PROBLEMS OF QUANTUM THEORY IN THE LIGHT OF THE THEORY OF PERTURBATIONS.

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SYNOPSIS.

1. It is shown in the following paper that the physical purport of *Delannay's* method in the theory of perturbations consists in successive approximation of a given motion by means of a set of conditionally periodic motions.

2. It is further shown, that at every degree of approximation two possible cases must be taken into account. In the one the motion of the system is periodic with respect to a certain variable, in the other this variable performs a libration.

3. In both cases formulæ for carrying through the calculations are given, fit for the purposes of the theory of quanta.

4. A modification of *Delaunay's* method, due to *Whittaker* is applicable to one of the two cases only and is therefore incomplete.

5. *Delaunay's* method (in our modification) quite automatically yields at every degree of approximation the special coördinates, looked for in the theory of quanta.

§ 1. INTRODUCTION.

THE method explained in the following pages was worked out by the author several years ago for the purpose of quantizing the helium atom.[1] Its principle, however, turned out not to be new[2] and to have been already the foundation of a procedure applied by Delaunay[3] in the theory of the moon. Though Delaunay did not use the notions of the "conditionally periodic system" and of the "angular variables," the introduction of these concepts changes only the formal side of the line of thought. We shall therefore in the following refer to our procedure as "*Delaunay's method*."

In the meantime methods of the theory of perturbations were introduced into the theory of quanta by others also: J. M. Burgers[4] adopted a manner of treatment due to Whittaker, while N. Bohr[5] dealt with the problem of finding the coördinates suitable for quantization in slightly disturbed systems. The rules given by Bohr were applied with great

[1] An exposition of the method with application to the problem of three bodies was given by the author in October, 1917, and the two following ones were communicated by him to the Association of Swiss Naturalists in 1919 (Cf. the brief report: Verh. der Schweis. Naturforschenden Ges., 2. Teil, p. 83, 1920).

[2] I am indebted for valuable quotations of literature to some letters of Mr. J. M. Burgers in fall of 1917.

[3] Delaunay, Théorie du mouvement de la lune (Mémoires de l' Académie des Sciences, Vol. 28, 29), Paris, 1860, 1867.

[4] J. M. Burgers, Verslagen Amsterdam, XXXVI., p. 115, 1917.

[5] N. Bohr, Kgl. Danske Vedensk. Selsk., Nat. Afd., 8. Raekke IV. 1, 1918.

success by Kramers[1] for explaining the gradual increase of the Stark effect with the electric field. This work however does not by any means make superfluous the publication of my modification of Delaunay's method. I believe on the contrary, that it can claim some interest for several reasons:

First, this procedure is particularly applicable to the theory of quanta because of the fact that the real motion can be regarded as *successively approximated by conditionally periodic motions.* A way is given for finding a set of conditionally periodical motions which approximate the real motion successively closer and closer. These systems are, however, not of the simplest type studied by Haeckel, and there therefore arises the problem of investigating a more general kind of conditionally period motion (§§ 4, 5). We shall see that all theorems which are valid in Haeckel's [2] case can be extended also to the new type. In particular there still remains the possibility of describing the motion by angular variables (§ 6). The principles of the theory of quanta, as they were laid down by the author[2] for conditionally periodic systems, can therefore be applied unchanged to whatever degree of approximation one desires.

Second, at any step of approximation there may appear two essentially different types of motion which determine the method of passing to the next step. Both Delaunay himself and Poincaré [3] pointed out this alternative, but, as it seems, not with sufficient emphasis; at least, this important discrimination is entirely ignored in Whittaker's modification[4] already referred to. Therefore this modification does not exhaust the problem and is in many cases inapplicable (§ 9). To show this we lay stress on making clear the physical meaning of the transformations used.

Third, the special forms, in which the theory of perturbations is carried through, start from the existence of a small constant parameter, by which the perturbation function may be developed. Such a parameter however is not available in all problems of the theory of quanta. Whittaker's method is free from this restriction, but comprises, as already mentioned, one of the two possible cases only. The center of gravity of our considerations lies therefore in showing how the numerical calculations can be carried through in the second case, not included in Whittaker's theory, without using a constant parameter (§§ 7, 8).

In so far as these three points are concerned, our investigation lies

[1] H. A. Kramers, ibidem, 8. Raekke, III. 3, 1919.

[2] P. S. Epstein, Ann. d. Phys., *50*, p. 815; *51*, p. 168, 1916.

[3] H. Poincaré, Les méthodes nouvelles de la mécanique celeste, Vol. II, §§ 200 and ff. Paris, 1893.

[4] E. T. Whittaker, Proc. London Math. Soc., *34*, p. 206, 1902; Analytical Dynamics, p. 404, Cambridge, 1917.

in the region of general dynamics and will be only partly new to the astronomer.

Fourth and last, the method opens interesting aspects of the theory of quanta, dealt with in § 10. It is shown there, that the problem of coördinates suitable for quantization, is solved quite automatically by the method: it is sufficient to go on from approximation to approximation in order to find always the right variables. In the special case when the undisturbed motion is periodic, the coördinates found by our method agree exactly with those resulting from the rules given, for this case, by Bohr. Also for some other cases discussed by Bohr our method gives closely similar results. There exists, however, a difference, for Bohr regards the possibility of exact quantization as an exception, and generally expects, for a disturbed motion, indeterminate values of energy; whereas from the point of view of our theory, exact quantization is the normal case, and indeterminate values of energy are at least much rarer than Bohr implies. The decision between the two points of view seems to lie within experimental possibilities.

§ 2. From the Transformation Theory of Dynamics.

We proceed to put together several theorems, to be used in this and in the following communication.

Let the differential equation of a mechanical system of f degrees of freedom be given in the canonical form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad (i = 1, 2, 3, \cdots f), \tag{1}$$

where the *Hamiltonian function* $H(p, q, t)$ depends upon the *momenta* p_i, upon the *coördinates of position* q_i and upon the time t. By the system of equations

$$\begin{cases} p_i = p_i(P_1, \cdots P_f;\ Q_1, \cdots Q_f;\ t), \\ q_i = q_i(P_1, \cdots P_f;\ Q_1, \cdots Q_f;\ t) \end{cases} \tag{2}$$

one may pass to the new variables P_i, Q_i. It remains to be determined, in which case the coördinates P_i, Q_i are again *canonical* variables.

In the following we shall have to deal with the two special cases, in which the functions (2) are either independent of time, or contain time in the form of a linear term $A_i t$, (resp. $B_i t$), where A_i and B_i are constants. The criterion for P_i, Q_i being canonical coördinates can then be expressed by means of the so-called "Lagrange's parenthesis"

$$(a, b) = \sum_{i=1}^{f} \left(\frac{\partial p_i}{\partial a} \cdot \frac{\partial q_i}{\partial b} - \frac{\partial q_i}{\partial a} \cdot \frac{\partial p_i}{\partial b} \right) \tag{3}$$

in the following way

$$(4)\qquad \begin{cases}(Q_j,\ Q_h) = 0, \\ (P_j,\ P_h) = 0,\end{cases} \qquad (P_j,\ Q_h) = \begin{cases}0 & \text{for } j \neq h, \\ 1 & \text{“} \quad j = h.\end{cases}$$
$$(h, j = 1, 2, \cdots f).$$

We shall discuss the above two cases separately:

1. A transformation, independent of time

$$(2')\qquad \begin{cases}p_i = p_i(P_1, \cdots P_f;\ Q_1, \cdots Q_f), \\ q_i = q_i(P_1, \cdots P_f;\ Q_1, \cdots Q_f),\end{cases}$$

satisfying conditions (4) is called a "*contact-transformation*."[1] The most interesting property of these transformations is, that both systems p_i, q_i and P_i, Q_i have the same Hamiltonian function: *the Hamiltonian function is invariant with respect to contact-transformations.* We see further from the form of equations (4) that they are entirely independent of the special form of the Hamiltonian function, so that they can be satisfied by a proper choice of the transformation equations (2′) alone. We can express this important circumstance by the following statement:

Let a system of coördinates p, q be given, canonical with respect to two Hamiltonian functions H and H^*. If we succeed in finding another system of coördinates P, Q, which are canonical with respect to H, they will be canonical with respect to H^* also.

By a well-known theorem of Jacobi's the contact transformation can be defined by one single function of the variables P_i and q_i

$$(5)\qquad W = W(P_1, \cdots P_f;\ q_1, \cdots q_f),$$

from which the transformation-equations can be derived in the form

$$(5')\qquad p_i = \frac{\partial W}{\partial q_i}, \qquad Q_i = \frac{\partial W}{\partial P_i}.$$

2. The case in which equations (2) assume the form

$$(2'')\qquad \begin{cases}p_i = p_i'(P_1, \cdots P_f;\ Q_1, \cdots Q_f) + A_i t, \\ q_i = q_i'(P_1, \cdots P_f;\ Q_1, \cdots Q_f) + B_i t,\end{cases}$$

where A_i and B_i are constants, and conditions (4) are satisfied, we shall name a "*Delaunay-transformation.*" The difference between this and the first case is that here there exists no invariancy of the Hamiltonian function. On the contrary, the new variables, P_i, Q_i, are canonical with respect to a *changed* Hamiltonian function

$$(6)\qquad H^* = H + \sum_{i=1}^{b}(A_i q_i - B_i p_i).$$

[1] E. T. Whittaker, Analytical Dynamics, Chapter XI. Cambridge, 1917.

A glance at conditions (4) shows us that they do not restrict the choice of the constants A_i, B_i. The applications below deal with the case in which the dependence on time only of the coördinates of position q_i is given by the values of the constants B_i ($i = 1, 2, \cdots f$). It follows from the remark just made that the choice of the quantities A_i is entirely at our discretion. In particular we can put these constants equal to zero, which leads to the following simplest form of the Delaunay-transformation

$$(7)\qquad \begin{cases} p_i = p_i^*(P_1, \cdots P_f;\ Q_1, \cdots Q_f), \\ q_i = q_i^*(P_1, \cdots P_f;\ Q_1, \cdots Q_f) + B_i t, \\ \qquad H^* = H - \sum_{i=1}^{b} B_i p_i. \end{cases}$$

p_i^*, q_i^* are functions which, taken alone (*i.e.*, putting B_i equal to zero) yield a contact-transformation.

§ 3. Delaunay's Method.

Let the motion of a system be given by the Hamiltonian function

$$H(p_1, \cdots p_f;\ q_1, \cdots q_f).$$

For the sake of simplifying our considerations we suppose forces to be conservative, and therefore time to be lacking as an explicit argument of H. But we shall see that these considerations can be enlarged without any essential change for certain forms of explicit dependence on time.

To carry through the integration of this problem, we shall, with Delaunay (l.c.), use the following method of approximation: we choose a conditionally periodic motion, having the Hamiltonian function H_1, in such a way that it is as near as possible to the given motion, and if we split function H in two terms

$$(8)\qquad H = H_1 + R_1,$$

so that the remainder R_1, or the "*perturbation function*," becomes as small as possible. The motion, defined by H_1, is called the "*first intermediate motion*."

One of the important features of conditionally periodic motions is that they can be described by so-called "*angular variables*," *i.e.*, it is possible to introduce by means of a contact-transformation

$$(9)\qquad \begin{cases} W = W(u_1, \cdots u_f;\ q_1, \cdots q_f), \\ p_i = \dfrac{\partial W}{\partial q_i}, \qquad w_i = \dfrac{\partial W}{\partial u_i}, \end{cases}$$

a new set of canonical coördinates (w_i) and momenta (u_i), possessing the following properties:[1]

1. The state of the system (for instance its original coördinates p_i, q_i) is a periodic function of the variables $w_1, w_2, \cdots w_f$, having the period 2π.

2. The variables w_i are linear functions of time

$$w_i = \Omega_i t + \delta_i. \tag{10}$$

The coefficient Ω_i is called the "*average motion.*"

3. The momenta u_i are constant and are proportional to Planck's quantum of action h:

$$u_i = \frac{h}{2\pi} n_i, \tag{11}$$

n_i meaning whole numbers, the so-called "*quantum-integers.*"

4. Transformation (9) being accomplished, H becomes a function of the momenta u_i only and is independent of the variables w_i.

The theory of conditionally periodic motions (cf. § 6) affords the means of establishing transformation (9) for the intermediate motion defined by function H_1. Using the theorem stated in § 2 we see, however, that the variables introduced by those transformation-equations prove to be canonical with respect to the whole Hamiltonian function H also. Of course they lose their physical meaning: referred to the system determined by H, neither are the quantities w_i linear functions of time, nor are the quantities u_i constant. But since the form of functional dependence is retained, p_i and q_i remain in a formal respect periodic functions of the variables w_i. If within the limits of variation of p_i and q_i the perturbation function is a regular function of these coördinates, as we shall suppose it to be, it will be itself a periodic function of the angular variables, and in most cases it may be expanded into an f-fold Fourier-series. At least the possibility of such an expansion is supposed in all that follows.

On the other hand, the term H_1 of the Hamiltonian function will continue to depend on the quantities u_i only:

$$H = H_1(u_1, \cdots u_f) + \sum_{m_1}, \sum_{m_2} \cdots \sum_{m_f} b_{m_1, m_2, \cdots m_f} \frac{\cos}{\sin} (m_1 w_1 + \cdots + m_f w_f), \tag{12}$$

where the coefficients b are functions of the momenta u_i.

Let b be that one of the coefficients which has the largest numerical value. We are to consider the system defined by the Hamiltonian function

$$H_2 = H_1(u_1, \cdots u_f) + b(u_1, \cdots u_f) \frac{\cos}{\sin} (m_1 w_1 + \cdots + m_f w_f). \tag{13}$$

[1] Cf. C. G. Charlier, Die Mechanik des Himmels, Vol. I., p. 94, Leipzig, 1902; also P. S. Epstein, Ann. d. Phys., *51*, p. 176, 1916.

We shall prove in §§ 4, 5 that such a function again represents a conditionally periodic system. This we can choose as the *second intermediate motion*, putting similarly to (8)

$$H = H_2(u, w) + R_2(u, w), \tag{8'}$$

and denoting by R_2 the remaining part of the *Fourier*-series. Instead of u, w we have now to introduce the angular variables u', w' of the conditionally periodic motion, given by (13); then H_2 is dependent on the quantities u' only, while R_2 must be rearranged in a new f-fold Fourier-series, proceeding by the arguments w'.

By this transformation the periodic term of the Fourier-series (12) having the largest numerical value is taken away and converted into a part of the aperiodic term. The same procedure applied to the periodic term, second in absolute value, leads to the *third intermediate* conditionally periodic motion, and takes away this term also. Going on successively in such a way, we can make the numerically important terms of the Fourier-series vanish one after another (retaining only their aperiodic parts), and we can proceed to conditionally periodic motions which approximate more and more closely the real motion, until the desired degree of precision is reached.

This procedure is particularly well adapted to the theory of quanta, operating with that particular function which is the most important one for that theory. In most applications, indeed, the problem is to find the expression of energy in terms of the quantum-integers (11), and to express in terms of the momenta u_i the Hamiltonian function of that intermediate motion by which one wishes to close the approximation.

Of course the question remains unsettled as to whether the procedure sketched in this paragraph is a convergent one. We can point out only that Delaunay used it in the theory of the moon, and arrived at a good agreement with observation, and that the theory of quanta does not by far aspire to the degree of precision achieved by Delaunay and his successors. We shall briefly revert to this question again in § 10.

§ 4. Extended Theory of Conditionally Periodic Motions.

We are now to supply the proof that the Hamiltonian function, given by equations (13) § 3 defines in fact a conditionally periodic motion. We confine ourselves to the case of the cosine, because that of the sine results from the first one by a simple shift of phase of the argument. We start from the equation of energy

$$H_2 = H_1(u_1, \cdots u_f) + b(u_1, \cdots u_f)\cos(m_1 w_1 + \cdots + m_f w_f) = \alpha, \tag{14}$$

because the Hamiltonian function, if independent of time, is known to express the energy α of the system.

Our first object is to bring about a separation of variables, and we succeed in this by a simple contact-transformation:

$$(15)\qquad \begin{cases} W = u_1'\cdot(m_1w_1 + \cdots + m_fw_f) + \sum_{i=2}^{f} u_i'w_i, \\ u_i = \dfrac{\partial W}{\partial w_i}, \qquad w_i' = \dfrac{\partial W}{\partial u_i'}, \end{cases}$$

from which,

$$(16)\qquad \begin{cases} u_1 = m_1u_1, \qquad u_2 = u_2' + m_2u_1', \qquad \cdots \qquad u_f = u_f' + m_fu_1', \\ w_1' = m_1w_1 + \cdots + m_fw_f, \qquad w_2' = w_2, \qquad \cdots \qquad w_f = w_f. \end{cases}$$

We introduce the quantities u_i', w_i', instead of u_i, w_i, into our equation (14)

$$(17)\qquad H_2 = H_1'(u_1', \cdots u_f') + b'(u_1', \cdots u_f') \cos w_1' = \alpha.$$

Now only one single coördinate of position appears in the energy equation; the other coördinates do not enter in the Hamiltonian function at all, and are therefore called "*cyclic variables*." It follows from this that their corresponding momenta are constant (in virtue of the canonical equations $u_i' = -\partial H/\partial w_i'$). *Thus the separation of variables is accomplished:* Equation (17) gives us the relation between the two variables u_1' and w_1', while the quantities α, u_2', $\cdots$ u_f' must be considered as constant.

We will now investigate equation (17) in order to determine if it represents a conditionally periodic motion. Some properties of that type of motion we have already mentioned in § 3. But their characteristic feature is this, that the coördinates, determining the position of the system, are either already of such a nature that the state of the system is *periodic* in them when they grow without limit (in the manner of a plane angle), or that they perform "*librations*," *i.e.*, swing backwards and forwards between two fixed limits. In the special case in which the Hamiltonian function is quadratic in terms of the momenta, the conditions for the occurrence of such motions and the properties of the same were studied by Haeckel (cf. § 3). In particular Haeckel showed that, separation of variables being possible, one can always choose the constants of the motion in such a way that it assumes the characteristics of conditional periodicity. We shall prove that the conditions in the case of the form of Hamiltonian function given by equation (17) are quite similar. This form is on the one hand more special than Haeckel's, the dependence on the coördinate w_1' being a fixed one, but on the other hand it is more general, leaving entirely open the dependence on the coördinate u_1'.

For the sake of simplifying our notation we shall again drop the accents on the letters. $u_2, u_3, \cdots u_f$ playing only the part of constants, we write for short

(17′) $$H = H_1(u_1) + b(u_1)\cdot\cos w_1 = \alpha.$$

This equation establishes a functional dependence between u_1 and w_1, of which, on account of the physical meaning of the problem and on account of the symbol u_1, referring to a mechanical momentum, we can conclude in advance as follows: (1) u_1 is an analytic, not infinitely many-valued function of $\cos w_1$. (2) It follows from this that we know the whole path of u_1 if this quantity is given between the limits $0 \leq w_1 \leq \pi$, the values of u_1 in the adjacent intervals $-\pi \leq w_1 \leq 0$, $\pi \leq w_1 \leq 2\pi$, etc., resulting from the former ones by reflection. (3) u_1 has, at least in part of the region between 0 and π, real values. We shall suppose it to be real in the vicinity of $w_1 = 0$, a restriction which does not limit the generality of our conclusions.

If in a diagram we plot w_1 as abscissa and u_1 as ordinate, the tangent of the angle at any point of the curve is

$$\frac{du_1}{dw_1} = -\frac{\dfrac{\partial H}{\partial w_1}}{\dfrac{\partial H}{\partial u_1}}.$$

The maxima and minima of the curve are given by the condition

(18) $$\frac{\partial H}{\partial w_1} = -b(u_1)\sin w_1 = 0,$$

the turning points (*i.e.*, the maxima and minima of w_1) by

(19) $$\frac{\partial H}{\partial u_1} = \frac{\partial H_1}{\partial u_1} + \frac{\partial b}{\partial u_1}\cos w_1 = 0.$$

To simplify our argument we shall suppose that b does not vanish in any point of the curve, and that u_1 assumes no infinite values, though our conclusions would be the same if these possibilities were admitted. The maxima and the minima of the curve lie at $\sin w_1 = 0$, *i.e.*, $w_1 = 0$ and $w_1 = \pi$. Starting from $w_1 = 0$, the value of u_1 changes in a monotonic way until the curve reaches $w_1 = \pi$, or returns to $w_i = 0$. There are several possible cases of this: (1) $\partial H/\partial u_1$ has no roots within the real interval $0 \leq w_1 \leq \pi$, then the variable w_1 increases also monotonically (Fig. 1, Curve I.). (2) $\partial H/\partial u_1$ has in the above interval *one* simple root at $w_1 = w_1{}^\circ$; then the curve turns back at this point (Curve

II.). (3) $\partial H/\partial u_1$ has *two* roots or one double root (Fig. 2, Curve III.).

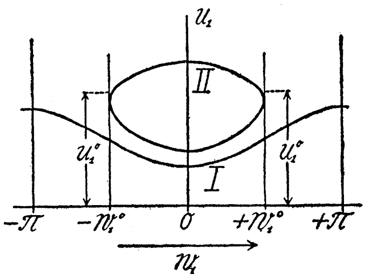

Fig. 1.

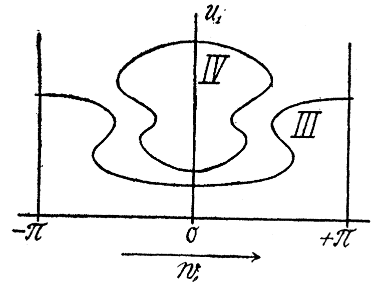

Fig. 2.

(4) $\partial H/\partial u_1$ has three roots (Curve IV.), and so on.[1]

The question now arises as to whether it is possible to realize by a proper choice of the constants α the cases of the curves I. and II. I have not succeeded in answering this question generally for any functions H_1 and b, but in every special case the investigation is carried out easily.

§ 5. THEORY OF CONDITIONALLY PERIODIC MOTIONS (CONTINUED).

We will discuss the above question in two instances typical for practical problems.

(*a*) Let equation (17′) be given in the form

$$-\frac{1}{2(u_1+u_2)^2}+\frac{\beta}{2}(u_1+u_2)^2\cos w_1=\alpha,$$

where β is a constant. According to (19) a turning point can only appear when condition

$$\frac{1}{(u_1+u_2)^3}+\beta(u_1+u_2)\cos w_1^0=0$$

is fulfilled.

The elimination of u_1+u_2 from these two equations yields for the position of the turning point

$$\cos w_1^0=-\frac{\alpha^2}{\beta}. \tag{20}$$

We thus arrive at the result: If $a^2>|\beta|$, there exists no turning point and the curve is of the type I., running monotonically from $w_1=0$ to $w_1=\pi$. If $\alpha^2<|\beta|$ there exists one and only one turning point between 0 and π, for $\cos w_1^2$ by (20) is uniquely determined for every value of α. Therefore we have the curve of type II. The cases III. and IV. cannot be realized by any choice of α.

[1] If the assumption, that u_1 is real for $w_1=0$, is omitted, there exist still two other types of curves for *one* and *two* roots of $\partial H/\partial u_1$.

(*b*) Let the Hamiltonian function be expressed by

$$H = -\frac{1}{2(u_1 + u_2)^2} - \omega u_1 + 2\beta\sqrt{u_1}\cos w_1 = \alpha,$$

and let β be a small quantity and u_1 be restricted also to small values only, ω being a positive constant.

Condition (19) for the turning point yields

$$\frac{1}{(u_1 + u_2)^3} - \omega + \frac{\beta}{\sqrt{u_1}}\cos w_1 = 0.$$

We expand both equations in powers of u_1 and confine ourselves to terms of the first order only

$$(21)\qquad \begin{aligned} -\frac{1}{2u_1^2} + \left(\frac{1}{u_2^3} - \omega\right)u_1 + 2\beta\sqrt{u_1}\cos w_1^0 &= \alpha, \\ \left(\frac{1}{u_2^3} - \omega\right) + \frac{\beta}{\sqrt{u_1}}\cos w_1^0 &= 0. \end{aligned}$$

Eliminating u_1 and writing for short $\Omega = 1/u_2^3$, we obtain

$$\cos^2 w_1^0 = -\frac{1}{\beta^2}\frac{1}{\Omega - \omega}\left(\alpha + \frac{1}{2u_2^2}\right).$$

According to condition $0 \leq \cos^2 w_1^0 \leq 1$, which must be satisfied for all real values of w_1^0, we are led to the result (putting $\Omega > \omega$):
In the cases

$$\alpha < -\left[\beta^2(\Omega - \omega) + \frac{1}{2u_2^2}\right] \quad \text{and} \quad \alpha > -\frac{1}{2u_2^2}$$

there does not exist any turning point; in the opposite case

$$-\frac{1}{2u_2^2} \leqq \alpha \leqq -\left[\beta^2(\Omega - \omega) + \frac{1}{2u_2^3}\right]$$

there is a turning point.

On the contrary, if we put $\Omega < \omega$, the signs $>$ and $<$ in the inequalities must be exchanged. That *no more than one* turning point can exist, results from the following consideration: supposition $u_1 = 0$ would lead to $\alpha = -1/2u_2^2$ and cannot be generally satisfied. Therefore $\sqrt{u_1}$ has always the same sign, and according to equation (21), $\cos w_1^0$ is also restricted to one sign. The limits, within which u_1 is contained, if the case of Curve II. is considered, with both suppositions $\Omega > \omega$ and $\Omega < \omega$ become

$$(23)\qquad 0 \leq u_1 \leq \beta^2.$$

The Curves III. and IV. are not to be obtained in either of our instances. They appear indeed to belong to rare exceptions, and the author did not meet with them in any application studied by him. We can therefore leave these possibilities out of the following considerations. Our result therefore may be stated thus: *in a certain interval of values of the constant α the case of Curve I. is valid, for all other values the case is that of Curve II.*

In the first case the variable w_1 increases without limit, for, according to the canonical equations, velocity w_1 can change its sign only if the condition $\partial H/\partial u_1 = 0$ is fulfilled. The momentum u_1 is then a periodic function of the coördinate w_1 and can be represented by a Fourier-series.

$$u_1 = c_0 + c_1 \cos w_1 + c_2 \cos 2w_1 + \cdots. \tag{24}$$

We shall call this case the "*case of periodicity.*"

In the second case w_1 varies between two fixed limits, or, using an expression of *Charlier*, w_1 performs "a librational motion." We call this behavior therefore the "*case of libration.*" One sees best the analytical character of the function u_1 if he supposes its dependence on w_1 to hold for complex values of w_1 also, and if he accordingly studies the distribution of u_1 in the complex w_1-plane (Fig. 3). In the segment of the real axis between the points $w_1 = -w_1^2$ and $w_1 = +w_1^2$ the variable u_1 has *two* real values, as results from Curve II. (Fig. 1). In the vicinity of the turning points (u_1^0, w_1^0) and $(u_1^0, -w_1^0)$ Curve II. can be approximated by the two parabolæ

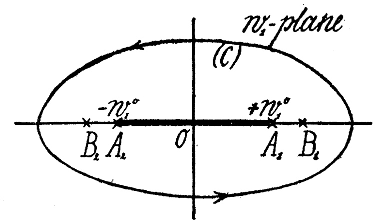

Fig. 3.

$$u_1 - u_1^0 = c\sqrt{w_1 - w_1^0} \qquad \text{and} \qquad u_1 - u_1^0 = c\sqrt{w_1 + w_1^0},$$

the contact being a simple one by supposition. This means that the function u_1 of the complex variable w_1 has, within the interval $-\pi \leq w_1 \leq +\pi$ of the real axis, *two branch-points* $w_1 = w_1^0$ and $w_1 = -w_1^0$, each one having the exponent 1/2, so that the totality of all values of this function can be represented by means of a two-sheeted Riemann-surface. If one connects, in the usual way, the two branch-points by a strait cut, function u_1 assumes a set of real values at the lower bank of the cut and another set of real values at the upper bank, which joins continuously the first one at the branch-points. We can, therefore, regard the change of the variable w_1 from the value $-w_1^0$ to $+w_1^0$ and back again *as a circuit around the branch-cut.*

Summarizing we can say: according to the value attributed to the constant α two cases occur. In the *first* case w_1 increases without limit and u_1 is a periodic function of w_1, having the period 2π (*case of periodicity*). In the *second* case w_1 swings monotonically forward from the limit $-w_1^0$ to the limit w_1^0 and backward from w_1^0 to w_1^0, and the analytical behavior of u_1 in the complex w_1-plane can be represented by use of a two-sheeted Riemann-surface (*case of libration*). The circumstances thus agree with those found by Haeckel for the special form of kinetic energy studied by him; we are therefore also justified in naming systems, defined by the Hamiltonian function (17′) or (13) conditionally periodic systems.

By these considerations we have come to a knowledge of the path of the variables u_1 and w_1. We are now to say a few words about the cyclic variables $w_2, w_3, \cdots w_f$. In the usual case of a Hamiltonian function, depending on the momenta quadratically, the cyclic coördinates can increase monotonically only; whereas in our case matters are more complicated, and it is advisable, in order to elucidate them, to use Jacobi's function of action, having here the form

$$W = \int u_1 dw_1 + u_2 w_2 + \cdots + u_f w_f. \tag{25}$$

The equations of motion then are known to be

$$\frac{\partial W}{\partial u_i} = \bar{w}_i \qquad (i = 2, 3, \cdots f),$$

the symbols $\bar{w}_i$ being used to indicate f new constants; or from (25)

$$w_i - \bar{w}_i = - \int \frac{\partial u_1}{\partial u_i} dw_1.$$

In the *first* (periodic) case, when expression (24) for u_1 holds, it follows that

$$w_i - \bar{w}_i = - \frac{\partial c_0}{\partial u_i} w_1 - \frac{\partial c_1}{\partial u_i} \sin w_1 - \frac{1}{2} \frac{\partial c_2}{\partial u_i} \sin 2w_1 + \cdots.$$

The first term of the right side increases monotonically and without limit. To this one-sided increase a periodic change is superposed, having the rhythm of the variable w_1, so that dw_i/dw_1 may have different signs for different values of w_1, from which only the average change of w_i is one-sided. In the special case, where c_0 is independent of u_i ($\partial c_0/\partial u_i = 0$), the average change of w_i vanishes, *i.e.*, we arrive at an oscillation between two fixed limits, or a libration. The period of this libration coincides with the period of the variable w_1, whence we draw the conclusion that this case occurs in so-called "*degenerate*" systems only.

Not very different from these are the conditions in the *second* case, when w_1 performs a librational motion. u_1, and consequently $\partial u_1/\partial u_i$, can then be divided into a *regular* part, having the same value on both banks of the branch-cut of Fig. 3, and into a *branched* part, having opposite signs on the two banks. The contribution of the second part to the integral is permanently positive, while that of the first one is periodic and changes its sign synchronously with w_1. Librational motions result, when the branched part does not depend on u_i. Here, too, this possibility is restricted to cases of degeneration.

It is known that degenerate systems can always be reduced to non degenerate ones with fewer degrees of freedom.[1] We suppose this reduction to be accomplished, and we are then permitted to put aside the possibility of cyclic coördinates, performing a libration. We arrive then at the statement that, though these variables do not always increase monotonically, their eventual fluctuations are of a regularly rhythmical kind, so that their behavior is in agreement with their designation as, "conditionally periodic motions."

§ 6. Introduction of Angular Coördinates.

As one of the chief properties of conditionally periodic motions, we pointed out the possibility of describing them by angular coördinates. We now proceed to show that these coördinates and the corresponding canonical momenta can be found in our more generalized case in just the same way as in the case studied by Haeckel. We shall, moreover, show that everyone of the properties of these variables enumerated in § 2 continues to hold. To avoid the asymmetry of notations coming from the fact that the momentum u_1 is invariable while the other momenta are constant, we shall consider the general case, denoting the coördinates of position by $g_1, g_2, \cdots g_f$, the momenta by $p_1, \cdots p_f$ and the constants of integration by $\alpha_1, \alpha_2, \cdots a_f$ (α_1 meaning the energy of the system). We start by considering the function of action

$$W(q_1, \cdots q_f;\ \alpha_1, \cdots \alpha_f) = \sum_{i=1}^{f} \int p_i(q_i;\ \alpha_1, \cdots \alpha_f)dq_i, \tag{26}$$

in which separation of variables is accomplished. Hence the equations of motion follow by Jacobi's method of integration in the following form

$$\left\{ \begin{aligned} t + \beta_1 &= \frac{\partial W}{\partial \alpha_1} = \sum_{i=1}^{f} \int \frac{\partial p_i}{\partial \alpha_1} dq_i, \\ \beta_h &= \frac{\partial W}{\partial \alpha_h} = \sum_{i=1}^{f} \int \frac{\partial p_i}{\partial \alpha_h} dq_i, \quad (h = 2, 3, \cdots f), \end{aligned} \right. \tag{26'}$$

β_i denoting f additional constants.

[1] Cf. P. S. Epstein, l.c., p. 179.

For the behavior of every individual variable g_i we found in § 5 two different possibilities: either the state of the system is a periodic function of the same with the period 2π (*i.e.*, the physical region of variability of q_i equals 2π); or the variable q_i performs a libration, while the corresponding momentum p_i is a function of q_i branched in a certain manner (Fig. 3). In the last case the partial derivatives $\partial p_i/\partial \alpha_h$ evidently belong also to the same branching type. Therefore while q_i performs a libration and comes back to the initial value (resp. while q_i increase by 2π), the integral

$$s_{ih} = \int \frac{\partial p_i}{\partial \alpha_h} dq_i, \qquad (i, h = 1, 2, \cdots f), \tag{27}$$

changes by the "modulus of periodicity"

$$w_{ih} = \oint \frac{\partial p_i}{\partial \alpha_h} dq_i. \tag{28}$$

The path of integration, symbolized by the circle at the sign of integration is, in the case of a branched integrand (*libration*), an arbitrary circuit about the branch-cut of the integrand in the complex q_i-plane which, however, is not permitted to include other singularities of the integrand.[1] In the case of unlimited increase of q_i (*periodicity*) it is the real path of integration from 0 to 2π.

On the other hand, if one inverts equation (27) and regards the variable q_i as a function of s_{ih}, q_i turns out as a *periodic function* of this quantity with the period ω_{ih}.[2] This kind of dependence makes it possible to introduce new variables by the relations

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 + \frac{1}{2\pi}\sum_{i=1}^{f} \omega_{i1} w_i,$$
$$\frac{\partial W}{\partial \alpha_h} = \quad \beta_h \quad = \frac{1}{2\pi}\sum_{i=1}^{f} \omega_{ih} w_i,$$

due to Weierstrass, in such a way, that the state of the system becomes periodic with respect to the same, having the period 2π. These are the angular coördinates looked for, which, as follows from equations (29), are in this case also linear functions of time.

Also the canonical momenta, corresponding to the angular coördinates, are expressed in the same way as in Haeckel's case. For it was shown by the author (l.c.), that there they are given by the expressions

$$u_i = \frac{1}{2\pi} \oint p_i dq_i, \qquad (i = 1, 2, \cdots f), \tag{30}$$

[1] Such an integral was called by Riemann a "*complete*" integral.

the symbol of integration having the same meaning as in (28). These equations can be regarded as f simultaneous relations between the constants $\alpha_1, \alpha_2, \cdots \alpha_f$, on which the momenta p_i are dependent, and can be resolved with respect to these constants. It follows that

$$\alpha_i = \alpha_i(u_1, u_2, \cdots u_f), \qquad (i = 1, 2, \cdots f), \tag{31}$$

where the first of these expressions gives the Hamiltonian function as we shall see immediately.

We now multiply the equations (29) by $\partial\alpha_h/\partial u_k$ respectively and add all these equations

$$\sum_{h=1}^{f} \frac{\partial W}{\partial \alpha_h} \cdot \frac{\partial \alpha_h}{\partial \alpha_k} = \frac{1}{2\pi} \sum_{i=1}^{f} w_i \sum_{i=1}^{f} \omega_{ih} \frac{\partial \alpha_h}{\partial u_k} \cdot \tag{32}$$

The path of integration in the formulæ (28) and (29) can be regarded as a fixed one, independent of the constants α_i, from which results:

$$\omega_{ih} = 2\pi \frac{\partial u_i}{\partial \alpha_h} \cdot$$

The second sum on the right of (32) hence becomes

$$\sum_{h=1}^{f} \frac{\partial u_i}{\partial \alpha_h} \cdot \frac{\partial \alpha_h}{\partial u_k} = \frac{\partial u_i}{\partial u_k},$$

while the left side is reduced to $\partial W/\partial u_k$:

$$\frac{\partial W}{\partial u_k} = \sum_{i=1}^{f} \frac{\partial u_i}{\partial u_k} w_i.$$

The quantities u_i are independent, all derivatives vanish, therefore, with the exception of the case $i = k$, in which the derivative becomes equal to 1. From the sum on the right there remains the single term w_i. We can therefore draw the following conclusion: *if in the expression* (26) *of the Hamiltonian function, the constants α_i are replaced by the new constants u_i by means of equations* (31), *the following relations follow*

$$\begin{cases} W = W(q_1, \cdots q_f; u_1, \cdots u_f), \\ p_i = \dfrac{\partial W}{\partial q_i}, \qquad w_i = \dfrac{\partial W}{\partial u_i} \cdot \end{cases} \tag{33}$$

According to § 2 these equations give the analytical expression of a contact-transformation, arranging the transition from the coördinates u_i, w_i. It results from this, that these latter variables are *canonical coördinates* of the system. We have already pointed out, that the coördinates w_i are linear functions of time, and that the momenta are constants. Lastly, the first of relations (31) shows, that the energy α_1 is expressed by the momenta u_i only, and does not depend on the variables w_i.

§ 7. Computation in the Case of Periodicity.

The carrying through of the calculations given in § 6 generally presents difficulties. In the special case, however, where the second term of the Hamiltonian function is small compared with the first one (which is always the case in the theory of perturbations), it can be carried out by successive approximation. We shall show this first to be true in the case of periodicity. Returning to the notations of § 4, let the original coördinates be u_i, w_i. By contact-transformation (15) we arrive at the system u_i', w_i', while u_i'', w_i'' are the angular variables.

To begin with, we shall show how the momenta u_i' can be expressed by the new ones u_i'' and how the energy α can be calculated in terms of the momenta u_i''. We shall however restrict ourselves to the degree of precision required for the applications of Part II. (dealing with the theory of dispersion).

The cyclic variables w_2', w_3', $\cdots w_f'$ are disposed of in a few words. Since librational motions are excluded (cf. the end of § 4), the integration in (30) goes from 0 to 2π, and the momenta u_i' being constant, this equation immediately yields

$$u_i' = u_i'', \qquad (i = 2, 3, \cdots f). \tag{34}$$

This result remains true for the case of librations dealt with in the next paragraph.

We now turn to the non-cyclic variable w_1' and consider the case in which the corresponding momentum u_1' is periodic. The limits of integration of equation (30) are again 0 and 2π; if we introduce for u_i' expression (24), there follows

$$u_i' = c_0$$

and

$$u_1' = u_1'' + c_1 \cos w_1' + c_2 \cos 2w_1' + \cdots, \tag{35}$$

where the coefficients c_i depend on the constants u_1'', $\cdots u_f''$.

We now make use of our supposition that the quantity b' in the Hamiltonian function (17′) is small compared to H_1'. This means that even H_1', neglecting the second term, gives a certain approximation to the real motion, at least for short times, and that momentum u_1'' differs only by a small quantity of the order b'/H_1' from the momentum u_1' defined by the first term

$$u_1' = u_1'' + \delta, \tag{36}$$

where

$$\delta = c_1 \cos w_1' + c_2 \cos 2w_1' + \cdots. \tag{36′}$$

We suppose functions H_1' and b' in the vicinity of $u_1' = u_1''$ to be developable in powers of the small quantity δ

$$(37)\qquad \begin{cases} H_1'(u_1') = H_1'(u_1'' + \delta) = a_0 + a_1\delta + a_2\delta^2 + \cdots, \\ \;b'(u_1') = \;b'(u_1'' + \delta) = b_1 + b_2\delta + b_3\delta^2 + \cdots, \end{cases}$$

where we put for short

$$(38)\qquad a_n = \frac{1}{n!}\left(\frac{\partial^n H_1'}{\partial u_1'^n}\right)_{u_1'=u_1''}; \qquad b_{n+1} = \frac{1}{n!}\left(\frac{\partial^n b'}{\partial u_1'^n}\right)_{u_1'=u_1''}$$

The indices are so chosen as to indicate the order of magnitude of the term in question.

When inserted in equation (17) this leads to

$$a_0 + a_1\delta + a_2\delta^2 + \cdots + (b_1 + b_2\delta + \cdots)\cos w_1' = \alpha.$$

We substitute series (36′) for δ and arrange the expression according to cosines of the multiples of w_1':

$$\left(a_0 - \alpha + \frac{a_2c_1}{2} + \frac{b_2c_1}{2} + \cdots\right) + (a_1c_1 + b_1 + \cdots)\cos w_1' + \left(\frac{a_2c_1}{2} + \frac{b_2c_1}{2} + a_1c_2 + \cdots\right)\cos 2w_1' + \cdots = 0.$$

This relation being valid for any value of w_1', the coefficients must vanish individually, and this yields an infinite series of equations for the calculation of c_1, c_2, c_3, $\cdots$ and α. Restricting ourselves to terms of the first and second order, we have

$$(39)\qquad c_1 = -\frac{b_1}{a_1}, \qquad c_2 = \frac{b_1}{2a_1^3}(a_1b_2 - a_2b_1),$$

$$(40)\qquad \alpha = a_0 + \frac{b_1}{2a_1^2}(a_2b_1 - a_1b_2) + \cdots.$$

The problem of expressing α and u_1' in terms of u_1'' is thus solved, but there remains that of expressing coördinates w_i' in terms of the angular variables w_i'' and of the corresponding momenta u_i''. According to § 6 we have to apply contact-transformation (33), while Jacobi's function (26) assumes the form

$$W = \int u_1' dw_1' + \sum_{i=2} u_i'' w_i',$$

or by use of formulæ (34) and (36)

$$(41)\qquad W = \sum_{i=1} u_i'' w_i'' + c_1 \sin w_1' + c_2 \sin 2w_1' + \cdots.$$

Contact-transformation (33) yields besides the relations (34) and (36), already known to us, the new ones

$$(42)\quad w_i'' = w_i' + \frac{\partial c_1}{\partial u_i''}\sin w_1' + \frac{1}{2}\frac{\partial c_2}{\partial u_i''}\sin 2w_1' + \cdots \qquad (i=1, 2, \cdots f),$$

whence by inversion (neglecting terms of the second order)

$$(43)\qquad w_i' = w_i'' - \frac{\partial c_1}{\partial u_i''}\sin w_1''.$$

The transformations, hitherto used, are well fitted for studying the character of the motion, but they have the disadvantage of treating the pair of coördinates w_1, u_1 in a different way from the rest. By this an asymmetry is carried into the final formula which is not justified by physical reasons.

For kinematical and dynamical conclusions it obviously does not matter what kind of coördinates are used in describing the motion, but this is not so for the theory of quanta: as soon as quantum conditions are involved an arbitrary procedure may have dangerous consequences.
The asymmetry should therefore be avoided, and this is easily accomplish
The asymmetry should therefore be avoided, and this is easily accomplished by use of a third transformation which is the inverse of (15):

$$(44)\qquad \begin{cases} W = u_1''(m_1 w_1''' + \cdots + m_f w_f''') + \sum_{i=2} u_i'' w_i''', \\ \dfrac{\partial W}{\partial w_i'''} = u_i''', \qquad \dfrac{\partial W}{\partial u_i''} = w_i''. \end{cases}$$

The formulæ for the direct transition from u_i, w_i to u_i''', w_i''' can be drawn from (15), (35), (36), (42) and (44). Using the abbreviations

$$(45)\qquad \begin{aligned} -\vartheta &= m_1 w_1 + m_2 w_2 + \cdots + m_f w_f, \\ -\varphi(u_1''', \cdots u_f'''; \vartheta) &= c_1 \sin\vartheta + \tfrac{1}{2}c_2 \sin 2\vartheta + \tfrac{1}{3}c_3 \sin 3\vartheta + \cdots, \end{aligned}$$

where coefficients c_i are to be regarded as functions of the new constants u_i'''. The formulæ looked for are found to have the perfectly symmetric form

$$(46)\qquad u_i = u_i''' + \frac{\partial\varphi}{\partial w_i}, \qquad w_i''' = w_i + \frac{\partial\varphi}{\partial u_i'''}, \quad (i = 1, 2, \cdots f).$$

These equations form a contact-transformation, which may be written thus

$$(47)\qquad \begin{cases} W = \sum_{i=1}^{f} u_i''' w_i + \varphi(u_1''', \cdots u_f'''; \vartheta), \\ \dfrac{\partial W}{\partial u_i'''} = w_i''', \qquad \dfrac{\partial W}{\partial w_i} = u_i. \end{cases}$$

In fact this transformation was introduced by Poincaré and adopted by Whittaker, the form of the function φ being left provisionally indeterminate. Whittaker's results agree completely with the formulæ of this paragraph, but assumption (45) for the form of function φ is made by him quite arbitrarily, and it does not appear from his treatment whether this assumption is always admissible; whereas from our treatment it follows that this assumption is possible for a limited region of values of the constant α only, but that it is necessary in this region.

Equations (39) and (40) continue to hold also if transformation (44) is applied, a_n and b_n now meaning

$$(48)\qquad \begin{cases} a_n = \dfrac{1}{n!}\left[\left(m_1\dfrac{\partial}{\partial u_1} + \cdots + m_f\dfrac{\partial}{\partial u_f}\right)^n H_1\right]_{u_i=u_i'''}, \\ b_{n+1} = \dfrac{1}{n!}\left[\left(m_1\dfrac{\partial}{\partial u_1} + \cdots m_f\dfrac{\partial}{\partial u_f}\right)^n b\right]_{u_i=u_i'''}. \end{cases}$$

§ 8. Computation in the Case of Libration.

The problem becomes more complicated than in the last paragraph if the variable w_1' performs librational motions. Theoretically it is always possible to represent the integrand of (25) in terms of a single variable, introducing u_1' by means of the equation of energy (17'):

$$w_1' = \arccos\frac{\alpha - H_1'(u_1)}{b'(u_1)},$$

whence

$$\int u_1' dw_1' = \int u_1' d \arccos\frac{\alpha - H_1'(u_1')}{b'(u_1')}.$$

But practically this way turns out to be very tedious. In computations of moderate precision one succeeds much more quickly by using a method explained in the following lines which forms an important part of our communication. But I have had no experience as to which procedure is more convenient in calculations of great accuracy.

To make clear the idea of our method, we shall consider the special form, under which the case of libration appears both in the theory of three bodies and in the theory of dispersion (keeping in mind that the scope of application of this method is a much larger one). In the problems mentioned, librations occur only when the equation of energy (17') assumes the special form

$$(49)\qquad H_1' + 2\beta\sqrt{u_1'}\cdot\cos w_1' = \alpha,$$

where u_1' has a small numerical value and H_1' and β are functions developable in powers of u_1', which may depend on the constants $u_2', \cdots u_f'$ as well.

We make use of the expansion

$$\begin{cases} H_1'(u_1') = a_0 + a_1 u_1' + a_2 u_2'^2 + \cdots, \\ \beta(u_1') = \beta_1 + \beta_2 u_1' + \beta_3 u_1'^2 + \cdots, \end{cases} \tag{50}$$

where β is not equal to zero. It follows from the equation of energy that:

$$(a_0 - \alpha + 2\beta_1 \sqrt{u_1'} \cos w_1' + a_1 u_1') + (2\beta_1 \sqrt{u_1'} \cos w_1' + a_2 u_1')u_1' + \cdots = 0. \tag{51}$$

This series converging more or less rapidly, we can provisionally neglect terms of higher order in u_1 in order to get an approximate value of u_1 by solving the remaining equation, and then build up an expansion of u_1 by taking into account the neglected terms one after another. We have seen in § 5, that, looked at as a function of the complex variable w_1, the momentum u_1 has two branch-points with the exponent 1/2. *The point of our method is to form the expansion in such a way that every individual term of it shall be of this type of branching.*

We easily arrive at such an expansion in the following way: We denote the totality of all terms of relation (51), with the exception of the first parenthesis, by Δ

$$\Delta = (2\beta_2 \sqrt{u_1'} \cos w_1 + a_2 u_1')u_1' + \cdots \tag{52}$$

and thus write that equation in the form

$$a_0 - \alpha + \Delta + 2\beta_1 \sqrt{u_1'} \cos w_1' + a_1 u_1' = 0,$$

which can be formally solved for $\sqrt{u_1}$

$$\sqrt{u_1'} = -\frac{\beta_1}{a_1} \cos w_1' \pm \frac{1}{a_1} \sqrt{\beta_1^2 \cos^2 w_1' + a_1(\alpha - a_0) - a_1\Delta}. \tag{53}$$

If squared, and at the same time expanded in powers of the small quantity Δ, this equation gives

$$u_1' = \frac{1}{a_1^2} \left\{ \Phi + \beta_1^2 \cos^2 w_1' \mp 2\beta_1 \sqrt{\Phi} \cos w_1' - a_1 \left(1 \mp \frac{\beta_1 \cos w_1'}{\sqrt{\Phi}} \right) \Delta \pm \frac{1}{4} \frac{a_1^2 \beta_1 \cos w_1'}{\Phi^{312}} \Delta^2 + \cdots \right\}, \tag{54}$$

with the abbreviation:

$$\Phi = \beta_1^2 \cos^2 w_1' + a_1(\alpha - a_0) = \beta_1^2 + a_1(\alpha - a_0) - \beta_1^2 \sin^2 w_1'. \tag{55}$$

One gets the first approximation by entirely neglecting the quantity Δ

$$u_1' = \frac{1}{a_1^2} \{\Phi + \beta_1^2 \cos^2 w_1' \mp 2\beta_1 \sqrt{\Phi} \cos w_1'\}, \tag{56}$$

$$\sqrt{u_1'} = -\frac{\beta_1}{a_1} \cos w_1' \pm \frac{1}{a_1} \sqrt{\Phi}. \tag{56'}$$

This approximation is sufficient for treating the case given in Communication III. The following point in these equations is remarkable: u_1' being real by its physical meaning, $\sqrt{\Phi}$ cannot become imaginary, and consequently $\sqrt{u_1'}$ always remains real also. It follows from this that u_1' is always positive and never passes through the value $u_1' = 0$.

The *second* approximation results, if the first one is carried into the correction term of the first order with respect to Δ, and if correction terms of higher order are neglected. To obtain the *third*, the second must be carried into the terms of first order and the first into those of second order, and so on. In any case u_1 is represented by a series, expanded in integer powers of $\sqrt{\Phi}$, and this latter function $\sqrt{\Phi}$ is evidently of the type of branching desired.

At every step of approximation the momentum acquires the form

$$u_1' = F(\cos w_1', \sqrt{\Phi}), \tag{57}$$

where F is a *rational* function of the two arguments. Hence, according to (30), momentum u_1'', corresponding to the angular variable results in the form

$$u_1'' = \frac{1}{2\pi} \oint F(\cos w_1', \sqrt{\Phi}) dw_1'. \tag{58}$$

The path of integration in the complex w_1'-plane is here a circuit around the two branch-points A_1, A_2 of the exact function u_1' which do not coincide exactly, but only approximately with those (B_1, B_2) of the square root $\sqrt{\Phi}$ (Fig. 3[1]). A closed curve (c), however, which embraces the first two branch-points, if plotted in a suitable way, encloses also the latter two. We can therefor regard (58) at any degree of approximation as a "complete" integral, encircling the branch-cut of function $\sqrt{\Phi}$, which means a considerable simplification of the considerations.

Equation (58) gives us therefore u_1'' as a function of energy α. By inversion (*e.g.*, using successive approximation) the looked-for dependence of energy on the momentum u_1'' is found. For the purpose of Communication II., the first approximation given by equation (56) is sufficient: the part which is rational with respect to $\cos w_1'$ yields no contribution to the integral, and this reduces to

$$u_1'' = \frac{\beta_1}{\pi a_1^2} \oint \sqrt{\beta_1^2 + a_1(\alpha - a_0) - \beta_1^2 \sin^2 w_1'} \cdot d \sin w_1'. \tag{59}$$

Account is already taken in this expression of the double sign of the square root, because, proceeding along the path of integration, the sign changes automatically at the moment of passing from the lower to the

[1] We recall that in § 5 the accents at the letters were dropped.

upper bank of the branch cut. We shall agree, that u_1'' is positive, which is true, if, β_1 being positive, we attribute to the roots (56) and (56′) the lower sign on the lower bank of the branch cut[1]. The integration is easily accomplished and yields:

$$u_1'' = \frac{\beta_1^2 + a_1(\alpha - a_0)}{a_1^2}, \tag{60}$$

from which the expression of energy looked for is obtained in the form

$$\alpha = a_0 - \frac{\beta_1^2}{a_1} + a_1 u_1''. \tag{61}$$

Without entering into the particulars of the deduction, we will give the second approximation of the energy also:

$$\begin{aligned}\alpha = a_0 - \frac{\beta_1^2}{a_1} - \frac{\beta_1^3}{a_1^4}(2a_1\beta_2 - a_2\beta_1) + a_1 u_1''& \\ - 4\frac{\beta_1}{a_1^4}(a_1\beta_2 - a_2\beta_1)u_1'' + \frac{5a_2}{8a^7}u_1''^2&.\end{aligned} \tag{62}$$

It must be kept in mind that, for the momenta $u_2', u_3', \cdots u_f'$ corresponding to cyclic coördinates, relations (34) remain valid

$$u_i' = u_i'', \qquad (i = 2, 3, \cdots f) \tag{34}$$

and that the quantities a_0, α, β_1 are expressed in terms of the constants $u_2', u_3', \cdots u_f'$. By (61) or by (62) the energy α is therefore given as a function of the momenta $u_1''(i = 1, 2, \cdots f)$.

In order to pass from the variables w_i', u_i' to the angular coördinates w_i'', we make use again of the contact-transformation (33). The function of action

$$W = \int u_1' dw_1' + \sum_{i=2}^{f} u_1' w_1'$$

is now to be expressed in terms of the variables u_i'' and w_i' which yields according to (56), (61) and (34):

$$\begin{aligned}W = \sum_{i=2}^{f}{}_i\, u_i'' w_i' + \frac{\beta_1^2}{a_1}\cos 2w_1'& \\ + \frac{2\beta^1}{a_1^2}\int \sqrt{a_1^2 u_1'' - \beta_1^2 \sin^2 w_1'}\cdot d\sin w_1'&,\end{aligned} \tag{63}$$

$$w_i'' = \frac{\partial W}{\partial u_i''}, \qquad u_i' = \frac{\partial W}{\partial w_i'}. \tag{64}$$

[1] If β_1 is negative, in all following formulæ of this paragraph the sign of β_1 must be changed.

From the first of the equations (64) there follows for $i = 1$

$$w_1'' = w_1' - \arcsin\left(\frac{\beta_1}{a_1\sqrt{u_1''}}\sin w_1'\right),^1 \tag{65}$$

or

$$\frac{\beta_1}{a_1\sqrt{u_1'}}\sin w_1' = -\sin w_1''\cdot\cos w_1' + \cos w_1''\cdot\sin w_1',$$

whence

$$\begin{cases} \sin w_1' = -\dfrac{a_1\sqrt{u_1''}\ \sin w_1''}{\sqrt{\beta_1^2 + a_1^2 u_1'' - 2a_1\beta_1\sqrt{u_1''}\cos w_1''}}, \\ \cos w_1' = \dfrac{\beta_1 - a_1\sqrt{u_1''}\cos w_1''}{\sqrt{\beta_1^2 + a_1^2 u_1'' - 2a_1\beta_1\sqrt{u_1''}\cos w_1''}}. \end{cases} \tag{66}$$

Introducing these expressions into equation (56′) and taking into account our convention with respect to the sign, we have

$$\sqrt{u_1'} = -\frac{1}{a_1}\sqrt{\beta_1^2 + a_1^2 u_1'' - 2a_1\beta_1\sqrt{u_1''}\cos w_1''}. \tag{67}$$

Our relations (66) assume therefore the form

$$\begin{aligned} \sqrt{u_1'}\sin w_1' &= \sqrt{u_1''}\sin w_1'', \\ \sqrt{u_1'}\cos w_1' &= -\frac{\beta_1}{a_1} + \sqrt{u_1''}\cos w_1''. \end{aligned} \tag{68}$$

If β_1 is not positive, but negative, the upper sign of formulæ (56) and (56′) must be chosen. Then in equations (59), (65), (66) the constant β_1 and in (67) the square root also undergo a change of sign, whereas the final formulæ remain unchanged.

By means of the transformation (63), (64) it is also easy to express the variables $w_2', w_3', \cdots w_f'$ in terms of the angular coördinates. This, however, is not necessary for the applications of Communication II.

By the special form of the energy equation (49) coördinate u_1' is really emphasized so that the asymmetry involved in contact-transformation (15) is physically justified. A further transformation of these equations is therefore superfluous in the case of this paragraph.

§ 9. Whittaker's Modification of Delaunay's Method.

In our introduction we have already mentioned an investigation on the theory of perturbations, due to Whittaker. The chief assumption of his method consists in using expansion (45) for function φ of the

[1] We write (−), making use of the abitrariness of sign at $\sqrt{u_2''}$. The sign of $\sqrt{u_1'}$ being negative according to (56′), we shall ascribe to $\sqrt{u_1''}$ also the negative sign.

contact-transformation (47). As we showed in § 7 this amounts to supposing the momentum u_1' to be always developable into the Fourier-series

$$u_1' = c_0 + c_1 \cos w_1' + c_2 \cos 2w_1' + \cdots.$$

Whittaker thus makes the assumption that momentum u_1' is always a periodic function of the variable w_1', whereas we showed in §§ 4, 5, that besides this possibility there exists the other one of librational motion and that in every individual case it depends on the numerical values of the constants whether the one or the other alternative is obtained. We see from these facts, *that Whittaker's method exhausts the problem not totally but only in part*.

That the case of libration is by no means of minor importance we conclude from its occurring both in the problem of three bodies and in the theory of dispersion (cf. Communications II. and IV.). Using Whittaker's procedure of approximation uncritically one always runs the risk of applying expression (40) outside the limits of its validity, for these limits cannot be determined from that procedure, but only from general considerations such as those of our §§ 4, 5. We will discuss the simple instance of the physical pendulum in order to show to what mistakes one is liable to fall a victim.

Let l be the reduced length of the pendulum in its mass, α the energy, g the acceleration of the field of gravitation, and ϑ the angle of displacement measured from the center of oscillation. Then the energy equation is

$$p^2 + \mu \cos \vartheta = C, \tag{69}$$

putting, for short, $\mu = 2m^2gl^3$ and $C = 2ml^2\alpha$. According to (30) the momentum corresponding to the angular coördinate is then

$$u = \frac{1}{2\pi} \oint p\,d\vartheta = \frac{1}{2\pi} \oint \sqrt{C - \mu \cos \vartheta}\, d\vartheta. \tag{70}$$

Even in this simplest case the two paths of integration, mentioned in §§ 4, 5 must be discriminated:

1. $C > \mu$, *i.e.*, the pendulum swings over and performs a rotational motion. This is the *case of periodicity*, for p is then a periodic function of the angle ϑ and the integration must be extended from 0 to 2π. We reduce the elliptic integral (70) to its normal form

$$u = \sqrt{\frac{C+\mu}{\pi}} \int_0^{2\pi} \sqrt{1 - k^2 \sin^2 \frac{\vartheta}{2}}\, \frac{d\vartheta}{2}, \quad k = \sqrt{\frac{2\mu}{C+\mu}}\,.$$

With the assumptions made k is less than unity and the integral by a usual method becomes

$$u = \sqrt{C + \mu}[1 - \tfrac{1}{4}k^2 + \cdots],$$

whence

$$C = u^2 + \frac{\mu^2}{4u^2} + \cdots. \tag{71}$$

2. $C < \mu$, this is the more usual *case of the librational motion* or oscillation of the pendulum. As shown in § 6 the integral must then be taken in the complex plane around the branch cut of the integral. The modulus k is larger than 1, and it is advisable to transform the integral, introducing a new variable by the substitution $k \sin \vartheta/2 = \sin \psi$:

$$u = \frac{C + \mu}{\pi\sqrt{2\mu}}\int_0^{2\pi} \frac{\cos^2 \psi d\psi}{\sqrt{1 - \frac{1}{k^2}\sin^2 y}} = \frac{C + \mu}{\sqrt{2\mu}}\left[1 + \frac{1}{8k^2} + \cdots\right]$$

and

$$C = \sqrt{2\mu}\, u\left(1 - \frac{u}{8\sqrt{2\mu}} + \cdots\right) - \mu. \tag{72}$$

We have therefore in the case $C > \mu$ an expansion in decreasing powers of u^2, in the case $C < \mu$ one in increasing powers of u. It is here quite evident that the case of libration should not be overlooked and that applying the expansion in decreasing powers to the case $C < \mu$ is not permissible. In most problems of the theory of perturbations the circumstances are not so transparent and must be carefully analyzed.

We admit that this case, discussed for illustration because of its simplicity in calculations, does not entirely correspond to the conditions of the perturbation theory: librational motion only occurs when the term $\mu \cos \vartheta$ is no longer small compared with p^2 but has the same order of magnitude. One can therefore scarcely regard the oscillation of the pendulum in the field of gravity as a perturbation of the rotational motion without field. There are however enough cases in the theory of perturbations in which a very small term produces a decisive change of the analytical character of the whole problem. We have seen in § 4 that the derivative $\partial b/\partial u_1$ discriminates between periodicity and libration. Now just in the special case studied in § 8, we have $b = 2\beta\sqrt{u_1}$ and $\partial b/\partial u_1 = \beta/\sqrt{u_1}$ (neglecting small terms). Thus, if u_1 has a very small numerical value, a very small perturbational term may have a numerically large derivative and may exercise considerable effect.

10. Aspects of the Theory of Quanta Opened up by this Method.

In our preceding considerations we tacitly supposed that the undisturbed motion, given by the Hamiltonian function H_1 of equation

(12), does not belong to the class of "*degenerate*" motions. With this supposition we arrived at the conclusion *that in passing from one approximation to the next one the momenta, suitable for quantization according to* (11), *change by small correction terms only.*

We will now drop this restriction and extend our considerations to degenerate systems. The property of such systems which is most important to us[1] is the existence of one or several commensurabilities between the *average motions* (10), having the form

$$k_1\Omega_1 + k_2\Omega_2 + \cdots + k_f\Omega_f = 0, \tag{73}$$

where $k_1, k_2, \cdots k_f$ are whole numbers. The number of existing commensurabilities we shall call the "*degree of degeneration*." The highest degree possible is obviously $f - 1$, only one of the variables w_i remaining independent. This highest degree of degeneration is reached by periodic motions.

Among the terms of perturbation function (12) such may occur which have just the argument

$$k_1w_1 + k_2w_2 + \cdots + k_fw_f,$$

where by k_i are denoted the same numbers as in (73). Such terms can properly be called "*degenerate terms*." The special properties of these degenerate terms appear, when transformation (16) is applied, for the canonical equations yield

$$\frac{\partial H_1}{\partial u_1'} = w_1' = k_1\Omega_1 + \cdots + k_f\Omega_f = 0. \tag{74}$$

Function H_1 is thus altogether independent of u_1' and is therefore a constant, all other arguments $(u_2', u_3', \cdots u_f')$ of this function being constant. We can combine this constant with the energy and write

$$\alpha' = \alpha - H_1, \tag{75}$$

whence (17′) assumes the simpler form

$$b(u_1)\cos w_1 = \alpha'. \tag{76}$$

Now in this equation the term $b(u_1)\cos w_1$ is no longer a small correction term, but the only variable term. This circumstance requires *that a degenerate term produces not a small correction of the special coördinates justified for quantization, but a decisive change of the same.*

For the convenience of the following communications we shall put together the formulæ for the treatment of degenerate terms. The methods used in §§ 7, 8 cannot be applied here, and we will follow the

[1] Cf. P. S. Epstein, Ann. d. Phys., *51*, p. 179, 1916.

procedure briefly mentioned in the beginning of § 8. We obtain

$$dw_1 = -\frac{\frac{1}{b}\frac{\partial b}{\partial u_1}du_1}{\sqrt{b^2 - \alpha'^2}} \tag{77}$$

and according to (30) and (34)

$$u_1' = -\frac{1}{2\pi}\oint\frac{\frac{1}{b}\frac{\partial b}{\partial u_1}du_1}{\sqrt{b^2 - \alpha'^2}}, \qquad u_i' = u_i, \quad (i = 2, 3, \cdots f). \tag{78}$$

By means of these relations α_1 must be determined in terms of the quantities u_i'. The angular coördinates then result from the contact-transformation

$$W = \int u_1 dw_1 + \sum_{i=2}^{f} u_i' w_i$$

$$w_i' = \frac{\partial W}{\partial u_i'}, \qquad u_i = \frac{\partial W}{\partial w_i},$$

where u_1 must be regarded as a function of u_i' and w_1 given by equations (76) and (78). In this manner we obtain

$$\begin{cases} w_1' = \int \frac{\partial u_1}{\partial u_1'} dw_1, \\ w_i' = w_i + \int \frac{\partial u_1}{\partial u_i'} dw_1, \qquad (i = 2, 3, \cdots f). \end{cases} \tag{79}$$

Taking the logarithm of equation (76), we have

$$\log b = \log \alpha' - \log \cos w_1,$$

and differentiating partially with respect to u_i'

$$\frac{1}{b}\frac{\partial b}{\partial u_1}\frac{\partial u_1}{\partial u_i'} + \frac{1}{b}\frac{\partial b}{\partial u_i'} = \frac{1}{\alpha'}\frac{\partial \alpha'}{\partial u_i'}.$$

In particular: $\partial b/\partial u_1' = 0$, from which

$$\begin{cases} \frac{\partial u_1}{\partial u_1'} = \frac{b}{\frac{\partial b}{\partial u_1}} \frac{1}{\alpha'}\frac{\partial \alpha'}{\partial u_1'}, \\ \frac{\partial u_1}{\partial u_i'} = \frac{b}{\frac{\partial b}{\partial u_i}}\left(\frac{1}{\alpha'}\frac{\partial \alpha'}{\partial u_1'} - \frac{1}{b}\frac{\partial b}{\partial u_i'}\right), \qquad (i = 2, 3, \cdots f). \end{cases} \tag{80}$$

Finally:

$$
(81)\qquad \begin{cases} w_1' = \dfrac{1}{\alpha'}\dfrac{\partial\alpha'}{\partial u_1'}\displaystyle\int \dfrac{du_1}{\sqrt{b^2-\alpha'^2}}, \\[2ex] w_i' = w_i + \dfrac{\dfrac{\partial\alpha'}{\partial u_i}}{\dfrac{\partial\alpha'}{\partial u_1'}} w_1' + \displaystyle\int \dfrac{\dfrac{1}{b}\dfrac{\partial b}{\partial u_i'}}{\sqrt{b^2-\alpha'^2}}\,du_1. \end{cases}
$$

From the above it appears that the degenerate terms are the most important for determining the coördinates of quantization. It is therefore desirable to have a method of separating them from the perturbation function when the Fourier-expansion of the same is not yet found. To such a separation the following simple consideration leads: Let the perturbation function of equation (8) be formally expressed in terms of the angular coördinates u_i, w_i of the first intermediate motion. Ascribe to the variables u_i, w_i that dependence on time which they possess in the first intermediate motion, *i.e.*: $u_i = \text{const.}$, $w_i = \Omega_i t + \delta_i$. If you form the time average of R_1 for an infinitely long time, you obtain besides the constant term of R_1, independent of the variables w_i, just the sum of the degenerate terms which also are constant, the time dropping out from them according to relation (73).

In this way we write

$$
Q = \bar{R}_1 = \lim_{T=\infty}\frac{1}{T}\int_0^T R_1 dt.
$$

The time average being independent of the choice of coördinates, we can express Q by any other set of coördinates instead of the angular variables provided that we ascribe to them that dependence on time which they have in the first intermediate motion. In particular the number of coördinates of a degenerate conditionally periodic motion can always be reduced by the degree of degeneration and there can be found a special system of $f - s$ separation variables.[1] Let these special coordinates be denoted by $q_1, q_2, \cdots q_{f-s}$, and the corresponding canonical momenta by $p_1, p_2, \cdots p_{f-s}$, so that $p_i = p_i(q_i;\ \alpha_1, \alpha_2, \cdots \alpha_{f-s})$ where α_i denote the constants of integration. Then the time average (82), according to a theorem due to *Burgers*,[2] can be expressed in the following form, convenient for computation

$$
(83)\qquad Q = \bar{R}_1 = \frac{1}{\Delta}\oint\oint\cdots\oint R_1 F dq_1 dq_2 \cdots dq_{f-s},
$$

[1] P. S. Epstein, l.c., p. 179.

[2] J. M. Burgers, Verslagen Amsterdam, 1917.

where F and Δ are the two determinants

$$(84)\qquad \begin{cases} F = \left| \dfrac{\partial p_i}{\partial \alpha_h} \right|, \\ \Delta = \left| \oint \dfrac{\partial p_i}{\partial \alpha_h} dq_i \right|. \quad (i, h = 1, 2, \cdots f - s). \end{cases}$$

When the highest degree of degeneration appears the undisturbed motion is a periodic one, and the average for a long time can be replaced by the average for a period. In this case our function Q is identical with Bohr's function Ψ by which the choice of coördinates is determined according to his rules. *Though Bohr's rules are derived from an entirely different point of view, they agree substantially with ours.*[1]

Also in the other cases discussed by Bohr the directions given by him are closely similar to ours. If the undisturbed motion is *non-degenerate*, the effect of a *very small* perturbation depends essentially on the aperiodic term (*i.e.*, the constant term, independent of $w_1, \cdots w_f$); for according to formulæ (40) and (61) all other terms yield changes of energy, containing the small quantity b_1 (resp. β_1) quadratically. This aperiodic term, if quantities of the second order are neglected, must be expressed by the original coördinates of the undisturbed motion so that the changes in the motion produced by the perturbation need not be investigated at all in this case.

In motions *entirely or partially degenerate* the degenerate terms must be discarded one after another in the way described. If we denote by s the degree of degeneration, the problem is reduced to a non-degenerate one in s steps.[2] Therefore every motion to which the general theory of this paper is applicable is in principle liable to rigorous quantization; and in this consists the chief difference between our view and that of Bohr, which makes us expect rigorous quantization to be impossible in most cases, and therefore, spectral lines to become diffuse. It is, however, to be pointed out that the convergence conditions of the procedure become extremely unfavorable if several degenerate terms exist; so that the determination of quantization coördinates may become impracticable, owing to difficulties of computation. But in any case much is already

[1] We believe that in the text of this paragraph general and unambiguous directions as to how such terms are to be treated are for the first time given. Bohr's (l.c., p. 55) assertion that the integral must be taken between the limits θ and 2π of his variables βi appears to be a mistake, for generally βi has not the dimension of an angle. In discussing instances Bohr and Kramers make use of special artificial methods, applicable to the cases in question only.

[2] The case may occur that the disturbed motion is also degenerate, having a degree of degeneration S ($< s$). Then the reduction is accomplished in $s - S$ steps. This involves no change in our conclusions, a degenerate system being formally reducible to a non-degenerate one of fewer degrees of freedom.

gained if a method is known by which the treatment can be attempted generally, and carried through in part of the problems. In Communication III. we shall discuss an instance of practical importance for this case: the combined effect of magnetic and electric fields on a hydrogen-like atom.[1]

It is an important question as to whether the resulting coördinates are independent of the arbitrariness, lying in the unrestricted choice of the first intermediate motion and of the order in which the degenerate terms are taken into account. As a matter of fact by a rather simple consideration this independence can at least be made very probable. We shall, however, return to this problem at a later opportunity, and will only mention here that in most of the applications the first intermediate function has a physical meaning: it represents the original or normal motion of the system, while the perturbation function gives the effect of some external source of disturbance. By this the arbitrariness is practically removed. It is moreover obvious that the order of treatment is of no consequence if a greater number of operations is carried through than the number giving the degree of degeneration; the system is no more degenerate at this step of approximation and has therefore one set of separation variables only.

As mentioned in the introduction, the above considerations were worked out in 1917. The lack of agreement with experiment in the case of the helium spectrum caused me however to reject the whole theory.[2] But when Bohr in his above mentioned papers established nearly the same quantization rules which follow from my method, my confidence in it was revived. The scope of this method seems however to be confined to motions of *a single electron in a stationary field.*

I should not like to conclude without expressing my sincere thanks to Mr. I. S. Bowen, who kindly looked through the manuscript of this paper, correcting and smoothing my English style.

CALIFORNIA INSTITUTE OF TECHNOLOGY, September, 1921.

[1] Bohr declared this system to be non-quantitable, l.c., p. 93, 94; Abhandlungen über Atombau, p. XVII, Braunschweig, 1294, if the directions of the two fields enclose a finite angle. But we shall see that it can be rigorously quantized from the point of view of our method.

[2] Cf. P. S. Epstein, Die Naturwissenschaften, p. 230, 1918.