RADIATION FROM A GROUP OF ELECTRONS.

BY LEIGH PAGE.

SYNOPSIS.

Electric Intensity and Radiation Flux at a Great Distance from a Group of Electrons.—(1) The *general equation* for the intensity is obtained from the field equations of classical electrodynamics by expanding the retarded expression for the intensity due to a point charge:

$$E = \frac{1}{4\pi Rc^2}\left[\left\{\frac{d}{dt}(\Sigma e\boldsymbol{v}) + \frac{1}{c}\frac{d^2}{dt^2}(\Sigma e\boldsymbol{r}\cdot\boldsymbol{M}\boldsymbol{v}) \cdots + \frac{1}{(n-1)!c^{n-1}}\frac{d^n}{dt^n}(\Sigma e\boldsymbol{r}\cdot\boldsymbol{M}^{n-1}\boldsymbol{v}) + \cdots\right\}\times\boldsymbol{M}\right]\times\boldsymbol{M},$$

where R is the distance from a fixed point in the group, $\boldsymbol{r}$ is the position vector of each electron, $\boldsymbol{v}$ its velocity, and $\boldsymbol{M}$ a unit vector in the direction of R. The Poynting flux is merely $c\boldsymbol{E}^2$. The terms in the series correspond to successive orders of radiation. The general condition for the absence of the nth order of radiation is evidently that $\Sigma e\boldsymbol{r}\cdot\boldsymbol{M}^{n-1}\boldsymbol{v} =$ constant. (2) *For a ring of evenly spaced electrons,* the radiation of all orders less than k, the number of electrons, vanishes; but the radiation of order k does not. In fact $R_k = C_k\beta^{2(k-1)}R_1$, where β is the speed relative to that of light and C_k is a constant which increases rapidly with k, so that for most atoms the higher order radiation from a ring with a number of electrons should be even greater than the radiation from the single electron of the hydrogen atom. Only in the case of a continuous distribution of electricity along the ring do all orders vanish.

Radiationless orbits of Bohr's atomic theory are seen to be in definite conflict with classical electrodynamics. Even the rotation of an asymmetrical positive nucleus would not neutralize the radiation from the hydrogen electron. Therefore, the advisability of seeking some other interpretation of the numerical relationships established by the theory is suggested.

CLASSICAL electrodynamics requires an irreversible radiation of energy from a solitary electron whenever it is accelerated. If, however, a number of electrons at distances from one another small compared to the wave-length emitted are accelerated, the radiations from the individual electrons may interfere in such a way as to annul the first order radiation from the group. This is the case when two or more evenly spaced electrons revolve in a single circular orbit. When the velocities of the electrons are high, however, the radiation of lowest order which does not vanish may be quite comparable with the first order radiation which would be emitted if all the electrons save one were removed. Therefore it has seemed worth while to investigate the higher order radiation from a group of electrons as required by the classical electrodynamics.

Choose as origin some point O in the group of electrons. Describe about O a sphere of radius R very large compared to the greatest distance between electrons in the group. Let $\boldsymbol{c}$ be a vector equal in magnitude to the velocity of light, and having a direction from O toward a point P on the surface of the sphere. The Poynting flux at P is

$$\boldsymbol{s} = c(\boldsymbol{E} \times \boldsymbol{H}).$$

But

$$\boldsymbol{H} = \frac{1}{c}(\boldsymbol{c} \times \boldsymbol{E}),$$

and, as both $\boldsymbol{E}$ and $\boldsymbol{H}$ are at right angles to $\boldsymbol{c}$,

$$\boldsymbol{s} = \boldsymbol{c}E^2. \tag{1}$$

The retarded electric intensity at a distance R from an electron[1] is

$$\boldsymbol{E} = \frac{e}{4\pi R^2 k^2 c\left(1 - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}\right)^3}\left[(\boldsymbol{c} - \boldsymbol{v}) + \frac{Rk^2}{c^3}\{\boldsymbol{f} \times (\boldsymbol{c} - \boldsymbol{v})\} \times \boldsymbol{c}\right]$$

in Heaviside-Lorentz rational units, where $\boldsymbol{v}$ is the velocity of the electron, $\boldsymbol{f}$ its acceleration, and $k^2 \equiv 1 - \beta^2$, where $\beta \equiv v/c$. By making R great enough, the first term can be made as small as desired compared to the second, and hence at P

$$\boldsymbol{E} = \frac{e}{4\pi R c^4\left(1 - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}\right)^3}\{\boldsymbol{f} \times (\boldsymbol{c} - \boldsymbol{v})\} \times \boldsymbol{c}. \tag{2}$$

Let $\boldsymbol{r}$ be the position vector of an electron in the group relative to the origin O at the time 0. Denote by $\boldsymbol{M}$ a unit vector in the direction of $\boldsymbol{c}$. Let $\boldsymbol{r}_e$ be the position vector of the electron at the time $\boldsymbol{r}_e\cdot\boldsymbol{M}/c$. Then $\boldsymbol{r}_e$ specifies the *effective position* of the electron. The radiations emitted by all the electrons when in their effective positions will reach P at the same time R/c. The electric intensity involved in this radiation will now be calculated.

Replacing $\boldsymbol{c}$ in (2) by $c\boldsymbol{M}$, and expanding, the electric intensity at P due to a single electron can be put in the form

$$\begin{aligned}\boldsymbol{E} &= \frac{e}{4\pi R c^2}\left[\left\{\frac{\boldsymbol{f}_e}{(1 - \boldsymbol{\beta}_e\cdot\boldsymbol{M})^2} + \frac{\boldsymbol{f}_e\cdot\boldsymbol{M}}{c}\frac{\boldsymbol{v}_e}{(1 - \boldsymbol{\beta}_e\cdot\boldsymbol{M})^3}\right\} \times \boldsymbol{M}\right] \times \boldsymbol{M}\\ &= \frac{e}{4\pi R c^2}\left[\left\{\frac{1}{1 - \boldsymbol{\beta}_e\cdot\boldsymbol{M}}\frac{d}{dt_e}\left(\frac{\boldsymbol{v}_e}{1 - \boldsymbol{\beta}_e\cdot\boldsymbol{M}}\right)\right\} \times \boldsymbol{M}\right] \times \boldsymbol{M}.\end{aligned}$$

[1] See the author's "Introduction to Electrodynamics," p. 29, or Abraham, "Theorie der Electrizität," Vol. II., p. 99.

Now,

$$t_e = \frac{\boldsymbol{r}_e \cdot \boldsymbol{M}}{c},$$

and after a time dt has elapsed at O (Fig. 1),

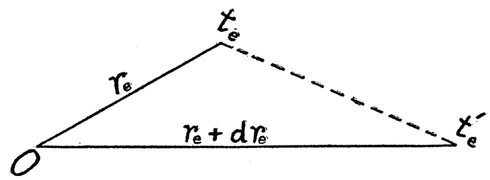

Fig. 1.

$$t_e' = dt + \frac{\boldsymbol{r}_e \cdot \boldsymbol{M}}{c} + \boldsymbol{\beta}_e \cdot \boldsymbol{M} dt_e.$$

Hence

$$\begin{aligned} dt_e &\equiv t_e' - t_e \\ &= dt + \boldsymbol{\beta}_e \cdot \boldsymbol{M} dt_e, \end{aligned}$$

or

$$\frac{1}{1 - \boldsymbol{\beta}_e \cdot \boldsymbol{M}} \frac{d}{dt_e} = \frac{d}{dt}. \tag{3}$$

Therefore

$$\begin{aligned} \boldsymbol{E} &= \frac{e}{4\pi R c^2} \left[\left\{ \frac{d}{dt} \left(\frac{\boldsymbol{v}_e}{1 - \boldsymbol{\beta}_e \cdot \boldsymbol{M}} \right) \right\} \times \boldsymbol{M} \right] \times \boldsymbol{M} \\ &= \frac{e}{4\pi R c^2} \left[\left\{ \frac{d}{dt} \left(\frac{1}{1 - \boldsymbol{\beta}_e \cdot \boldsymbol{M}} \frac{d\boldsymbol{r}_e}{dt_e} \right) \right\} \times \boldsymbol{M} \right] \times \boldsymbol{M} \\ &= \frac{e}{4\pi R c^2} \left[\frac{d^2 \boldsymbol{r}_e}{dt^2} \times \boldsymbol{M} \right] \times \boldsymbol{M}. \end{aligned} \tag{4}$$

Put

$$\tau = \frac{\boldsymbol{r} \cdot \boldsymbol{M}}{c}, \qquad \tau_e = \frac{\boldsymbol{r}_e \cdot \boldsymbol{M}}{c}.$$

Then

$$\boldsymbol{r}_e = \boldsymbol{r} + \boldsymbol{v}\tau_e + \frac{1}{2} \dot{\boldsymbol{v}} \tau_e^2 \cdots + \frac{1}{(n-1)!} \boldsymbol{v}^{(n-2)} \tau_e^{n-1} \cdots \tag{5}$$

and

$$\begin{aligned} \tau_e &= \tau + \dot{\tau}\tau_e + \frac{1}{2} \ddot{\tau} \tau_e^2 \cdots + \frac{1}{(n-1)!} \tau^{(n-1)} \tau_e^{n-1} \cdots \\ \tau_e^2 &= \tau^2 + \dot{\overline{\tau^2}}\, \tau_e + \frac{1}{2} \ddot{\overline{\tau^2}}\, \tau_e^2 \cdots + \frac{1}{(n-1)!} \tau^{2(n-1)} \tau_e^{n-1} \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_e^m &= \tau^m + \dot{\overline{\tau^m}}\, \tau_e + \frac{1}{2} \ddot{\overline{\tau^m}}\, \tau_e^2 \cdots + \frac{1}{(n-1)!} \tau^{m(n-1)} \tau_e^{n-1} \cdots \end{aligned} \tag{6}$$

The solution of (6) is

$$\tau_e^m = \tau^m + \frac{m}{m+1}\,\dot{\overline{\tau^{m+1}}} + \frac{m}{2\cdot m+2}\,\ddot{\overline{\tau^{m+2}}}\cdots +$$

$$\frac{m}{(n-1)!\,m+n-1}\,\tau^{m+n-1\,(n-1)}\cdots \qquad (7)$$

The correctness of this solution may be proved by mathematical induction. For assume that the solution has this form for the first n terms of τ_e, for the first $n-1$ terms of τ_e^2, and in general for the first $n-m+1$ terms of τ_e^m for all m's less than $n+1$. Then by substituting the assumed terms in (6), the $n-m+2$ term of τ_e^m is found to be

$$\frac{1}{(n-m+1)!}\left[\tau^{m\,(n-m+1)}\,\tau^{n-m+1} + (n-m)\,\tau^{m\,(n-m)}\,\tau^{n-m+1\,(1)} + \text{etc.}\right],$$

which, by Leibnitz's theorem for the kth derivative of the product of two factors, is equal to

$$\frac{1}{(n-m+1)!}\frac{d^{n-m}}{dt^{n-m}}\left[\frac{d}{dt}(\tau^m)\,\tau^{n-m+1}\right]$$

$$= \frac{m}{(n-m+1)!}\frac{d^{n-m}}{dt^{n-m}}\left[\tau^n\dot{\tau}\right]$$

$$= \frac{m}{(n-m+1)!\,n+1}\,\tau^{n+1\,(n-m+1)},$$

which is the form predicted by (7) for the $n-m+2$ term.

Now the first term of τ_e is obviously τ. Hence all the terms of all the series are as given by (7).

Substituting in the expression (5) for $\boldsymbol{r}_e$ the values of τ_e^m given by (7), the nth term of $\boldsymbol{r}_e$ is found to be

$$\frac{1}{(n-1)!}\left[\boldsymbol{v}^{(n-2)}\tau^{n-1} + (n-2)\boldsymbol{v}^{(n-3)}\tau^{n-1\,(1)} + \text{etc.}\right]$$

$$= \frac{1}{(n-1)!}\frac{d^{n-2}}{dt^{n-2}}(\boldsymbol{v}\tau^{n-1}).$$

Hence

$$\tau_e = \boldsymbol{r} + \boldsymbol{v}\tau + \frac{1}{2}\frac{d}{dt}(\boldsymbol{v}\tau^2)\cdots + \frac{1}{(n-1)!}\frac{d^{n-2}}{dt^{n-2}}(\boldsymbol{v}\tau^{n-1})\cdots \qquad (8)$$

and substitution in (4) gives the electric intensity due to a single electron at a great distance R from the group in the direction $\boldsymbol{M}$. Summing up over all the electrons, and replacing τ by $\boldsymbol{r}\cdot\boldsymbol{M}/c$, the total electric intensity at P is seen to be

$$\boldsymbol{E} = \frac{1}{4\pi Rc^2}\left[\left\{\frac{d}{dt}(\Sigma e\boldsymbol{v}) + \frac{1}{c}\frac{d^2}{dt^2}(\Sigma e\boldsymbol{r}\cdot\boldsymbol{M}\boldsymbol{v}) + \frac{1}{2c^2}\frac{d^3}{dt^3}(\Sigma e\overline{\boldsymbol{r}\cdot\boldsymbol{M}}^2\boldsymbol{v}) \cdots + \frac{1}{(n-1)!\,c^{n-1}}\frac{d^n}{dt^n}(\Sigma e\overline{\boldsymbol{r}\cdot\boldsymbol{M}}^{n-1}\boldsymbol{v})\cdots\right\}\times\boldsymbol{M}\right]\times\boldsymbol{M}. \quad (9)$$

Now (1) shows that the condition for absence of radiation is that $\boldsymbol{E}$ shall vanish for all directions of $\boldsymbol{M}$. If the electrons are to remain in a group, and not become separated further and further as time goes on, this condition requires that each of the vector sums enclosed in parentheses in (9) shall remain constant in time. Therefore in scalar notation the conditions for absence of radiation of each order are:

Order of Radiation.	Conditions.	No. of Conditions for Three-dimensional Motion.	No. of Conditions for Two-dimensional Motion.
1st........	$\Sigma e\dot{x}_j$ = const.	3	2
2d.........	$\Sigma e x_a \dot{x}_j$ = const.	9	4
3d.........	$\Sigma e x_a x_b \dot{x}_j$ = const.	18	6
nth........	$\Sigma e x_a x_b \cdots x_{n-1}\dot{x}_j$ = const.	$\frac{3}{2} n(n+1)$	$2n$

where x_1, x_2, x_3 are rectangular coördinates, and the suffixes a, b, $c\cdots$ assume values corresponding to all independent combinations of 1, 2, and 3, j having successfully the values 1, 2, and 3. As Larmor[1] has pointed out, a necessary but not sufficient condition for absence of second order radiation is that the magnetic moment of the group remain constant.

An interesting application of the conditions for absence of radiation can be made to the case of k evenly spaced electrons revolving in a fixed circular orbit with constant speed. The conditions for absence of radiation of the nth order become

$$e\Omega a^n \sum_p \sin^\alpha (\theta + P)\cos^\beta (\theta + P) = \text{const.}, \quad (10)$$

where Ω is the common angular velocity of the electrons, a the radius of the orbit, α and β any positive integers whose sum is n, θ the angular displacement of the first electron, and

$$P \equiv \frac{2\pi}{k}(p - 1),$$

the summation over the electrons extending from $p = 1$ to $p = k$.

[1] Phil. Mag., *42*, 595, 1921.

Writing the sine and cosine in exponential form, expanding by the binomial theorem and multiplying, each term in (10) is easily seen to have the form

$$C\sum_{p} e^{i[n-2(s+t)][\theta+P]} = f(\theta)\frac{1 - e^{2\pi i[n-2(s+t)]}}{1 - e^{\frac{2\pi}{k}i[n-2(s+t)]}}$$

where s and t are positive integers such that

$$s + t \leqq n,$$

and C is a constant not containing θ.

When $n - 2(s + t)$ is zero, the term reduces to the constant pC, which is not a function of θ and hence does not vary with the time. Otherwise each term vanishes except when $n - 2(s + t)$ is equal to an integral multiple of k. But this cannot occur if n is less than k. Hence *the radiation of all orders less than the number of electrons in the ring vanishes.* Thus, if there are two electrons in the ring, the first order radiation will vanish; if there are eight electrons, radiation of the first seven orders will vanish.

Suppose that the number k of electrons in the ring is even. Then all odd orders of radiation vanish, for $n - 2(s + t)$ cannot equal k for n odd. Similarly, if the number of electrons in the ring is odd, all even orders of radiation less than $2k$ vanish.

If the number of electrons in the ring is infinite, the radiation of all orders vanishes. Hence a continuous ring of electricity of uniform density rotating about its axis of symmetry will not radiate—a well-known result.[1]

The radiation emitted by any specified group of electrons may easily be calculated from (9) and (1). Thus the total radiation of lowest order emitted by a pair of electrons revolving in a circle so as to be always at opposite ends of a diameter, is the second order radiation. In Heaviside-Lorentz rational units, it amounts to

$$R_2 = 9.6\beta^2 R_1, \tag{11}$$

where

$$R_1 \equiv \frac{e^2 f^2}{6\pi c^3}$$

is the first order radiation that would be emitted by one of the two electrons in the absence of the other. Therefore the second order radiation from a pair of electrons is greater than the first order radiation from a

[1] Schott, Phil. Mag., *36*, 255, 1918.

single electron traversing the same orbit at the same rate if

$$\beta > .323.$$

On Bohr's theory of the atom this occurs in the K rings of elements of atomic number greater than 44 (ruthenium). Furthermore, a simple calculation shows that the second order radiation from the pair of electrons in the K rings of elements of atomic number greater than 2 (helium) should be greater than the first order radiation from the single electron in the hydrogen atom when it is in the first Bohr ring.

In general, if k evenly spaced electrons are revolving in a circular orbit, the radiation of the kth order (which is the lowest order which does not vanish) is

$$R = C_k \beta^{2(k-1)} R_1,$$

where the numerical constant C_k increases rather rapidly with k, being 9.6 for two electrons and 173 + for four electrons.

The chief object of this paper is to emphasize the inconsistency between Bohr's theory of the atom and classical electrodynamics. In the case of the hydrogen or ionized helium atom, where the theory has been most successful, Bohr requires the single electron to revolve about the nucleus in a circular or elliptical orbit without radiating. Electrodynamics, on the other hand, allows no possibility of orbital motion of a solitary electron without radiation. Even in the case of a number of electrons following one another around the same orbit, the higher order radiation emitted should be greater, in the case of most atoms, than the first order radiation from hydrogen. Now the radiation of energy specified by the Poynting vector is one of the most fundamental properties of classical electrodynamics. Hence it is difficult to see how electrodynamics and Bohr's theory of the atom can ever find a common ground. This raises the question of the advisability of seeking for other than a dynamical interpretation of the numerical relationships which Bohr's theory has been so successful in establishing.

It might be thought that the radiation from the single electron in the hydrogen or ionized helium atom might be annulled by radiation from an symmetrical nucleus rotating at the same rate as the electron. Suppose, for instance, that the hydrogen nucleus consists of two small coincident circular rings of electricity, one positive and the other negative, rotating in opposite senses about their common center, and having charges of such densities that the net charge on the nucleus is one unit of positive electricity. Investigation shows that the asymmetric distribution of charge and the angular velocity of rotation can be chosen so as to satisfy the conditions of each order for absence of radiation from the atom. But the equal quantities of positive and negative electricity which must be distributed asymmetrically on the two rings to annul the radiation of each order become greater and greater as the order increases, as indeed is evident from the fact that the exponent of $\mathbf{r} \cdot \mathbf{M}$ in the terms of (9) increases with the ordinal number of the term. Hence the presence of infinite amounts of both positive and negative electricity within the nucleus would be required in order to make possible the entire annulment of radiation from the atom—a condition which makes the formal solution of the problem of no physical value.

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