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THEORY OF LONGITUDINAL VIBRATIONS OF VISCOUS RODS.

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SYNOPSIS.

Theory of Longitudinal Vibrations of Thin Rods, Taking into Account Damping.—Starting with the general differential equation of wave-motion in one dimension including a viscosity term, expressions are derived for the wave-velocity and logarithmic decrement in the case of *free vibrations* of the rod. The velocity is practically the same as for undamped vibrations, while the decrement is proportional to the viscosity and to the frequency. The equation is also solved for the case of *forced vibrations due to two simple harmonic forces at the ends,* equal in amplitude but opposite in phase. If the damping is small and frequency, $\omega/2\pi$, is near the resonance frequency, $\omega_0/2\pi$, the expression for the displacement of the end of the rod is very simple: $\xi = (4X_0l/\pi G\delta)\cos\theta\sin(\omega t - \theta)$, where $\tan\theta = -2\pi(\omega_0 - \omega)/\omega_0\delta$, δ is the logarithmic decrement per period, l the length of the rod, X_0 the maximum value of the periodic stress, and G Young's modulus. It is shown that this expression may also be obtained by reducing the rod to an equivalent system possessing one degree of freedom.

§ 1. *General Equations of Wave-Motion in Rods.*—Our starting point is the following equation, which is exactly analogous to that for plane waves in an extended medium:[1]

$$\frac{\partial^2\xi}{\partial t^2} = P\frac{\partial^2\xi}{\partial x^2} + Q\frac{\partial^3\xi}{\partial x^2\partial t}\cdot \tag{1}$$

ξ is the displacement, at the time t, of that cross-section of the rod whose undisturbed coördinate is x. P is defined by the equation $P = G/\rho$, where G is Young's modulus and ρ the density. P is therefore the square of the wave-velocity in the absence of damping. As long as lateral effects can be ignored, it does not matter whether the material is isotropic or not: G is in any case the modulus along the axis of the rod. For brevity, we call Q the "viscosity," and treat it as a constant of the material, implying thereby that it is independent of the frequency. Its

[1] See, for example, Lamb's Dynamical Theory of Sound, 1910, Chap. VI.; or Lamb's "Hydrodynamics," 1916, Chap. XI.

possible dependence upon frequency can be tested by experiment. We shall leave aside all consideration of the mechanism by which the energy is dissipated in the material, and assume nothing further than that the frictional force is proportional to the rate of deformation. The dimensions of Q are $[L^2T^{-1}]$.

Since in this paragraph we are concerned only with damped sine-waves of any length, we may write the solution of (1) thus:

$$\xi = A\epsilon^{-jkx-bt}, \tag{2}$$

where

$$b = \eta + j\omega \tag{2a}$$

and A is a quantity dependent upon boundary conditions. k is the wave-length constant, η the attenuation-constant (that is, attenuation with time: attenuation in space need not be considered). $\omega = 2\pi f$ is the frequency-constant, or angular velocity.

On substituting (2) and (2a) in (1), and equating real and imaginary parts, we derive the following relations. For the attenuation-constant, we find

$$\eta = \frac{Qk^2}{2}. \tag{3}$$

The wave-velocity is

$$c = \frac{\omega}{k} = \pm\sqrt{P - \frac{Q^2k^2}{4}}. \tag{4}$$

On substituting this value in (2), we find for the displacement at any point

$$\xi = A\epsilon^{-Q^2k^2t/2} \cos(kx \pm \omega t). \tag{5}$$

From (4) and (5) it follows that

$$k = \frac{2\pi}{\lambda}, \tag{6}$$

where $\lambda = c/f$ represents the wave-length.

From (3), (5), and (6), the logarithmic decrement per period is found to be

$$\delta = \frac{\eta}{f} = \frac{Qk^2}{2f} = \frac{\pi\omega Q}{c^2}. \tag{7}$$

In order that a system of stationary waves may exist, the length l of the rod must be equal to an integral number of half wave-lengths. If this number is even, the center of the rod must also be free. Hence the wave-length constant for the fundamental or any harmonic is, by eq. (6), $k = \pi m/l$, in which m is any positive integer. The value for $m = 1$

gives the fundamental. From (4) and (7) it is evident that the damping increases, while the wave-velocity decreases, with increasing order of harmonics. For all practical purposes the change in velocity may be neglected; hence (always ignoring lateral effects) to a very close degree of approximation the harmonics have frequencies 2, 3, 4 $\cdots$ times the fundamental frequency.

§ 2. *Forced Vibrations.*—We will first solve the problem for the motion of a rod whose center is fixed, and at whose opposite ends two longitudinal simple-harmonic forces of like amplitude but opposing phases are applied; for the present paper originated in connection with a study of high-frequency vibrations of piezo-electric crystals and mathematically, the problem stated is identical with the problem of piezo-electric excitation, in which an impressed alternating electric field produces an alternating longitudinal mechanical stress, which at any instant is uniform throughout the rod, and numerically equal to the fictitious stress at either end. The identity of the two problems follows from the fact that the terminal conditions are the same, being expressed by eq. (11) below.

Assuming throughout that a steady state of vibration has been reached, so that the decrement η/f is compensated by an equal and opposite *increment*, we write the solution of (1) in the form

$$\xi = A\epsilon^{-j\omega t}, \tag{8}$$

in which A is a complex function of x, involving both the amplitude of ξ and the phase-difference between ξ and the impressed forces.

After the usual differentiations and substitutions, we find

$$\frac{\partial^2 A}{\partial x^2} = \gamma^2 A, \tag{9}$$

in which

$$\gamma^2 = \frac{\omega^2}{-P + j\omega Q}\cdot \tag{10}$$

We take the origin at the center of the rod, so that the latter extends from $-l/2$ to $+l/2$. Let the impressed periodic stress at the ends of the rod have the form $X = X_0 \cos \omega t$, or, in exponential form, $X = X_0\epsilon^{-j\omega t}$ (dynes per cm.²). Then at the end, where $x = l/2$,

$$G\frac{\partial \xi}{\partial x} = X_0\epsilon^{-j\omega t}, \tag{11}$$

where G is Young's modulus as before.

Equation (9) is now integrated, the constants being determined from (11) and the fact that, when $x = 0$, $\xi = 0$ and $A = 0$. In order to save space, we omit the steps of the solution, which is most conveniently expressed in the form[1]

$$A = \frac{X_0}{G\gamma} \cdot \frac{\sinh \gamma x}{\cosh \frac{\gamma l}{2}} \cdot \tag{12}$$

Since, by eq. (10), γ is a function of the fundamental constants, it is evident that from (12) the amplitude and phase of the motion at any point along the rod can be derived.

From here on we shall be concerned only with the motion at the ends of the rod. The value assumed at the ends by A, which we will call A_0, is obtained by setting $x = l/2$ in eq. (12):

$$A_0 = \frac{X_0}{G\gamma} \tanh \frac{\gamma l}{2} \cdot \tag{13}$$

§ 3. In order to apply the last equation to actual cases, and in particular to use it for the determination of Q, it is necessary to reduce it to a more workable form. Upon reduction of (10), having regard to (3), (4), and (6), we find that, as long as Q is small,

$$\gamma = -\frac{\eta}{c} + jk = \frac{1}{c}(-\eta + j\omega). \tag{14}$$

This equation holds to a high degree of precision, even if the logarithmic decrement is as large as 0.1.

On substituting this value of γ in (13), and making obvious reductions and approximations, we arrive at the following expression for A_0, which is very accurate as long as Q is small enough to be ignored in (4) and n is small in comparison with ω:

$$A_0 = \frac{X_0}{Gk} \cdot \frac{4}{\frac{4\pi^2 n^2}{\omega_0^2} + \delta^2} \left(\frac{2\pi n}{\omega_0} + j\delta \right) = k_1 + jk_2, \tag{15}$$

in which the real coefficients k_1 and k_2 are written for brevity. ω_0 is the angular velocity at resonance (see footnote under § 4), and n denotes the difference $\omega_0 - \omega$, $\omega/2\pi$ being any frequency not far from resonance; n may therefore be regarded as a measure of the *dissonance* corresponding to any frequency.

From (15), together with (8), we readily find for the displacement at the end of the rod at any time t,

[1] In the abstract of this paper which appeared in the PHYSICAL REVIEW, Vol. 15, p. 146, 1920, the factor sinh γx was erroneously printed as sin γx.

$$\xi = \xi_0 \cos (\omega t - \varphi). \tag{16}$$

Here

$$\xi_0^2 = k_1^2 + k_2^2,$$

and

$$\tan \varphi = k_2/k_1 = \frac{\omega_0 \delta}{2\pi n} \cdot \tag{17}$$

The phase-angle φ is 90° at resonance, and passes through most of the range from 0° to 180° in the region close to resonance.

§ 4. *Equivalent System with One Degree of Freedom.*—The fact that, with stationary waves, all parts of the rod agree in phase, suggests that a simple method of analysis may be reached by substituting for the actual rod an equivalent system possessing only one degree of freedom. We will therefore consider only the fundamental frequency.

The transition is easily made. In accordance with well-known principles[1] the equivalent mass M is found to be equal to half the actual mass of the rod, or $M = \frac{1}{2}\rho lbe$, these symbols representing density, length, breadth, and thickness respectively.

In place of Young's modulus we use the coefficient of stability, or "equivalent stiffness" g, which, close to the resonant frequency, and when the damping is small, is expressed as

$$g = M\omega_0^2 = \frac{\pi^2 beG}{2l} \cdot$$

This equation follows from the resonance relation $\omega_0 = 2\pi f_0$, and the fact that $c = \sqrt{G/\rho} = 2f_0 l$. M and g correspond to L and $1/C$ in an electric circuit having concentrated, as contrasted with distributed, constants.[2]

The equation of motion is

$$M\frac{d^2x}{dt^2} + N\frac{dx}{dt} + gx = F_0 \cos \omega t. \tag{18}$$

The relation between x, the equivalent displacement, and the actual displacement ξ of the end of the rod, is given below. F_0 is the amplitude of the equivalent impressed force. The resistance factor, N, bears to the viscosity Q the relation $N = \pi^2 \rho beQ/2l$. This is proved by equating the decrement $N/2f_0 M$ with the value given in eq. (7).

[1] See, for example, Lamb's Dynamical Theory of Sound, 1910, p. 13.

[2] Strictly, ω_0 is the angular velocity when the amplitude of the *velocity* dx/dt of the equivalent mass M is a maximum under forced vibrations. The maximum amplitude of equivalent *displacement* x comes (under forced vibrations) at the angular velocity $\sqrt{(g/M) - (N^2/2M^2)}$, while the angular velocity of free vibrations is $\sqrt{(g/M) - (N^2/4M^2)}$. The distinction between these three values may under ordinary circumstances be ignored.

The steady-state solution of (18) is

$$x = x_0 \sin(\omega t - \theta), \tag{19}$$

in which the maximum displacement is

$$x_0 = \frac{F_0}{\omega\sqrt{N^2 + \left(\omega M - \frac{g}{\omega}\right)^2}} = \frac{F_0}{\omega N}\cos\theta, \tag{20}$$

and in which

$$\tan\theta = \frac{\omega M - (g/\omega)}{N} = -\frac{2\pi n}{\omega_0 \delta} \tag{21}$$

approximately.

Equation (19) expresses the motion of the rod on what may be termed the "concentrated mass" method, in distinction from the "distributed mass" method first considered. It is easy to show that, near resonance, x and ξ agree in phase (cf. eqs. (17) and (21)). In order to make the amplitudes of x and ξ identical, the force F must be suitably expressed in terms of the impressed stress X_0. To this end, we consider the expression for ξ_0 in (17), and making use of eqs. (7) and (15) and the expressions for P, k, and N, we find that at resonance $\xi_0 = 2X_0 be/\omega_0 N$. Upon equating this with the expression for maximum x_0 at resonance from eq. (20), namely $x_0 = F_0/\omega_0 N$, we see that F_0 must have the form

$$F_0 = 2X_0 be. \tag{22}$$

This establishes the validity of the method of "concentrated mass," for all cases in which the damping and the range in frequency are both small enough for the expression for g to be satisfied to the desired degree of precision. Equations (19) to (21) are, under the restrictions just named, as accurate as those under the more general theory, and are much more convenient. Their application in the solution of problems with piezo-electric quartz rods, and in particular their use in determining the value of the coefficient of viscosity, must be reserved until later.

Finally, the following simple expression for the displacement of the end of the rod, in terms of the fundamental constants, may be derived from equations (19), (20) and (22), together with the expression for N in terms of Q:

$$\xi = \frac{4X_0 l}{\pi G \delta}\cos\theta \sin(\omega t - \theta).$$

It is easily verified that this equation also follows from equations (16) and (17) according to the more rigorous method of "distributed mass."

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