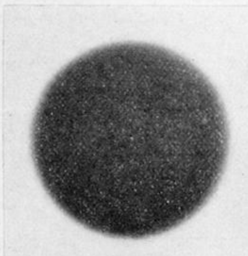
The agreement between the diffraction figures produced by an ellipse and a rotated circle casting the same geometrical shadow is shown in the table. These data were taken from Plate II., No. 6, and No. 9, enlarged.

x.	y (Observed).	
	Rotated Circle.	Ellipse.
.000	1.330	1.340
.056	1.035	1.038
.127	.875	.851
.190	.720	.735
.253	.615	.631
.380	.445	.455
.507	.300	.314
.633	.197	.200
.760	.108	.108
.887	.044	.041
1.032	.006	.000
1.050	.000	

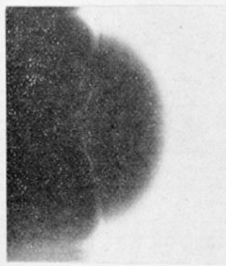
Attention may be called again to one of the important properties of the evolute. That is, the evolute is the *envelope of the normals* of the corresponding curve producing it. In the experiment with the straight-edge tangent to the elliptical plate it has been shown that for any given position of the straight-edge the median line of the corresponding pointed shadow is tangential to the diffraction figure and normal to the geometrical shadow of the plate. Hence, we may conclude that these diffraction figures constitute a family of envelopes, and corresponding to the caustics of reflection and refraction, that the figures here shown are the *caustics of diffraction*.

UNIVERSITY OF PITTSBURGH,
May, 1922.

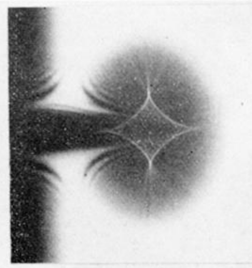

1
4
7
2
5
8
3
6
9



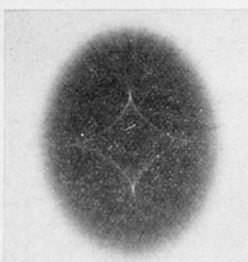
11



13



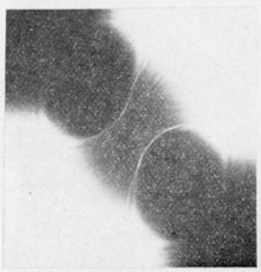
17



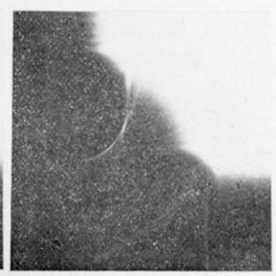
10

14

16



12



15

18