Theory of a Two-Dimensional Ising Model with Random Impurities. II. Spin Correlation Functions

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We continue our investigation of an Ising model with immobile random impurities by studying the spin-spin correlation functions. These correlations are not probability-1 objects and have a probability distribution. When the random bonds have the particular distribution function studied in the first paper of this series, we demonstrate that the average value and the second moment of the temperature derivatives of these correlations are infinitely differentiable but fail to be analytic at T_c, the temperature at which the observable specific heat fails to be analytic. When $T<T_c$, we consider $S_\infty(l)=\lim_{m\to\infty}\langle\sigma_{0,0}\sigma_{l,m}\rangle$. This limit is not independent of l. In the special case that the random bonds are symmetrically distributed about the lth row, the geometric mean of $S_\infty(l)$ is computed and shown to vanish exponentially rapidly when $T\to T_c-$. We contrast this with a lower bound that shows that the spontaneous magnetization can vanish no more rapidly than T_c-T, and present a description of how the local magnetization $S_\infty(l)^{\frac{1}{2}}$ behaves as $T\to T_c-$.

1. INTRODUCTION

In the first paper of this series[1] we study the effect of immobile random impurities on a magnetic phase transition by constructing a modification of the two-dimensional Ising model in which all vertical bonds $E_2(j)$ connecting the jth row to the $(j+1)$th row are the same, but $E_2(j)$ is allowed to vary randomly from row to row with a probability density $P(E_2)$. We explain in that paper the connection of this modification of the experimental situation,[2] set up a general formalism for computing the free energy, and, finally, for a particular narrow $P(E_2)$ of width N^{-1}, compute the terms in the specific heat that do not vanish as $N\to\infty$.

In 1944, Onsager[3] computed the free energy of a "pure" two-dimensional Ising model. The next properties of this lattice to be investigated were the spin-spin correlation functions which were studied by Kaufman and Onsager[4] in 1949. In this paper, we continue to follow this historical order of development and study the spin-spin correlation functions for our random Ising model.

Our model of random impurities is in reality a collection of models, each of which has a certain probability attached to it. This collection possesses a well-defined set of thermodynamic properties because, as shown in I, the free energies and hence the specific heats of each lattice in the collection possess with probability 1, the same thermodynamic limit. This basically results from the fact that the free energy is a property of the lattice as a whole and does not depend on the detailed arrangement of bonds $E_2(j)$ in any particular lattice. The spin-spin correlation functions, however, are quite different from the free energy in that they do depend on the detailed arrangement of bonds $E_2(j)$ relative to the locations of the two spins which are being correlated and therefore are not probability-1 objects. Thus, in contrast with Onsager's lattice, it is necessary, if one wants to characterize these spin-spin correlations completely, to compute not only their average value but also their probability distribution. The computation of these probability distributions is quite involved, however, and for the purpose of obtaining explicit results we will restrict our attention to the average value and the second moment. In other words, we carry out the program outlined in Sec. 5 D of I.

Since there are many features of the correlation functions of our random Ising model that may be contrasted with the correlation functions of Onsager's lattice, it seems appropriate to outline the comparisons to be made before carrying out the detailed calculations. As in I, we have obtained explicit results only for the particular distribution function

$$P(E_2)\frac{dE_2}{d\lambda}=\mu(\lambda)=N\lambda_0^{-N}\lambda^{N-1},\quad 0\le\lambda\le\lambda_0$$
$$=0,\qquad \text{otherwise}\quad(1.1)$$

where

$$\lambda=\tanh^2(E_2/kT),\qquad(1.2)$$

k is the Boltzmann's constant, and N is large. In this case the spin-spin correlation at fixed separation never deviates to order 1 from the corresponding correlation function for Onsager's lattice. This is a consequence of the fact that the result of Kaufman and Onsager[4] is a continuous function of T. Therefore we concentrate not on the spin-spin correlations themselves, but on their first temperature derivative. For Onsager's lattice these derivatives are known[4] to diverge when $T\to T_c$ as

[1] B. M. McCoy and T. T. Wu, Phys. Rev. 176, 631 (1968). This paper will henceforth be referred to as I.
[2] See also B. M. McCoy and T. T. Wu, Phys. Rev. Letters 21, 549 (1968).
[3] L. Onsager, Phys. Rev. 65, 117 (1944).
[4] B. Kaufman and L. Onsager, Phys. Rev. 76, 1244 (1949).

$\ln|T-T_c|$. In the random lattice specified by (1.1), the average values of these derivatives differ to order 1 from their Onsager values only if $T-T_c=O(N^{-2})$, where T_c is the temperature found in I where the specific heat fails to be analytic. For nearest neighbors these average values are given by (3.38), an expression which is infinitely differentiable but not analytic at $T=T_c$. A similar behavior is shown to obtain for other separations besides nearest neighbors.

The variance of $(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle$ is studied in Sec. 5. It is seen in (5.4) that this variance is of order 1, whereas (3.38) shows that the average values contain a term proportional to $\ln N^2$. We conclude Sec. 5 by demonstrating that the variance of $(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle$ is also infinitely differentiable but not analytic at $T=T_c$. We also show that the singularities of the second moment and of the square of the average value are different.

Perhaps the most interesting feature of the spin-spin correlation functions of Onsager's lattice is not the singularity at $T=T_c$ of $\langle\sigma_{0,0}\sigma_{l,m}\rangle$ for l and m fixed, but rather the behavior when T is fixed and the separation between the spins tends to infinity. When $T<T_c$ in Onsager's lattice,

$$\lim_{l^2+m^2\to\infty}\langle\sigma_{0,0}\sigma_{l,m}\rangle=M^2\,, \tag{1.3}$$

where M is the spontaneous magnetization. We investigate such limits in Sec. 6 and show that in sharp contrast with Onsager's lattice the correlation functions for any lattice in our collection will *not*, in general, approach a limit when $T<T_c$ and the separation tends to infinity. For the particular case of $\langle\sigma_{l,0}\sigma_{l,m}\rangle=S_m(l)$, a limit will exist when $m\to\infty$ but for any given lattice of our collection the value of this limit depends on l; in other words, S_∞ is not a probability-1 object. For technical reasons, this lack of a probability-1 limit prevents us from determining the spontaneous magnetization. We are, however, able to compute the average of $\ln S_\infty(l)$ in the subset of lattices which obeys the additional symmetry restriction

$$E_2(l+j)=E_2(l-j-1)\,. \tag{1.4}$$

In such a symmetrical row we show that as $T\to T_c-$,

$$\langle\ln S_\infty(l)\rangle_{E_2}=-\tfrac{1}{4}\ln N^2+N^{-2}(T-T_c)^{-1}C_1^{-1}\ln 2+C_2\ln[(T_c-T)N^2]+O(1)\,, \tag{1.5}$$

where C_1 is given by (3.1) and C_2 by (6.32) and $\langle\cdots\rangle_{E_2}$ denotes the average over all sets $\{E_2\}$. This result implies that the geometric mean of $S_\infty(l)$ vanishes exponentially rapidly as $T\to T_c-$, and we speculate that if we allow more randomness by totally violating (1.4) this geometric mean cannot vanish less rapidly. However, it must not be inferred from (1.5), that the spontaneous magnetization vanishes exponentially rapidly as $T\to T_c-$, for in the following paper[5] we demonstrate that the spontaneous magnetization is bounded below by

$$\text{const}\,N(T_c-T)\quad\text{as}\quad T\to T_c-\,. \tag{1.6}$$

Therefore, although the specific heat computed in I has an infinitely differentiable essential singularity at T_c, not all physical quantities behave so smoothly. This variety of singularities in physical quantities near T_c is further explored in the next paper of this series, where we use the methods of this paper to study the boundary magnetization and boundary spin correlation functions of a half-plane random Ising lattice. We are then able to obtain several more lower bounds like (1.6) which are extremely interesting because they imply that much of the usual "critical exponent" description of critical phenomena does not apply to our model.

The calculations needed to make precise the results just outlined are rather lengthy. For ease of reference, we have developed all the general formalism needed for the entire paper in Sec. 2, and we suggest the reader consult this formalism only as it is actually applied in the later sections. The remainder of the paper is devoted exclusively to the case (1.1). In Sec. 3 we study the average value of $(d/dT)\langle\sigma_{0,0}\sigma_{l,m}\rangle$ by combining the results of Sec. 2 and those of I. Section 4 and Appendix A are devoted to a rather lengthy and intricate analysis of a two-dimensional integral equation derived in Sec. 2. We advise the reader to omit this analysis until he sees how the final results are used in Secs. 5 and 6. We make this suggestion because approximations (4.70), (4.83), and (4.87), which we actually use in the sequel, are much simpler than the more refined analysis that is needed to justify them. Section 5 is then devoted to the study of the variance of $(d/dT)\langle\sigma_{0,0}\sigma_{l,m}\rangle$. We conclude in Sec. 6 with the computation leading to (1.5), and several speculations that lead to a qualitative picture of the behavior of $S_\infty(l)^{1/2}$ as $T\to T_c-$.

2. FORMULATION OF PROBLEM

The Hamiltonian of our Ising model of $2\mathfrak{M}$ columns and $2\mathfrak{M}+1$ rows is

$$\mathcal{E}=-E_1\sum_{j=-\mathfrak{M}}^{\mathfrak{M}}\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{j,k}\sigma_{j,k+1}-\sum_{j=-\mathfrak{M}}^{\mathfrak{M}-1}E_2(j)\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{j,k}\sigma_{j+1,k}\,, \tag{2.1}$$

where $\sigma_{j,k}=\pm1$ and j labels the row and k the column of a lattice site. We apply cyclic boundary conditions in the horizontal direction by identifying $k=\mathfrak{N}+1$ with $k=-\mathfrak{N}+1$, but we do not connect row $\mathfrak{M}$ with row $-\mathfrak{M}$.[6] We begin our study of spin-spin correlation functions by remarking that the cal-

[5] B. M. McCoy, following paper, Phys. Rev. 188, 1014 (1969).

[6] In I we numbered the rows $1\leq j\leq\mathfrak{M}$. The slight change in the present presentation is made for convenience in computing the correlation functions.

culation of Ising spin-spin correlation functions in terms of appropriate determinants given by Montroll, Potts, and Ward[7] may be applied to the system described by (2.1) for any set of energies $\{E_2\}$. Furthermore, because of the boundary conditions imposed, we may use the methods of IV [8] to calculate the elements of these determinants. We will sketch these developments in Sec. 2 A to establish a notation. For a thorough explanation of the techniques we refer the reader to these papers. In Sec. 2 B we combine these results with those of I to compute the average over the set of energies $\{E_2\}$ of the nearest-neighbor correlations $\langle\sigma_{0,0}\sigma_{1,0}\rangle$ and $\langle\sigma_{0,0}\sigma_{0,1}\rangle$. In Sec. 2 C we derive the more complicated expressions that are needed to study average spin-spin correlations other than nearest neighbor. Finally, in Sec. 2 D we study the probabilistic nature of these correlation functions by computing the second moments of $\langle\sigma_{0,0}\sigma_{1,0}\rangle$ and $\langle\sigma_{0,0}\sigma_{0,1}\rangle$, and of $(d/dT)\times\langle\sigma_{0,0}\sigma_{1,0}\rangle$ and $(d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle$.

A. General Correlation Functions

The work of Ref. 7 may be directly applied to any of our lattices to show

$$\langle\sigma_{0,0}\sigma_{l,m}\rangle=\pm z_1{}^m\prod_{j=0}^{l-1}z_2(j)Pf(y^{-1}+Q)Pf(y)\,, \quad (2.2)$$

where

$$z_1=\tanh\beta E_1\,,\quad z_2(j)=\tanh\beta E_2(j)\,, \quad (2.3)$$

with $\beta=(kT)^{-1}$. The matrix y is the nonsingular submatrix of the matrix $\delta=A'-A$, where A is the antisymmetric $8\mathfrak{M}(2\mathfrak{M}+1)\times 8\mathfrak{M}(2\mathfrak{M}+1)$ matrix whose Pfaffian is the generating function for polygon configurations drawn on the lattice of Fig. 1. The Pfaffian of the matrix A' is the corresponding generating function of the lattice obtained by drawing a path from the site (0,0) to the site (l,m) and replacing every bond z_i on that path by z_i^{-1}. Finally, Q is the submatrix of A^{-1} in the subspace determined by y.

To compute the matrix Q we recall from IV that A may be explicitly written as

$$A(j,k;\,j,k)=\begin{array}{c} \\ R\\ L\\ U\\ D\end{array}\!\!\begin{array}{c}\begin{array}{cccc}R & L & U & D\end{array}\\ \begin{bmatrix}0 & 1 & -1 & -1\\ -1 & 0 & 1 & -1\\ 1 & -1 & 0 & 1\\ 1 & 1 & -1 & 1\end{bmatrix}\end{array} \quad (2.4a)$$

for $-\mathfrak{M}\leq j\leq\mathfrak{M}$ and $-\mathfrak{N}+1\leq k\leq\mathfrak{N}$,

$$A(j,k;\,j,k+1)=-A^T(j,k+1;\,j,k)$$
$$=\begin{bmatrix}0 & z_1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \quad (2.4b)$$

for $-\mathfrak{M}\leq j\leq\mathfrak{M}$ and $-\mathfrak{N}+1\leq k\leq\mathfrak{N}-1$,

$$A(j,k;\,j+1,k)=-A^T(j+1,k;\,j,k)$$
$$=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & z_2(j)\\ 0 & 0 & 0 & 0\end{bmatrix} \quad (2.4c)$$

for $-\mathfrak{M}\leq j\leq\mathfrak{M}-1$ and $-\mathfrak{N}+1\leq k\leq\mathfrak{N}$, and

$$A(j,\mathfrak{N};\,j,-\mathfrak{N}+1)=-A^T(j,-\mathfrak{N}+1;\,j,\mathfrak{N})$$
$$=-A(j,0;\,j,1) \quad (2.4d)$$

for $-\mathfrak{M}\leq j\leq\mathfrak{M}$. All other matrix elements of A are zero [compare (2.6) of IV].

Because our lattice is translationally invariant in the horizontal direction, we may follow IV to find [analogous to (7.1) of IV]

$$A^{-1}(j,k;\,j',k')=(2\mathfrak{N})^{-1}\sum_\theta e^{i\theta(k-k')}[B^{-1}(\theta)]_{jj'}\,, \quad (2.5)$$

where the sum is over

$$\theta=\pi(2n-1)/2\mathfrak{N},\quad n=1,2,\cdots,2\mathfrak{N}$$

and $B(\theta)$ is a $4(2\mathfrak{M}+1)\times 4(2\mathfrak{M}+1)$ skew-Hermitian matrix defined by

$$B_{j,j}(\theta)=\begin{array}{c} \\ R\\ L\\ U\\ D\end{array}\!\!\begin{array}{c}\begin{array}{cccc}R & L & U & D\end{array}\\ \begin{bmatrix}0 & 1+z_1e^{i\theta} & -1 & -1\\ -1-z_1e^{-i\theta} & 0 & 1 & -1\\ 1 & -1 & 0 & 1\\ 1 & 1 & -1 & 0\end{bmatrix}\end{array} \quad (2.6a)$$

for $-\mathfrak{M}\leq j\leq\mathfrak{M}$,

$$B_{j,j+1}(\theta)=-B_{j+1,j}{}^T(\theta)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & z_2(j)\\ 0 & 0 & 0 & 0\end{bmatrix} \quad (2.6b)$$

for $-\mathfrak{M}\leq j\leq\mathfrak{M}-1$, and zero for all other values of j and j'. We are interested in the $\mathfrak{N}\to\infty$ limit where (2.5) becomes

$$A^{-1}(j,k;\,j',k')=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\; e^{i\theta(k-k')}[B^{-1}(\theta)]_{jj'}\,. \quad (2.7)$$

Rearrange the rows and columns of B so that all the R,L rows (columns) precede all U,D rows (columns) and call the resulting $4(2\mathfrak{M}+1)\times 4(2\mathfrak{M}+1)$ matrix

$$\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}.$$

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[7] E. W. Montroll, R. B. Potts, and J. C. Ward, J. Math. Phys. 4, 308 (1963).

[8] B. M. McCoy and T. T. Wu, Phys. Rev. 162, 436 (1967). This paper will henceforth be referred to as IV.

Then, following (7.3) of IV, we obtain

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1} = \begin{pmatrix} b_{11}'^{-1} & -b_{11}'^{-1}b_{12}'b_{22}'^{-1} \\ 0 & b_{22}'^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}, \quad (2.8)$$

where

$$[b_{11}']_{j,j} = \begin{array}{c} \\ R \\ L \end{array}\!\!\begin{array}{cc} R & L \\ \end{array}\begin{pmatrix} 0 & 1+z_1e^{i\theta} \\ -1-z_1e^{-i\theta} & 0 \end{pmatrix}, \quad (2.9a)$$

$$[b_{12}']_{j,j} = \begin{array}{c} R \\ L \end{array}\overset{\quad U \quad D}{\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}}, \quad (2.9b)$$

$$[b_{22}']_{j,j} = \begin{array}{c} U \\ D \end{array}\overset{\quad U \quad D}{\begin{pmatrix} -ia & -b \\ b & ia \end{pmatrix}}, \quad (2.9c)$$

$$[l_{21}]_{j,j} = \begin{array}{c} U \\ D \end{array}\overset{\qquad R \qquad\qquad L}{\begin{pmatrix} (1+z_1e^{i\theta})^{-1} & (1+z_1e^{-i\theta})^{-1} \\ -(1+z_1e^{i\theta})^{-1} & (1+z_1e^{-i\theta})^{-1} \end{pmatrix}}, \quad (2.9d)$$

for $-\mathfrak{M} \leq j \leq \mathfrak{M}$ and

$$[b_{22}']_{j,j+1} = -[b_{22}'^T]_{j+1,j} = \begin{array}{c} U \\ D \end{array}\overset{\quad U \quad D}{\begin{pmatrix} 0 & z_2(j) \\ 0 & 0 \end{pmatrix}} \quad (2.9e)$$

for $-\mathfrak{M} \leq j \leq \mathfrak{M}-1$. All the other matrix elements are

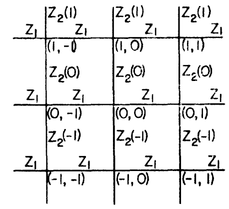

FIG. 1. Ising lattice with bond weights z_1 and $z_2(j)$.

zero and we have used the notation of I:

$$a = -2z_1 \sin\theta \, |1+z_1e^{i\theta}|^{-2}, \quad (2.10a)$$

$$b = (1-z_1^2)|1+z_1e^{i\theta}|^{-2}. \quad (2.10b)$$

The required inverse of b_{11}' is

$$[b_{11}'^{-1}]_{j,j'} = \begin{pmatrix} 0 & -(1+z_1e^{-i\theta})^{-1} \\ (1+z_1e^{i\theta})^{-1} & 0 \end{pmatrix}\delta_{j,j'}. \quad (2.11)$$

To compute $[b_{22}']^{-1}$ we define the $2(2\mathfrak{M}+1)\times 2(2\mathfrak{M}+1)$ matrix $C_{\{E_2\}}$ to be b_{22}' with U and D interchanged. Then from (2.8)

$$[B^{-1}]_{jl,j'l'} = [C^{-1}]_{jl,j'l'}, \quad \text{with } l=U,D, \; l'=U,D. \quad (2.12)$$

Explicitly,

$$C_{j,j} = \begin{array}{c} D \\ U \end{array}\overset{\quad D \quad U}{\begin{pmatrix} ia & b \\ -b & -ia \end{pmatrix}} \quad \text{for} \quad -\mathfrak{M} \leq j \leq \mathfrak{M} \quad (2.13a)$$

and

$$C_{j,j+1} = -[C^T]_{j+1,j} = \begin{pmatrix} 0 & 0 \\ z_2(j) & 0 \end{pmatrix}$$

$$\text{for} \quad -\mathfrak{M} \leq j \leq \mathfrak{M}-1, \quad (2.13b)$$

so that

$$C_{\{E_2\}} = \begin{array}{cc|cccccccccccc}
 & & -\mathfrak{M} & -\mathfrak{M} & -\mathfrak{M}+1 \cdots & -1 & -1 & 0 & 0 & 1 \cdots & \mathfrak{M}-1 & \mathfrak{M} & \mathfrak{M} \\
 & & D & U & D \cdots & D & U & D & U & D \cdots & U & D & U \\
\hline
-\mathfrak{M} & D & ia & b & & & & & & & & & \\
-\mathfrak{M} & U & -b & -ia & z_2(-\mathfrak{M}) & & & & & & & & \\
 & & & & \ddots & & & & & & & & \\
-\mathfrak{M}+1 & D & & -z_2(-\mathfrak{M}) & ia & & & & & & & & \\
\vdots & & & \ddots & \ddots & & & & & & & & \\
-1 & D & & & & ia & b & & & & & & \\
-1 & U & & & & -b & -ia & z_2(-1) & & & & & \\
0 & D & & & & & -z_2(-1) & ia & b & & & & \\
0 & U & & & & & & -b & -ia & z_2(0) & & & \\
 & & & & & & & & & \ddots & & & \\
1 & D & & & & & & & -z_2(0) & ia & & & \\
\vdots & & & & & & & & \ddots & \ddots & & & \\
\mathfrak{M}-1 & U & & & & & & & & & -ia & z_2(\mathfrak{M}-1) & 0 \\
\mathfrak{M} & D & & & & & & & & & -z_2(\mathfrak{M}-1) & ia & b \\
\mathfrak{M} & U & & & & & & & & & 0 & -b & -ia
\end{array} \quad . \quad (2.14)$$

We compute C^{-1} from the formula

$$[C^{-1}]_{jl,j'l'} = \text{cofactor} C_{j'l';jl} / \det C_{\{E_2\}}. \quad (2.15)$$

To evaluate the required cofactors, we define $C(j,j')$ as the determinant of the $2(j'-j)\times 2(j'-j)$ matrix obtained from $C_{\{E_2\}}$ by deleting all rows and columns with an index less than j or greater than j', $iD(j,j')$ as the determinant of the $[2(j'-j)-1]\times[2(j'-j)-1]$ matrix obtained by deleting the last row and column from the matrix defining $C(j,j')$ and $i\bar{D}(j,j')$ as the determinant of the $[2(j'-j)-1]\times[2(j'-j)-1]$ matrix obtained by deleting the first row and column from the matrix defining $C(j,j')$. Because $C_{\{E_2\}}$ has only three nonvanishing diagonals, we readily may use these definitions to obtain for $-\mathfrak{M}\leq j'\leq j\leq\mathfrak{M}$

$$[B^{-1}]_{jD,j'D}=-[B^{-1}]_{j'D,jD}{}^{*}=ib^{j-j'}\prod_{n=j'}^{j-1}z_2(n)C(-\mathfrak{M},j'-1)\times\bar{D}(j,\mathfrak{M})/C(-\mathfrak{M},\mathfrak{M}),\quad(2.16\text{a})$$

$$[B^{-1}]_{jU,j'U}=-[B^{-1}]_{j'U,jU}{}^{*}=ib^{j-j'}\prod_{n=j'}^{j-1}z_2(n)D(-\mathfrak{M},j')\times C(j+1,\mathfrak{M})/C(-\mathfrak{M},\mathfrak{M}),\quad(2.16\text{b})$$

$$[B^{-1}]_{jU,j'D}=-[B^{-1}]_{j'D,jU}{}^{*}=b^{j-j'+1}\prod_{n=j'+1}^{j}z_2(n)C(-\mathfrak{M},j'-1)\times C(j+1,\mathfrak{M})/C(-\mathfrak{M},\mathfrak{M})\quad(2.16\text{c})$$

and for $-\mathfrak{M}\leq j'<j\leq\mathfrak{M}$

$$[B^{-1}]_{jD,j'U}=-[B^{-1}]_{j'U,jD}{}^{*}=-b^{j-j'-1}\prod_{n=j'}^{j-1}z_2(n)D(-\mathfrak{M},j')\times\bar{D}(j,\mathfrak{M})/C(-\mathfrak{M},\mathfrak{M}).\quad(2.16\text{d})$$

All other inverse matrix elements are now readily obtained by combining (2.16) with (2.8). In particular, for the later study of spin correlations in the horizontal direction we need

$$[B^{-1}]_{jR,jR}=[B^{-1}]_{jL,jL}=i|1+z_1e^{i\theta}|^{-2}\times\{C(-\mathfrak{M},j-1)\bar{D}(j,\mathfrak{M})+D(-\mathfrak{M},j)C(j+1,\mathfrak{M})\}/C(-\mathfrak{M},\mathfrak{M})\quad(2.17\text{a})$$

and

$$[B^{-1}]_{jL,jR}=-[B^{-1}]_{jR,jL}{}^{*}=(1+z_1e^{i\theta})^{-1}\times\{1+(1+z_1e^{i\theta})^{-1}[iD(-\mathfrak{M},j)C(j+1,\mathfrak{M})-iC(-\mathfrak{M},j-1)\bar{D}(j,\mathfrak{M})-2bC(-\mathfrak{M},j-1)\times C(j+1,\mathfrak{M})]/C(-\mathfrak{M},\mathfrak{M})\}.\quad(2.17\text{b})$$

B. Average Nearest-Neighbor Spin-Spin Correlations

To compute $\langle\sigma_{0,0}\sigma_{1,0}\rangle$ from this formalism, we join the site (0,0) to the site (1,0) by a straight line. Then

$$y=\begin{array}{c|cc} & \begin{matrix}00\\U\end{matrix} & \begin{matrix}10\\D\end{matrix}\\ \hline 00\;U & 0 & z_2^{-1}(0)-z_2(0)\\ 10\;D & -[z_2^{-1}(0)-z_2(0)] & 0\end{array},\quad(2.18)$$

and (2.2) gives for any set of interactions $\{E_2\}$

$$\langle\sigma_{0,0}\sigma_{1,0}\rangle=[1-z_2{}^2(0)](2\pi)^{-1}\times\int_{-\pi}^{\pi}d\theta\,B^{-1}(\theta)_{1D,0U}+z_2(0),\quad(2.19)$$

with $[B^{-1}]_{1D,0U}$ given by (2.16). Similarly,

$$\langle\sigma_{0,0}\sigma_{0,1}\rangle=(1-z_1{}^2)(2\pi)^{-1}\times\int_{-\pi}^{\pi}d\theta\,e^{i\theta}B^{-1}(\theta)_{0L,0R}+z_1,\quad(2.20)$$

with $[B^{-1}]_{0L,0R}$ given by (2.17). These correlation functions clearly are functions of all the $E_2(j)$ but the dependence on $E_2(j)$ is not the same for all j, so, in contrast to the specific heat, they are not probability-1 objects, even as $\mathfrak{N}\to\infty$. For example, $\langle\sigma_{0,0}\sigma_{1,0}\rangle$ is zero if $E_2(0)=0$ and is 1 if $E_2(0)=\infty$, regardless of what the rest of the energies $E_2(j)$ are. Thus, $\langle\sigma_{0,0}\sigma_{1,0}\rangle$ and $\langle\sigma_{0,0}\sigma_{0,1}\rangle$ themselves are random objects and to characterize them, we need to know their probability distribution functions. Even these single-probability distributions will not completely characterize these nearest-neighbor correlations because, for example, $\langle\sigma_{j,0}\sigma_{j+1,0}\rangle$ and $\langle\sigma_{j',0}\sigma_{j'+1,0}\rangle$ are not independent random functions for $j\neq j'$. To characterize the spin-spin correlations completely, we need a joint probability distribution function involving all possible spin correlations in the system. In this paper, however, we are not interested in all the information contained in these probability functions and confine ourselves in this section to the average value and the average of the square of the spin-spin correlation functions.

To study these averages, it is convenient to define the ratios

$$C(j,j')/D(j,j')=x(j,j';\theta)\quad(2.21\text{a})$$

and

$$C(j,j')/\bar{D}(j,j')=-\bar{x}(j,j';\theta),\quad(2.21\text{b})$$

where the dependence on θ will be suppressed unless needed. We may then use the identities

$$\begin{aligned}C(-\mathfrak{M},\mathfrak{M})&=C(-\mathfrak{M},j)C(j+1,\mathfrak{M})-z_2{}^2(j)D(-\mathfrak{M},j)\bar{D}(j+1,\mathfrak{M})\\&=-D(-\mathfrak{M},j)\bar{D}(j,\mathfrak{M})+b^2C(-\mathfrak{M},j-1)C(j+1,\mathfrak{M})\end{aligned}\quad(2.22)$$

and the recursion relations

$$\begin{pmatrix}C(-\mathfrak{M},j+1)\\D(-\mathfrak{M},j+1)\end{pmatrix}=\begin{pmatrix}a^2+b^2&a\\a&1\end{pmatrix}\times\begin{pmatrix}1&0\\0&z_2{}^2(j)\end{pmatrix}\begin{pmatrix}C(-\mathfrak{M},j)\\D(-\mathfrak{M},j)\end{pmatrix}\quad(2.23\text{a})$$

and

$$\begin{pmatrix}C(j-1,\mathfrak{M})\\-\bar{D}(j-1,\mathfrak{M})\end{pmatrix}=\begin{pmatrix}a^2+b^2&a\\a&1\end{pmatrix}\times\begin{pmatrix}1&0\\0&z_2{}^2(j-1)\end{pmatrix}\begin{pmatrix}C(j,\mathfrak{M})\\-\bar{D}(j,\mathfrak{M})\end{pmatrix}\quad(2.23\text{b})$$

to write

$$B^{-1}(\theta)_{1D,0U}=z_2(0)\{z_2^2(0)+x(-\mathfrak{M},0;\theta)\bar{x}(1,\mathfrak{M};\theta)\}^{-1} \quad (2.24a)$$

and

$$B^{-1}(\theta)_{0L,0R}=(1+z_1e^{i\theta})^{-1}(1+(1+z_1e^{i\theta})^{-1}\times\{i[\bar{x}(0,\mathfrak{M};\theta)+x(-\mathfrak{M},0;\theta)-2a]-2b^{-1}[\bar{x}(0,\mathfrak{M};\theta)-a][x(-\mathfrak{M},0;\theta)-a]\}\times\{b^2+[\bar{x}(0,\mathfrak{M};\theta)-a][x(-\mathfrak{M},0;\theta)-a]\}^{-1}). \quad (2.24b)$$

The existence of the thermodynamic limit is equivalent to the existence of $\lim_{\mathfrak{M}\to\infty}\bar{x}(j,\mathfrak{M})$ and $\lim_{\mathfrak{M}\to\infty}\times x(-\mathfrak{M},j)$. These limits will in general be different for the different lattices of our collection. We also see from the recursion relation (2.23) that $\lim_{j\to-\infty}\bar{x}(j,\mathfrak{M})$ and $\lim_{j\to\infty}x(-\mathfrak{M},j)$ will not exist. However, we saw in I that the work of Furstenberg[9] may be applied to the recursion relation (2.23a) to show that for $\mathfrak{M}$ fixed and $\mathfrak{M}+j\to\infty$

$$x(-\mathfrak{M},j;\theta)\to x\,, \quad (2.25)$$

where x is a random variable whose distribution function $\nu(x)$ is independent of the boundary conditions imposed on the recursion relation and satisfies

$$\nu(x)=\frac{b^2}{(x-a)^2}\int_{-\infty}^{\infty}dx'\,|x'|\mu\left(x'\frac{a^2+b^2-ax}{x-a}\right)\nu(x'). \quad (2.26)$$

Here, as in I, we have defined

$$\mu(\lambda)d\lambda=P(E_2)dE_2 \quad (2.27)$$

and

$$\lambda=\tanh^2\beta E_2\,. \quad (2.28)$$

Similarly, we see from (2.23b) that for $\mathfrak{M}$ fixed and $\mathfrak{M}+j\to\infty$

$$\bar{x}(j,\mathfrak{M};\theta)\to\bar{x}\,, \quad (2.29)$$

where $\bar{x}$ is a random variable whose distribution function $\bar{\nu}(\bar{x})$ satisfies

$$\bar{\nu}(\bar{x})=\nu(\bar{x})\,. \quad (2.30)$$

Therefore, if a function depends on the collection of energies $E_2(j)$ for $-\mathfrak{M}\le j\le j_0$ and $j_1\le j\le\mathfrak{M}$ with $j_0\le j_1$ only through the ratios $x(-\mathfrak{M},j_0)$ and $\bar{x}(j_1,\mathfrak{M})$, we may average this function over these energies in the $\mathfrak{M}\to\infty$ limit by replacing $x(-\mathfrak{M},j_0;\theta)$ by x and $\bar{x}(j_1,\mathfrak{M};\theta)$ by $\bar{x}$ and averaging the resulting expression over x and $\bar{x}$ using $\nu(x)$ and $\nu(\bar{x})$. Since $z_2(0)$, $x(-\mathfrak{M},0)$, and $\bar{x}(1,\mathfrak{M})$ [and similarly $x(-\mathfrak{M},0)$ and $\bar{x}(0,\mathfrak{M})$] depend on independent subsets of $\{E_2\}$, we use this argument to obtain the desired results

$$\langle\langle\sigma_{0,0}\sigma_{1,0}\rangle\rangle_{E_2}=\int_0^{\infty}dE_2\,P(E_2)z_2\left\{1+(1-z_2^2)(2\pi)^{-1}\times\int_{-\pi}^{\pi}d\theta\int_{-\infty}^{\infty}dx\,\nu(x)\int_{-\infty}^{\infty}d\bar{x}\,\nu(\bar{x})[z_2^2+x\bar{x}]^{-1}\right\}, \quad (2.31a)$$

where $z_2(0)=z_2=\tanh\beta E_2$, and

$$\langle\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,e^{i\theta}(1+z_1e^{-i\theta})\times(1+z_1e^{i\theta})^{-1}\int_{-\infty}^{\infty}dx\,\nu(x)\int_{-\infty}^{\infty}d\bar{x}\,\nu(\bar{x})\times\left(1+\frac{ib(x+\bar{x}-2a)-2(x-a)(\bar{x}-a)}{b^2+(x-a)(\bar{x}-a)}\right). \quad (2.31b)$$

Finally, it should be remarked that these two average neighbor spin-spin correlation functions are closely related to the free energy studied in I. In particular, from

$$F_r=-\beta^{-1}\underset{\mathfrak{M}\to\infty}{\text{l.i.m.}}\,(2\mathfrak{M}+1)^{-1}\ln Z\,, \quad (2.32)$$

where

$$Z=\sum_{\text{all }\sigma}e^{-\beta\mathcal{E}} \quad (2.33)$$

and $\mathcal{E}$ is given by (2.1), we have

$$-\frac{\partial\beta F_r}{\partial\beta}=-E_1\langle\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2}-\langle E_2\langle\sigma_{0,0}\sigma_{1,0}\rangle\rangle_{E_2}. \quad (2.34)$$

The derivative of (2.34) with respect to T is the specific heat and because both spin correlations in (2.34) are monotonic nonincreasing, we conclude that the leading singularity of the temperature derivative of the nearest-neighbor correlation functions at T_c must be the same as that of $C_v{}^r$. This will be seen in more detail in Sec. 3, where we study the derivatives of these functions for the special case (1.1).

C. Other Average Spin-Spin Correlations

To study spin correlations other than nearest neighbor, we need more information than that provided by $\nu(x)$. To see this, consider $\langle\sigma_{0,0}\sigma_{0,2}\rangle$. Using the formalism of A, we find for any collection $\{E_2\}$

$$\langle\sigma_{0,0}\sigma_{0,2}\rangle=(1-z_1^2)^2\,\mathrm{Pf}\begin{vmatrix}A^{-1}(0,0;0,1)_{RR} & A^{-1}(0,0;0,1)_{RL}-(z_1^{-1}-z_1)^{-1} & A^{-1}(0,0;0,2)_{RL}\\ & A^{-1}(0,1;0,1)_{RL} & A^{-1}(0,1;0,2)_{RL}-(z_1^{-1}-z_1)^{-1}\\ & & A^{-1}(0,1;0,2)_{LL}\end{vmatrix}, \quad (2.35)$$

where

$$A^{-1}(0,k;0,k')_{RL}-(z_1^{-1}-z_1)^{-1}\delta_{k'-k-1,0}=-(1-z_1^2)^{-1}(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,e^{i(k'-k)\theta}(1+z_1e^{-i\theta})(1+z_1e^{i\theta})^{-1}\times\left(1+\frac{ib[\bar{x}(0,\mathfrak{M})+x(-\mathfrak{M},0)-2a]-2[\bar{x}(0,\mathfrak{M})-a][x(-\mathfrak{M},0)-a]}{b^2+[\bar{x}(0,\mathfrak{M})-a][x(-\mathfrak{M},0)-a]}\right) \quad (2.36a)$$

[9] H. Furstenberg, Trans. Am. Math. Soc. 108, 377 (1963).

and

$$A^{-1}(0,k;0,k')_{RR}=A^{-1}(0,k;0,k')_{LL}=(1-z_1^2)^{-1}(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i(k-k')\theta}ib[\bar{x}(0,\mathfrak{M})-x(-\mathfrak{M},0)]\times\{b^2+[\bar{x}(0,\mathfrak{M})-a][x(-\mathfrak{M},0)-a]\}^{-1}. \quad (2.36b)$$

This expression is markedly different from the expressions for $\langle\sigma_{0,0}\sigma_{0,1}\rangle$ and $\langle\sigma_{0,0}\sigma_{1,0}\rangle$ because it involves not just a single integral over a function of $x(-\mathfrak{M},0;\theta)$, but a product of such integrals which may be rewritten as a double integral over a function of $x(-\mathfrak{M},0;\theta_1)$ and $x(-\mathfrak{M},0;\theta_2)$. The ratios $x(-\mathfrak{M},0;\theta_1)$ and $x(-\mathfrak{M},0;\theta_2)$ involve exactly the same subset of $\{E_2\}$, so they are not statistically independent, and, in fact, are identical if $\theta_1=\theta_2$. Therefore, we cannot average (2.35) over $\{E_2\}$ by replacing these ratios by x and averaging over $\nu(x)$.

To study averages of functions of $x(-\mathfrak{M},0;\theta_1)$ and $x(-\mathfrak{M},0;\theta_2)$ over $\{E_2\}$ as $\mathfrak{M}\to\infty$, consider the direct sum of two two-dimensional vector spaces, one vector space containing the vectors

$$\begin{pmatrix}C(-\mathfrak{M},j;\theta_1)\\D(-\mathfrak{M},j;\theta_1)\end{pmatrix}$$

and the other containing the vectors

$$\begin{pmatrix}C(-\mathfrak{M},j;\theta_2)\\D(-\mathfrak{M},j;\theta_2)\end{pmatrix}.$$

In this four-dimensional space, we have the matrix recursion relation analogous to (2.23a)

$$\begin{bmatrix}C(-\mathfrak{M},j+1;\theta_1)\\D(-\mathfrak{M},j+1;\theta_1)\\C(-\mathfrak{M},j+1;\theta_2)\\D(-\mathfrak{M},j+1;\theta_2)\end{bmatrix}=\begin{bmatrix}a_1^2+b_1^2 & a_1 & 0 & 0\\ a_1 & 1 & 0 & 0\\ 0 & 0 & a_2^2+b_2^2 & a_2\\ 0 & 0 & a_2 & 1\end{bmatrix}\times\begin{bmatrix}1&0&0&0\\0&z_2^2(j)&0&0\\0&0&1&0\\0&0&0&z_2^2(j)\end{bmatrix}\begin{bmatrix}C(-\mathfrak{M},j;\theta_1)\\D(-\mathfrak{M},j;\theta_1)\\C(-\mathfrak{M}-1,j;\theta_2)\\D(-\mathfrak{M}-1,j;\theta_2)\end{bmatrix} \quad (2.37)$$

and a similar set of equations analogous to (2.23b) involving $-\bar{D}$ instead of D. Here we have defined

$$a(\theta_i)=a_i\,,\quad b(\theta_i)=b_i\,,\quad i=1,2. \quad (2.38)$$

We may generalize our previous treatment by defining

$$C(j,j';\theta_i)/D(j,j';\theta_i)=x_i(j,j') \quad (2.39a)$$

and

$$C(j,j';\theta_i)/\bar{D}(j,j';\theta_i)=-\bar{x}_i(j,j')\,,\quad i=1,2. \quad (2.39b)$$

The work of Furstenberg[9] guarantees that as $j+\mathfrak{M}\to\infty$, the ratios $x_i(-\mathfrak{M},j)$ approach the random variables x_i which are described by a joint probability density function $\nu(x_1,x_2;\theta_1,\theta_2)$. The recursion relation (2.37) implies

$$x_i(-\mathfrak{M},j+1)=\frac{(a_i^2+b_i^2)x_i(-\mathfrak{M},j)+a_iz_2^2(j)}{a_ix_i(-\mathfrak{M},j)+z_2^2(j)}. \quad (2.40)$$

The distribution $\nu(x_1,x_2)$ is characterized by the property that if we transform x_1 and x_2 according to (2.40) and average over E_2 with the probability density $P(E_2)$, then $\nu(x_1,x_2)$ will transform into itself. Therefore, as in I,

$$\nu(x_1,x_2)=\int_0^1 dx\,\mu(\lambda)\int_{-\infty}^{\infty}dx_1'\int_{-\infty}^{\infty}dx_2'\times\delta\left(x_1-\frac{(a_1^2+b_1^2)x_1'+a_1\lambda}{a_1x_1'+\lambda}\right)\times\delta\left(x_2-\frac{(a_2^2+b_2^2)x_2'+a_2\lambda}{a_2x_2'+\lambda}\right)\nu(x_1',x_2'). \quad (2.41)$$

Because $\nu(x_1,x_2)$ is a probability distribution,

$$\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\,\nu(x_1,x_2)=1. \quad (2.42)$$

More specifically, if we integrate (2.41) over x_1 (x_2), we recover the equation for $\nu(x_2;\theta_2)$ [$\nu(x_1;\theta_1)$], and thus conclude that

$$\int_{-\infty}^{\infty}dx_1\,\nu(x_1,x_2;\theta_1,\theta_2)=\nu(x_2;\theta_2) \quad (2.43a)$$

and

$$\int_{-\infty}^{\infty}dx_2\,\nu(x_1,x_2;\theta_1,\theta_2)=\nu(x_1;\theta_1). \quad (2.43b)$$

We also notice that if $\theta_1=\theta_2=\theta$, then (2.41) is solved by

$$\nu(x_1,x_2;\theta,\theta)=\delta(x_1-x_2)\nu(x_1). \quad (2.44)$$

Such a relation is expected since from (2.37) we see that when $\theta_1=\theta_2$ the ratios $x_1(j,j')$ and $x_2(j,j')$ are identical for all sets $\{E_2\}$.

We may now average a function of $x_1(-\mathfrak{M},0)$ and $x_2(-\mathfrak{M},0)$ over $\{E_2\}$ by replacing $x_1(-\mathfrak{M},0)$ by x_1 and $x_2(-\mathfrak{M},0)$ by x_2 and averaging the resulting expression over x_1 and x_2, using $\nu(x_1,x_2)$. In particular, we may apply this procedure to $\langle\sigma_{0,0}\sigma_{0,2}\rangle$ to find

$$\langle\langle\sigma_{0,0}\sigma_{0,2}\rangle\rangle_{E_2}=(2\pi)^{-2}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\,\nu(x_1,x_2)\int_{-\infty}^{\infty}d\bar{x}_1\int_{-\infty}^{\infty}d\bar{x}_2\,\nu(\bar{x}_1,\bar{x}_2)$$
$$\times\mathrm{Pf}\left|\begin{matrix}F_2(x_1,\bar{x}_1;\theta_1) & F_1(x_1,\bar{x}_1;\theta_1) & e^{i\theta_1}F_1(x_1,\bar{x}_1;\theta_1)\\ & e^{-i\theta_2}F_1(x_2,\bar{x}_2;\theta_2) & F_1(x_2,\bar{x}_2;\theta_2)\\ & & F_2(x_2,\bar{x}_2;\theta_2)\end{matrix}\right|, \quad (2.45)$$

where

$$F_1(x,\bar{x};\theta)=-e^{i\theta}(1+z_1e^{-i\theta})(1+z_1e^{i\theta})^{-1}\times\left(1+\frac{ib(x+\bar{x}-2a)-2(x-a)(\bar{x}-a)}{b^2+(x-a)(\bar{x}-a)}\right) \quad (2.46a)$$

and

$$F_2(x,\bar{x};\theta)=ie^{-i\theta}b(x-\bar{x})[b^2+(x-a)(\bar{x}-a)]^{-1}. \quad (2.46b)$$

These above considerations may also be applied to the averages $\langle\langle\sigma_{0,0}\sigma_{2,0}\rangle\rangle_{E_2}$ and $\langle\langle\sigma_{0,0}\sigma_{1,1}\rangle\rangle_{E_2}$. In general, to compute an average spin correlation such as $\langle\langle\sigma_{0,0}\sigma_{0,m}\rangle\rangle_{E_2}$ one needs a distribution function $\nu(x_1,\cdots,x_m)$ which satisfies the equation

$$\nu(x_1,\cdots,x_m)=\int_0^1 d\lambda\mu(\lambda)\int_{-\infty}^{\infty}\prod_{l=1}^{m}\left[dx_l'\times\delta\left(x_l-\frac{(a_l^2+b_l^2)x_l'+a_l\lambda}{a_lx_l'+\lambda}\right)\right]\nu(x_1',\cdots,x_m'), \quad (2.47)$$

where if we integrate a ν in m variables over one of its variables, we obtain a corresponding ν in $m-1$ variables.

D. Second Moment of Nearest-Neighbor Spin-Spin Correlation

It is a simple matter to compute $\langle\langle\sigma_{0,0}\sigma_{1,0}\rangle^2\rangle_{E_2}$ and $\langle\langle\sigma_{0,0}\sigma_{0,1}\rangle^2\rangle_{E_2}$ by squaring (2.19) and (2.20) and using $\nu(x_1,x_2)$ to average over the appropriate random variables. However, as mentioned in Sec. 2 B, we are in this paper concentrating on the temperature derivatives of the nearest-neighbor correlation functions, and therefore need to study $\langle[(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle]^2\rangle_{E_2}$ and the analogous quantity for the horizontal direction. From (2.31) we may see that

$$\left(\frac{d}{dT}\langle\sigma_{0,0}\sigma_{1,0}\rangle\right)^2=\left((2\pi)^{-1}\int_{-\pi}^{\pi}d\theta_1\frac{d}{dT_1}\{[1-z_2(0)^{(1)2}]z_2(0)^{(1)}[z_2(0)^{(1)2}+x_1^{(1)}(-\mathfrak{M},0)x_1^{(1)}(1,\mathfrak{M})]^{-1}+z_2(0)^{(1)}\}\right)$$
$$\times\left((2\pi)^{-1}\int_{-\pi}^{\pi}d\theta_2\frac{d}{dT_2}\{[1-z_2(0)^{(2)2}]z_2(0)^{(2)}[z_2(0)^{(2)2}+x_2^{(2)}(-\mathfrak{M},0)x_2^{(2)}(1,\mathfrak{M})]^{-1}+z_2(0)^{(2)}\}\right)_{T_1=T_2}, \quad (2.48)$$

where the superscripts (1) and (2) mean $T=T_1$ and $T=T_2$, respectively. We may average this product of integrals over $\{E_2\}$ much as we did in the last section, by defining a joint probability function $\tilde{\nu}(x_1,x_2)$ which satisfies an equation similar to (2.41). Explicitly,

$$\tilde{\nu}(x_1,x_2)=\int_0^1 dE_2\,P(E_2)\int_{-\infty}^{\infty}dx_1'\times\int_{-\infty}^{\infty}dx_2'\,\delta\left(x_1-\frac{(\tilde{a}_1^2+\tilde{b}_1^2)x_1'+\tilde{a}_1\lambda_1}{\tilde{a}_1x_1'+\lambda_1}\right)\times\delta\left(x_2-\frac{(\tilde{a}_2^2+\tilde{b}_2^2)x_2'+\tilde{a}_2\lambda_2}{\tilde{a}_1x_2'+\lambda_2}\right)\tilde{\nu}(x_1',x_2'), \quad (2.49)$$

where

$$\tilde{a}_j=-2z_1^{(j)}\sin\theta_j\,|1+z_1^{(j)}e^{i\theta_j}|^{-2}, \quad (2.50a)$$

$$\tilde{b}_j=(1-z_1^{(j)2})\,|1+z_1^{(j)}e^{i\theta_j}|^{-2}, \quad (2.50b)$$

and

$$\lambda_j=z_2^{(j)2}. \quad (2.50c)$$

We also have the relations

$$\int_{-\infty}^{\infty}dx_1\,\tilde{\nu}(x_1,x_2;\theta_1\theta_2)=\nu(x_2;\theta_2,T_2) \quad (2.51a)$$

and

$$\int_{-\infty}^{\infty}dx_2\,\tilde{\nu}(x_1,x_2;\theta_1,\theta_2)=\nu(x_1,\theta_1,T_1). \quad (2.51b)$$

The argument of the last section allows us to replace an average over $\{E_2\}$ by an appropriate average over $\tilde{\nu}(x_1,x_2)$ and we obtain

$$\langle[(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle]^2\rangle_{E_2}=(2\pi)^{-2}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2\int_0^{\infty}dE_2\,P(E_2)\int_{-\infty}^{\infty}dx_1\times\int_{-\infty}^{\infty}dx_2\int_{-\infty}^{\infty}d\bar{x}_1\int_{-\infty}^{\infty}d\bar{x}_2\frac{\partial^2}{\partial T_1\partial T_2}\{\tilde{\nu}(x_1,x_2)\tilde{\nu}(\bar{x}_1,\bar{x}_2)$$
$$\times[(1-z_2^{(1)2})z_2^{(1)}(z_2^{(1)2}+x_1\bar{x}_1)^{-1}+z_2^{(1)}]\times[(1-z_2^{(2)2})z_2^{(2)}(z_2^{(2)2}+x_2\bar{x}_2)^{-1}+z_2^{(2)}]\}. \quad (2.52)$$

Higher moments may similarly be computed by use of joint probability functions of more than two variables. We confine ourselves to the two-variable function $\tilde{\nu}(x_1,x_2)$. In Sec. 4, we will study (2.49) and use these results in Sec. 5 to extract explicit information about (2.52).

3. AVERAGE TEMPERATURE DERIVATIVE OF SPIN-SPIN CORRELATIONS

Throughout the rest of this paper we confine our attention to the power-law distribution (1.1) where N is considered to be large. Furthermore, though all of our results may be taken over to the more general case, for the sake of concreteness we confine ourselves to the ferromagnetic case $z_1>0$. In I we showed that, for all T with θ much greater than N^{-2} and for all θ with $|T-T_c|$ much greater than N^{-2}, $\nu(x)$ is well approximated by $\delta(x-x_m)$, where x_m is found from (4.10) of I. It is thus seen for $|T-T_c|\gg N^{-2}$ that $\langle\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2}$ differs from the value it has for a comparable Onsager lattice with the same T_c and E_1 only by a term of order N^{-1}. In this paper, we are not interested in such small effects and concentrate on the region near T_c where δ as defined by (4.18) of I is of order 1. Explicitly, we recall [(4.19) of I] that to leading order in N^{-1}

$$\delta=C_1(T-T_c)N^2=(T-T_c)N^2 4k\beta_c^2(1+z_{2c}^0)z_{2c}^{0-1}\times[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)+O(N^{-1})], \quad (3.1)$$

which defines the constant C_1. Here the subscript c means $T=T_c$ and we recall from (4.1) of I that T_c is determined from

$$\ln[z_{2c}^{0-1}(1-z_{1c})/(1+z_{1c})]=\tfrac{1}{2}N^{-1}. \quad (3.2)$$

Recall also the definitions of ϕ [(4.16) of I]

$$\phi=-8\lambda_0^{-1/2}z_{1c}(1+z_{1c})^{-2}N^2\theta, \quad (3.3a)$$

so that

$$a=\tfrac{1}{4}N^{-2}\lambda_0^{1/2}\phi+O(N^{-4}), \quad (3.3b)$$

and of the auxiliary variable [(3.12) of I]

$$\eta=(x-x_0)/(\lambda_0 x_0^{-1}+x), \quad (3.4)$$

where [(3.3) of I]

$$x(\theta;\lambda_0)\equiv x_0(\theta)=(2a)^{-1}\{a^2+b^2-\lambda_0+[(a^2+b^2-\lambda_0)^2+4\lambda_0 a^2]^{1/2}\}\sim\tfrac{1}{4}N^{-1}\lambda_0^{1/2}\phi. \quad (3.5)$$

The last relation is valid only when δ and ϕ are of order 1. Then, with the definition

$$X(\eta)=\nu(x)(dx/d\eta), \quad (3.6)$$

we find from (4.6) and (4.23) of I that when $\phi>0$,

$$X(\eta)\sim C_N\eta^{\delta-1}e^{-2N\eta-\phi^2/8N\eta}, \quad (3.7)$$

where

$$C_N^{-1}=2(\phi/4N)^\delta K_\delta(\phi) \quad (3.8)$$

and $K_\delta(\phi)$ is the modified Bessel function of the third kind of order δ. Finally, using the variable

$$\xi=4N\phi^{-1}\eta \quad (3.9)$$

and defining

$$W(\xi)=X(\eta)(d\eta/d\xi)=\tfrac{1}{4}N^{-1}\phi X(\eta), \quad (3.10)$$

we have

$$W(\xi)=\tfrac{1}{2}[K_\delta(\phi)]^{-1}\xi^{\delta-1}e^{-\frac{1}{2}\phi(\xi+\xi^{-1})} \quad (3.11)$$

and

$$x=\lambda_0^{1/2}\xi+O(N^{-1}). \quad (3.12)$$

To study $\langle(d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2}$ when $\delta=0(1)$, we note that following the same order-of-magnitude arguments of Sec. 4 of I, we may show that if the θ integration in the T derivative of (2.31b) is restricted to the region $\phi=0(1)$, we retain all the dependence on δ. This angular restriction destroys the δ-independent constant but this term can be regained by noting that when $\delta\to\pm\infty$ we must regain the known behavior near T_c of $\langle\sigma_{0,0}\sigma_{0,1}\rangle_o$, the nearest-neighbor spin-spin correlation function in the Onsager lattice with the same E_1 and T_c. In this lattice $E_2=\bar{E}_2$, where $\bar{E}_2$ satisfies

$$z_{1c}^{-1}(1-\bar{z}_{2c})/(1+\bar{z}_{2c})=1. \quad (3.13)$$

We may find $\langle\sigma_{0,0}\sigma_{0,1}\rangle_o$ either by using the fact that when $\mu(\lambda)=\delta(\lambda-\bar{\lambda})$, $\nu(x)=\delta[x-x(\theta;\bar{\lambda})]$ or by using the work of previous authors[4,7] to write

$$\langle\sigma_{0,0}\sigma_{0,1}\rangle_o=(2\pi)^{-1}\int_0^{2\pi}d\theta\,[(1-\alpha_1e^{i\theta})(1-\alpha_2e^{-i\theta})\times(1-\alpha_1e^{-i\theta})^{-1}(1-\alpha_2e^{i\theta})^{-1}]^{1/2}, \quad (3.14)$$

where

$$\alpha_1=z_1(1-\bar{z}_2)/(1+\bar{z}_2),\quad \alpha_2=z_1^{-1}(1-\bar{z}_2)/(1+\bar{z}_2), \quad (3.15)$$

and the square root is defined positive at $\theta=\pi$. In particular, it is easily shown that near T_c

$$\frac{d}{dT}\langle\sigma_{0,0}\sigma_{0,1}\rangle_o\sim k\beta_c^2\pi^{-1}((1+z_{1c}^{-1})\times[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)]\{\ln(|T-T_c|/T_c)+\ln\tfrac{1}{8}\beta_c[E_1(z_{1c}+z_{1c}^{-1})+E_2^0(z_{2c}^0+z_{2c}^{0-1})]\}+E_1z_{1c}^2(1-z_{1c}^2)^2\,\mathrm{gd}2\beta_cE_1+4E_2^0), \quad (3.16)$$

where gd stands for the Gudermanian ($\mathrm{gd}2x=2\tan^{-1}\times\tanh x$) and to $O(N^{-1})$ we have been able to replace $\bar{E}_2$ by E_2^0. We obtain the similar expression for $\langle\sigma_{0,0}\sigma_{1,0}\rangle_o$ by the replacement $E_1\leftrightarrow E_2^0$. This is also correct to leading order in N^{-1}.

If we now recall that

$$\nu(x;-\theta)=\nu(-x;\theta), \quad (3.17)$$

we may combine the preceding results of I with the temperature derivative of (2.31b) and recall that when $\delta=0(1)$,

$$b^2=\lambda_0+O(N^{-1}) \quad (3.18)$$

and

$$\tfrac{1}{2}(1+z_{2c}^0)(1-z_{1c})z_{2c}^{0-1}=1+O(N^{-1}), \quad (3.19)$$

to find

$$\langle (d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2} = 2C_{01}\int_0^{N^2} d\phi \frac{\partial}{\partial\delta}\int_0^\infty d\xi \int_0^\infty d\bar{\xi}\, W(\xi)W(\bar{\xi}) \times[1+\xi\bar{\xi}]^{-1}+\bar{K}_{01}+O(1), \quad (3.20)$$

where

$$C_{01}=\tfrac{1}{4}k\beta_c^2\pi^{-1}(1+z_{2c}^0)^2(1+z_{1c})^2 z_{1c}^{-1}z_{2c}^{0-1}(1-z_{1c}) \times[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)] = k\beta_c^2\pi^{-1}(1+z_{1c}^{-1})[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)]. \quad (3.21)$$

The N^2 dependence of (3.20) may be made explicit by noting from (3.11) that when $\phi\to\infty$ and δ is fixed [see the beginning of Sec. 4 of I],

$$W(\xi)\sim\delta(\xi-1-\delta/\phi). \quad (3.22)$$

Therefore, as $\phi\to\infty$,

$$\int_0^\infty d\xi\int_0^\infty d\bar{\xi}\, W(\xi)W(\bar{\xi})[1+\xi\bar{\xi}]^{-1} \to \tfrac{1}{2}(1+\delta/\phi)^{-1}+O(\delta^2/\phi^2), \quad (3.23)$$

and we obtain

$$\langle (d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2} = 2C_{01}\left(\int_0^\infty d\phi\left[\frac{\partial}{\partial\delta}\int_0^\infty d\xi W(\xi)\int_0^\infty d\bar{\xi}W(\bar{\xi})(1+\xi\bar{\xi})^{-1} +\tfrac{1}{2}(\phi+1)^{-1}\right]-\tfrac{1}{2}\ln N^2\right)+\bar{K}_{01}+o(1), \quad (3.24)$$

which, using

$$(1+\xi\bar{\xi})^{-1}=\tfrac{1}{2}[(1-\xi\bar{\xi})(1+\xi\bar{\xi})^{-1}+1], \quad (3.25)$$

may be reexpressed as

$$\langle (d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2} = C_{01}\left(\int_0^\infty d\phi\left[\frac{\partial}{\partial\delta}\int_0^\infty d\xi W(\xi)\int_0^\infty d\bar{\xi}W(\bar{\xi})(1-\xi\bar{\xi}) \times(1+\xi\bar{\xi})^{-1}+(\phi+1)^{-1}\right]-\ln N^2\right)+\bar{K}_{01}+o(1)$$

$$=C_{01}[\bar{R}(\delta)-\ln N^2]+K_{01}+o(1), \quad (3.26a)$$

where the last line defines the function $\bar{R}(\delta)$. Clearly $\bar{R}(\delta)$ is an even function of δ.

In an identical manner we may show that

$$\langle (d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle\rangle_{E_2} = C_{10}[\bar{R}(\delta)-\ln N^2]+\bar{K}_{10}+o(1), \quad (3.26b)$$

where C_{10} is obtained from C_{01} by the replacement $E_1\leftrightarrow E_2^0$. This relationship would be exact in Onsager's lattice because it would follow from symmetry alone and thus would be valid at any temperature. However, our lattice does not have the symmetry under $E_1\leftrightarrow E_2^0$, which Onsager's lattice does. Therefore the symmetry of (3.26) is only an approximate one which is not expected to hold to all orders in N^{-1} at all temperatures.

Because of the general identity (2.34), it is possible to express $\bar{R}(\delta)$ in terms of the function introduced in (4.33) of I:

$$R(\delta)=\int_0^\infty d\phi\left(\frac{\partial^2}{\partial\delta^2}\ln K_\delta(\phi)-(\phi+1)^{-1}\right). \quad (3.27)$$

Indeed, it is not without interest to verify directly for our special case that

$$C_v^r=-E_1\langle (d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2} -\langle E_2(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle\rangle_{E_2}, \quad (3.28)$$

where C_v^r is given by (4.39) of I. To verify this we demonstrate that

$$\bar{R}(\delta)=1-R(\delta). \quad (3.29)$$

This relationship may be seen if we first use

$$(1-\xi\bar{\xi})(1+\xi\bar{\xi})^{-1} =(\bar{\xi}^{-1}-\xi)(\bar{\xi}^{-1}+\xi)^{-1} =(\bar{\xi}^{-1}-\xi)\int_0^\infty dx\exp[-x(\bar{\xi}^{-1}+\xi)] \quad (3.30)$$

to write

$$J(\delta)=\int_0^\infty d\xi\int_0^\infty d\bar{\xi}\,\xi^{\delta-1}\exp[-\tfrac{1}{2}\phi(\xi+\xi^{-1})]\bar{\xi}^{\delta-1} \times\exp[-\tfrac{1}{2}\phi(\bar{\xi}+\bar{\xi}^{-1})](1-\xi\bar{\xi})(1+\xi\bar{\xi})^{-1}$$

$$=\int_0^\infty dx\int_0^\infty d\xi\int_0^\infty d\bar{\xi}\,(\bar{\xi}^{-1}-\xi)\xi^{\delta-1} \times\exp[-(\tfrac{1}{2}\phi+x)\xi-\tfrac{1}{2}\phi\xi^{-1}] \times\bar{\xi}^{\delta-1}\exp[-\tfrac{1}{2}\phi\bar{\xi}-(\tfrac{1}{2}\phi+x)\bar{\xi}^{-1}]$$

$$=\int_0^\infty dx\,4\phi^{1/2}(\phi+2x)^{-1/2}K_\delta[\phi^{1/2}(\phi+2x)^{1/2}] \times\{K_{-\delta+1}[\phi^{1/2}(\phi+2x)^{1/2}] -K_{-\delta-1}[\phi^{1/2}(\phi+2x)^{1/2}]\} \quad (3.31)$$

and use the recurrence relation

$$K_{-\delta+1}[\phi^{1/2}(\phi+2x)^{1/2}]-K_{-\delta-1}[\phi^{1/2}(\phi+2x)^{1/2}] =-2\delta\phi^{-1/2}(\phi+2x)^{-1/2}K_\delta[\phi^{1/2}(\phi+2x)^{1/2}] \quad (3.32)$$

to obtain

$$J(\delta)=-8\int_0^\infty dx\,\delta(\phi+2x)^{-1}K_\delta^2[\phi^{1/2}(\phi+2x)^{1/2}] =-8\delta\int_\phi^\infty dy\,y^{-1}K_\delta^2(y). \quad (3.33)$$

This last integral is readily evaluated[10] to give

$$J(\delta)=4\Big[K_\delta^2(\phi)+\phi\Big(K_{\delta+1}(\phi)\frac{\partial K_\delta(\phi)}{\partial\delta}-K_\delta(\phi)\frac{\partial K_{\delta+1}(\phi)}{\partial\delta}\Big)\Big]. \quad (3.34)$$

Therefore

$$\bar{R}(\delta)=\int_0^\infty d\phi\Big\{\frac{\partial}{\partial\delta}\Big[\phi[K_\delta(\phi)]^{-2}\times\Big(K_{\delta+1}(\phi)\frac{\partial K_\delta(\phi)}{\partial\delta}-K_\delta(\phi)\frac{\partial K_{\delta+1}(\phi)}{\partial\delta}\Big)\Big]+(\phi+1)^{-1}\Big\}. \quad (3.35)$$

We now may use

$$\delta K_\delta(\phi)-dK_\delta(\phi)/d\phi=K_{\delta+1}(\phi) \quad (3.36)$$

to find

$$\bar{R}(\delta)=\int_0^\infty d\phi\Big\{\frac{\partial}{\partial\delta}\Big[\phi[K_\delta(\phi)]^{-2}\times\Big(K_\delta(\phi)\frac{\partial^2K_\delta(\phi)}{\partial\delta\partial\phi}-\frac{\partial K_\delta(\phi)}{\partial\phi}\frac{\partial K_\delta(\phi)}{\partial\delta}\Big)\Big]+(\phi+1)^{-1}\Big\}, \quad (3.37)$$

from which (3.29) is obtained by integrating the first term by parts.

Having verified (3.29), we may now rewrite (3.26) as

$$\langle(d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle\rangle_{E_2}=-C_{01}[R(\delta)+\ln N^2]+K_{01}+o(1), \quad (3.38a)$$

$$\langle(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle\rangle_{E_2}=-C_{10}[R(\delta)+\ln N^2]+K_{10}+o(1). \quad (3.38b)$$

We know from (4.36) of I that as $\delta\to\infty$,

$$R(\delta)+\ln N^2=\ln N^2|\delta|^{-1}+\ln 2-\tfrac{1}{8}\delta^{-2}+O(|\delta|^{-3}), \quad (3.39)$$

so by comparison with (3.16) we obtain

$$K_{01}=k\beta_c\pi^{-1}\{(1+z_{1c}^{-1})[E_1(1-z_{1c})+E_2^0(1+z_{2c}^0)]\times\ln\tfrac{1}{32}(z_{1c}^{-1}+z_{1c})+E_1z_{1c}^2(1-z_{1c}^2)^2\ \mathrm{gd}2\beta_cE_1+4E_2^0\} \quad (3.40a)$$

and

$$K_{10}=k\beta_c^2\pi^{-1}\{(1+z_{2c}^{0-1})[E_1(1-z_{1c})^{-1}E_2^0(1-z_{2c}^0)]\times\ln\tfrac{1}{32}(z_{1c}^{-1}+z_{1c})+E_2^0z_{2c}^{02}(1-z_{2c}^{02})^2\ \mathrm{gd}2\beta_cE_2^0+4E_1\}. \quad (3.40b)$$

Then if we recall that

$$z_{1c}^{-1}z_{2c}^{0-1}(1-z_{2c}^{02})(1-z_{1c}^2)=4+O(N^{-1}) \quad (3.41a)$$

and

$$E_1(z_{1c}^{-1}+z_{1c})+E_2^0(z_{2c}^{0-1}+z_{2c}^0)=(z_{1c}^{-1}+z_{1c})(1-z_{1c})^{-1}\times[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)]+O(N^{-1}), \quad (3.41b)$$

we see that (3.28) is verified.

The analyticity properties of $R(\delta)$ have been studied in I and from that discussion and from (3.38) we conclude that the average nearest-neighbor spin correlation functions are infinitely differentiable functions of T even at T_c, where they possess an essential singularity.

A discussion of any of the other correlation functions requires use of an appropriate many variable distribution function satisfying (2.47). However, as long as the separation between the spins is small compared to N^2, it is possible to study $\langle(d/dT)\langle\sigma_{0,0}\sigma_{0,m}\rangle\rangle_{E_2}$ in terms of $\nu(x)$ alone. Indeed, from the facts that $\nu(x_1,\cdots,x_k)$ is non-negative and that if we integrate over one variable we get the corresponding ν function of one less variable we conclude that if $\nu(x_j;\theta_j)$ is sharply peaked at some value of x_j, $\nu(x_1,\cdots,x_j,\cdots,x_k;\theta_1,\cdots,\theta_j,\cdots,\theta_k)$ as a function of x_j is also sharply peaked. We also know from I that $\nu(x_j;\theta_j)$ is sharply peaked about $x_j=x_0(\theta_j)$ if $|T-T_c|\gg N^{-2}$ or $|\theta_j|\gg N^{-2}$. We therefore conclude that if all θ_j except one are much larger than N^{-2}, then

$$\nu(x_1,\cdots,x_k)\sim\prod_{j=1}^k\nu(x_j), \quad (3.42)$$

an approximation which may be verified by substitution in (2.47). To apply this observation first consider splitting up the integrals in (2.36) which define the inverse matrix elements $A^{-1}(0,k;0,k')_{RL}$ and $A^{-1}(0,k;0,k')_{RR}$ into two parts, one coming from integration over $\theta\gg N^{-2}$ and one from $\theta\sim N^{-2}$. At least as long as

$$|k-k'|\ll N^2 \quad (3.43)$$

the first region will give an order-1 contribution that depends on $k-k'$ and is independent of δ, but the second region's contribution is of order N^{-2}, independent of $|k-k'|$, and does depend on δ. These order-of-magnitude estimates hold for any collection of bonds consistent with (1.1). We therefore may calculate $\langle(d/dT)\times\langle\sigma_{0,0}\sigma_{0,m}\rangle\rangle_{E_2}$ when $\delta=0(1)$ and

$$m\ll N^2 \quad (3.44)$$

from expressions analogous to (2.45) by using the approximation (3.42), since the region in θ_j space where (3.42) fails does not contribute to leading order in N^{-1}. However, it is easily seen that this procedure amounts to replacing inverse matrix elements A^{-1} by $\langle A^{-1}\rangle_{E_2}$ in the formulas for $\langle\langle\sigma_{0,0}\sigma_{0,m}\rangle\rangle_{E_2}$. It is clear from (2.36) and the preceding analysis of this section that

$$\langle A^{-1}(0,k;0,k')_{RL}\rangle_{E_2}=A_1(k-k')+\bar{A}_1N^{-2}R(\delta)+O(N^{-2}) \quad (3.45a)$$

and

$$\langle A^{-1}(0,k;0,k')_{RR}\rangle_{E_2}=0, \quad (3.45b)$$

[10] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. 2, p. 90, Eq. (11).

where A_1 and $\bar{A}_1$ may be computed. A similar analysis can be carried out for the more general case of two spins in different rows and we conclude that the analyticity properties of $\langle (d/dT)\langle \sigma_{0,0}\sigma_{l,m}\rangle\rangle_{E_2}$ at T_c are the same (at least to leading order in N^{-1} when $l^2+m^2 \ll N^4$) as those of the previously considered case $l=0$, $m=1$. More precisely, we may obtain the behavoir of $\langle (d/dT)\langle \sigma_{0,0}\sigma_{l,m}\rangle\rangle_{E_2}$ when $\delta = O(1)$ from its Onsager ($N \to \infty$) limit by replacing $\ln|1-T/T_c|$ by $R(\delta)$ and adding a suitable constant to make the $\delta \to \infty$ behavior in the random lattice match the $T \sim T_c$ behavior of the correlation functions of Onsager's lattice.

4. TWO-VARIABLE INTEGRAL EQUATION

To extract further explicit information from the general formalism of Sec. 2 for the power-law distribution (1.1), we need to study the integral equation (2.47) in the case where at least two of the variables θ_l are of the order N^{-2}. In I we have found that when $\theta \sim N^{-2}$ and $\delta \sim 1$, the integral equation (2.26) for $\nu(x)$ could be approximated by a linear first-order differential equation which could be exactly solved. For the multivariable case (2.47) or (2.49), however, the corresponding approximations lead to a second-order partial differential equation which we are unable to solve exactly. In this section, we therefore concentrate on the two-variable function $\tilde{\nu}(x_1,x_2)$ and study the integral equation (2.49) in detail. Many of our considerations may be taken over to (2.47), but, since they are not needed for the following developments, these more complicated equations will not be further considered.

As remarked in I, the $\mu(\lambda)$ of (1.1), while it does not correspond to a temperature-independent $P(E_2)$, differs when $\delta = O(1)$ from a temperature-independent $P(E_2)$ by negligible terms of order N^{-1}. Similarly, because because T_1-T_2 is infinitesimal, (2.49) for (1.1) is equivalent to

$$\nu(x_1,x_2) = \int_0^1 dy\, N y^{N-1} \int_{-\infty}^{\infty} dx_1' \int_{-\infty}^{\infty} dx_2' \times \delta\left(x_1 - \frac{\lambda_{01}^{-1} x_1'(a_1^2+b_1^2)+a_1 y}{\lambda_{01}^{-1} a_1 x_1' + y} \right) \times \delta\left(x_2 - \frac{\lambda_{02}^{-1} x_2'(a_2^2+b_2^2)+a_2 y}{\lambda_{02}^{-1} a_2 x_2' + y} \right) \nu(x_1',x_2'), \quad (4.1)$$

where for notational convenience all tildes have been omitted and $\lambda_{0j} = \tanh^2 \beta_j E_2^0$. In a manner identical to I, we conclude that $\nu(x_1,x_2)$ vanishes unless

$$a_j x_{0j}|_{\lambda_{0j}=1} < a_j x_j < a_j^2 + b_j^2, \quad (4.2)$$

where x_{0j} is given by (3.5) evaluated at θ_j and T_j. It is also clear that

$$\nu(-x_1,\, x_2;\, -\theta_1,\, \theta_2) = \nu(x_1,x_2;\, \theta_1,\theta_2), \quad (4.3)$$

and similarly for x_2. It is therefore sufficient to consider $a_j > 0$. Then when (4.2) holds, we integrate over y and obtain

$$\nu(x_1,x_2) = \int_{x_{01}}^{\min[\lambda_{01}(x_1-a_1)/(a_1^2+b_1^2-a_1x_1),\,(a_1^2+b_1^2)/a_1]} dx_1' \int_{x_{02}}^{\min[\lambda_{02}(x_2-a_2)/(a_2^2+b_2^2-a_2x_2),\,(a_2^2+b_2^2)/a_2]} dx_2'$$
$$\times N[\lambda_{01}^{-1}x_1'(a_1^2+b_1^2-a_1x_1)/(x_1-a_1)]^{N-1} \lambda_{01}^{-1}x_1' b_1^2 (x_1-a_1)^{-2} \lambda_{02}^{-1} x_2' b_2^2 (x_2-a_2)^{-2}$$
$$\times \delta\left(\lambda_{01}^{-1} x_1' \frac{a_1^2+b_1^2-a_1x_1}{x_1-a_1} - \lambda_{02}^{-1} x_2' \frac{a_2^2+b_2^2-a_2x_2}{x_2-a_2} \right) \nu(x_1',x_2'). \quad (4.4)$$

We transform variables as we did in the one-variable equation of I by defining

$$\eta_j = (x_j - x_{0j})/(\lambda_{0j} x_{0j}^{-1} + x_j) \quad (4.5)$$

and

$$B_j^2 = \lambda_{0j}(x_{0j} - a_j) x_{0j}^{-1} (\lambda_{0j} + a_j x_{0j})^{-1} \quad (4.6)$$

so that

$$0 \le \eta_j \le B_j^2 \le 1. \quad (4.7)$$

Also define $X(\eta_1,\eta_2)$ by

$$\nu(x_1,x_2) = X(\eta_1,\eta_2)(d\eta_1/dx_1)(d\eta_2/dx_2) = X(\eta_1,\eta_2)(\lambda_{01}x_{01}^{-1}+x_{01})(\lambda_{02}x_{02}^{-1}+x_{02})(1-\eta_1)^{-2}(1-\eta_2)^{-2}, \quad (4.8)$$

so that

$$\int_0^{B_1^2} \int_0^{B_2^2} X(\eta_1,\eta_2)\, d\eta_2\, d\eta_1 = 1. \quad (4.9)$$

Then (4.4) may be written as

$$X(\eta_1,\eta_2)\left[\frac{\partial}{\partial\eta_1}\left(\frac{B_1{}^2-\eta_1}{\lambda_{01}{}^{-1}x_{01}B_1{}^2+\eta_1 x_{01}{}^{-1}}\right)^N\right]^{-1}\left[\frac{\partial}{\partial\eta_2}\left(\frac{B_2{}^2-\eta_2}{\lambda_{02}{}^{-1}x_{02}B_2{}^2+\eta_2 x_{02}{}^{-1}}\right)\right]^{-1}$$

$$=\int_0^{\min[B_1{}^2,\eta_1B_1{}^{-2}]}d\eta_1'\int_0^{\min[B_2{}^2,\eta_2B_2{}^{-2}]}d\eta_2'\,X(\eta_1',\eta_2')\lambda_{01}{}^{-N}\lambda_{02}{}^{-1}\left(\frac{x_{01}+\eta_1'\lambda_{01}x_{01}{}^{-1}}{1-\eta_1'}\right)\left(\frac{x_{02}+\eta_2'\lambda_{02}x_{02}{}^{-1}}{1-\eta_2'}\right)$$

$$\times\delta\left[\lambda_{01}{}^{-1}\left(\frac{x_{01}+\eta_1'\lambda_{01}x_{01}{}^{-1}}{1-\eta_1'}\right)\left(\frac{B_1{}^2-\eta_1}{\lambda_{01}{}^{-1}x_{01}B_1{}^2+\eta_1x_{01}{}^{-1}}\right)-\lambda_{02}{}^{-1}\left(\frac{x_{02}+\eta_2'\lambda_{02}x_{02}{}^{-1}}{1-\eta_2'}\right)\left(\frac{B_2{}^2-\eta_2}{\lambda_{02}{}^{-1}x_{02}B_2{}^2+\eta_2x_{02}{}^{-1}}\right)\right]. \quad (4.10)$$

In writing (4.10) we have explicitly used the fact that (1.1) is of power-law form by moving all factors involving η_1 or η_2 to the left-hand side of the equation. Because of this factorization, we may convert (4.10) into a partial-differential difference equation by differentiating along the curves:

$$C=\left(\frac{B_1{}^2-\eta_1}{\lambda_{01}{}^{-1}x_{01}B_1{}^2+\eta_1x_{01}{}^{-1}}\right)\bigg/\left(\frac{B_2{}^2-\eta_2}{\lambda_{02}{}^{-1}x_{02}B_2{}^2+\eta_2x_{02}{}^{-1}}\right). \quad (4.11)$$

Specifically, we apply the operator

$$\frac{(1-B_1{}^{-2}\eta_1)(\lambda_{01}{}^{-1}x_{01}{}^2B_1{}^2+\eta_1)}{\lambda_{01}{}^{-1}x_{01}{}^2+1}\frac{\partial}{\partial\eta_1}+\frac{(1-B_2{}^{-2}\eta_2)(\lambda_{02}{}^{-1}x_{02}{}^2B_2{}^2+\eta_2)}{\lambda_{02}{}^{-2}x_{02}{}^2+1}\frac{\partial}{\partial\eta_2} \quad (4.12)$$

to (4.10) and if $\eta_1<B_1{}^{-4}$ and $\eta_2<B_2{}^{-4}$, we obtain

$$NB_1{}^{-2}B_2{}^{-2}X(\eta_1B_1{}^{-2},\eta_2B_2{}^{-2})$$
$$=(\lambda_{01}{}^{-1}B_1{}^2x_{01}+\eta_1x_{01}{}^{-2})^{-2}(\lambda_{02}{}^{-1}B_2{}^2x_{02}+\eta_2x_{02}{}^{-1})^{-2}\left(\frac{(1-B_1{}^{-2})(\lambda_{01}{}^{-1}x_{01}{}^2B_1{}^2+\eta_1)}{\lambda_{01}{}^{-1}x_{01}{}^2+1}\frac{\partial}{\partial\eta_1}\right.$$
$$\left.+\frac{(1-B_2{}^{-2}\eta_2)(\lambda_{02}{}^{-1}x_{02}{}^2B_2{}^2+\eta_2)}{\lambda_{02}{}^{-1}x_{02}{}^2+1}\frac{\partial}{\partial\eta_2}+N-1\right)[(\lambda_{01}{}^{-1}B_1{}^2x_{01}+\eta_1x_{01}{}^{-1})^2(\lambda_{02}{}^{-1}B_2{}^2x_{02}+\eta_2x_{02}{}^{-1})^2X(\eta_1,\eta_2)]$$
$$=\left(\frac{(1-B_1{}^{-2}\eta_1)(\lambda_{01}{}^{-1}x_{01}{}^2B_1{}^2+\eta_1)}{\lambda_{01}{}^{-1}x_{01}{}^2+1}\frac{\partial}{\partial\eta_1}+\frac{(1-B_2{}^{-2}\eta_2)(\lambda_{02}{}^{-1}x_{02}{}^2B_2{}^2+\eta_2)}{\lambda_{02}{}^{-1}x_{02}{}^2+1}\frac{\partial}{\partial\eta_2}\right.$$
$$\left.+\frac{2(1-B_1{}^{-2}\eta_1)}{\lambda_{01}{}^{-1}x_{01}{}^2+1}+\frac{2(1-B_2{}^{-2}\eta_2)}{\lambda_{02}{}^{-1}x_{02}{}^2+1}+N-2\right)X(\eta_1,\eta_2). \quad (4.13)$$

If we also make the exponential change of variables made in I:

$$\eta_j=e^{-\tau_j},\quad j=1,2 \quad (4.14)$$

and

$$U(\tau_1,\tau_2)=X(\eta_1,\eta_2)(d\eta_1/d\tau_1)(d\eta_2/d\tau_2), \quad (4.15)$$

we obtain

$$N[U(\tau_1,\tau_2)-U(\tau_1+\ln B_1{}^2,\,\tau_2+\ln B_2{}^2)]$$
$$=\frac{\partial}{\partial\tau_1}\left(\frac{(1-B_1{}^{-2}e^{-\tau_1})(B_1{}^2x_{01}{}^2\lambda_{01}{}^{-1}e^{\tau_1}+1)}{(1+\lambda_{01}{}^{-1}x_{01}{}^2)}U(\tau_1,\tau_2)\right)$$
$$+\frac{\partial}{\partial\tau_2}\left(\frac{(1-B_2{}^{-2}e^{-\tau_2})(B_2{}^2x_{02}{}^2\lambda_{02}{}^{-1}e^{\tau_2}+1)}{(1+\lambda_{02}{}^{-1}x_{02}{}^2)}U(\tau_1,\tau_2)\right). \quad (4.16)$$

We know from (2.51) that

$$\int_{-\ln B_2}^{\infty}d\tau_2\,U(\tau_1,\tau_2)=U(\tau_1). \quad (4.17)$$

If we integrate (4.16) over τ_2, we find that if we require

$$\lim_{\tau_1\to\infty}e^{\tau_1}U(\tau_1,\tau_2)=0, \quad (4.18a)$$

and similarly for τ_2

$$\lim_{\tau_2\to\infty}e^{\tau_2}U(\tau_1,\tau_2)=0, \quad (4.18b)$$

then

$$N[U(\tau_1)-U(\tau_1+\ln B_1{}^2)]$$
$$=\frac{\partial}{\partial\tau_1}\left[\frac{(B_1{}^2-e^{-\tau_1})(B_1{}^2x_{01}\lambda_{01}{}^{-1}e^{\tau_1}+1)}{B_1{}^2(1+\lambda_{01}{}^{-1}x_{01}{}^2)}\right]U(\tau_1,\tau_2). \quad (4.19)$$

This is just the τ_1 derivative of (3.28) of I. Therefore, we conclude that the boundary condition (4.18) holds.

So far our analysis has been exact. To make further progress, we must make explicit use of the fact that we are interested in $\theta \sim N^{-2}$ and $|T-T_c| \sim N^{-2}$ by defining, in analogy with I,

$$\theta_j = -8\lambda_0^{-1/2} z_{1c}(1+z_{1c})^{-2} N^2 \theta_j. \quad (4.20)$$

We further know from I that $T=T_c$ if

$$\ln B_j^2(0) = -N^{-1}, \quad (4.21)$$

and so we define δ_j by

$$\lambda_{0j}^{-1}(1-z_1^{(j)})^2(1+z_1^{(j)})^{-2} - e^{-N^{-1}} = \tfrac{1}{2}N^{-2}\delta_j, \quad (4.22)$$

where, as $N\to\infty$, δ is to be of order 1. Explicitly δ is given by (3.1) with $T=T_j$. We note that

$$\ln B_j^2 = -N^{-1}[1-(2N)^{-1}\delta_j] + O(N^{-3}) \quad (4.23a)$$

and

$$x_{0j} \sim \tfrac{1}{4}N^{-1}\lambda_0^{1/2}\theta_j \quad (4.23b)$$

and further define

$$q_j = \tau_j - \ln 4N\phi_j^{-1}, \quad (4.24a)$$

with

$$U(\tau_1,\tau_2) = \hat{U}(q_1,q_2), \quad (4.24b)$$

to find that when $\phi_j = O(1)$ and $\delta_j = O(1)$, (4.16) is approximated by

$$\begin{aligned} &N\hat{U}(q_1,q_2) \\ &\quad -\hat{U}(q_1-N^{-1}[1-(2N)^{-1}\delta_1],\, q_2-N^{-1}[1-(2N)^{-1}\delta_2]) \\ &\quad = \frac{\partial}{\partial q_1}[(1-\tfrac{1}{4}N^{-1}\phi_1 e^{-q_1})(1+\tfrac{1}{4}N^{-1}\phi_1 e^{q_1})\hat{U}(q_1,q_2)] \\ &\qquad + \frac{\partial}{\partial q_2}[(1-\tfrac{1}{4}N^{-1}\phi_2^{-q_2})(1+\tfrac{1}{4}N^{-1}\phi_2 e^{q_2}) \\ &\qquad\qquad \times \hat{U}(q_1,q_2)]. \quad (4.25) \end{aligned}$$

It is now convenient to define

$$p_1 = \tfrac{1}{2}(q_1+q_2), \quad p_2 = \tfrac{1}{2}(q_1-q_2) \quad (4.26a)$$

and

$$V(p_1,p_2)dp_1\, dp_2 = \hat{U}(q_1,q_2)dq_1\, dq_2. \quad (4.26b)$$

Then if we expand

$$\begin{aligned} &2\hat{U}(q_1-N^{-1}[1-\tfrac{1}{2}N^{-1}\delta_1],\, q_2-N^{-1}[1-\tfrac{1}{2}N^{-1}\delta_2]) \\ &\quad = V(p_1-N^{-1}[1-N^{-1}\tfrac{1}{4}(\delta_1+\delta_2)],\, p_2+N^{-2}\tfrac{1}{4}(\delta_1-\delta_2)) \\ &\quad = V(p_1,p_2) - N^{-1}[1-N^{-1}\tfrac{1}{4}(\delta_1+\delta_2)]\frac{\partial V(p_1,p_2)}{\partial p_1} \\ &\qquad + N^{-2}\tfrac{1}{4}(\delta_1-\delta_2)\frac{\partial V(p_1,p_2)}{\partial p_2} + \tfrac{1}{2}N^{-2}\frac{\partial^2}{\partial p_1^2} \\ &\qquad\qquad \times V(p_1,p_2) + O(N^{-3}), \quad (4.27) \end{aligned}$$

we find the approximate partial-differential equation

$$\begin{aligned} &\frac{\partial^2}{\partial p_1^2}V(p_1,p_2) + \frac{\partial}{\partial p_1}[\tfrac{1}{2}(\delta_1+\delta_2) - \tfrac{1}{4}e^{-p_1}(\phi_1 e^{-p_2}+\phi_2 e^{p_2}) \\ &\qquad + \tfrac{1}{4}e^{p_1}(\phi_1 e^{p_2}+\phi_2 e^{-p_2})]V(p_1,p_2) \\ &\quad + \frac{\partial}{\partial p_2}[\tfrac{1}{2}(\delta_1-\delta_2) - \tfrac{1}{4}e^{-p_1}(\phi_1 e^{p_2}-\phi_2 e^{p_2}) \\ &\qquad + \tfrac{1}{4}e^{p_1}(\phi_1 e^{p_2}-\phi_2 e^{-p_2})]V(p_1,p_2) = 0. \quad (4.28) \end{aligned}$$

To complete the determination of our approximation to $V(p_1,p_2)$, we need a set of "boundary conditions" for (4.28). One condition is obviously

$$V(p_1,p_2) \geq 0. \quad (4.29)$$

The others can be obtained by noting that the exact equations (2.51) should also hold in the approximation we are considering. Therefore, we have

$$\begin{aligned} \int_{-\infty}^{\infty} dq_1\, \hat{U}(q_1,q_2) &= \hat{U}(q_2) = \tfrac{1}{2}[K_{\delta_2}(\phi_2)]^{-1} \\ &\quad \times \exp[-\delta_2 q_2 - \tfrac{1}{2}\phi_2(e^{q_2}+e^{-q_2})] \quad (4.30a) \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} dq_2\, \hat{U}(q_1,q_2) &= \hat{U}(q_1) = \tfrac{1}{2}[K_{\delta_1}(\phi_1)]^{-1} \\ &\quad \times \exp[-\delta_1 q_1 - \tfrac{1}{2}\phi_1(e^{q_1}+e^{-q_1})]. \quad (4.30b) \end{aligned}$$

If we integrate (4.28) with respect to q_1, we find

$$\begin{aligned} &\frac{\partial^2}{\partial q_2^2}\hat{U}(q_2) + \frac{\partial}{\partial q_2}[\delta_2 + \tfrac{1}{2}\phi_2(e^{q_2}-e^{-q_2})]\hat{U}(q_2) \\ &\quad = -\tfrac{1}{2}\phi_1[\lim_{q_1\to\infty} e^{q_1}\hat{U}(q_1,q_2) + \lim_{q_1\to-\infty} e^{-q_1}\hat{U}(q_1,q_2)]. \quad (4.31) \end{aligned}$$

Clearly (4.30) will hold only if

$$\lim_{q_1\to\pm\infty} (e^{q_1}+e^{-q_1})\hat{U}(q_1,q_2) = 0, \quad (4.32a)$$

and similarly, by integrating with respect to q_1,

$$\lim_{q_2\to\pm\infty} (e^{q_2}+e^{-q_2})\hat{U}(q_1,q_2) = 0. \quad (4.32b)$$

The function $V(p_1,p_2;\, \delta_1,\delta_2)$ has a useful symmetry property. If $p_j\to -p_j$ and $\delta_j \to -\delta_j$, Eq. (4.28) is left invariant. Furthermore, this transformation leaves the subsidiary conditions (4.30) invariant. Therefore, we conclude that

$$V(-p_1, -p_2;\, -\delta_1, -\delta_2) = V(p_1,p_2;\, \delta_1,\delta_2), \quad (4.33)$$

and similarly for $\hat{U}$.

Thus far, we have paralleled the analysis given in I for the approximate function $U(\tau)$ and (4.28) is analogous to (4.3) of I, the principal difference being that (4.3) of I is a first order ordinary differential equation we may solve exactly, whereas (4.28) is a second-order partial differential equation which we are unable to solve. However, in I we did not need the complete solution of (4.3) in order to study the $N\to\infty$, $\delta\to\infty$, or $\delta\to 0$ behavior of C_v^r. The complete solution was only needed if in addition to these qualitative results we desired to be able to plot C_v^r numerically. Therefore, even though we cannot analytically solve (4.28), we can study the limiting cases:

(i) $\phi_1\to\infty$, $\phi_2\to\infty$ with ϕ_1/ϕ_2 and δ_1 and δ_2 fixed;

(ii) $\delta_1\to\infty$ and $\delta_2\to\infty$ such that δ_1/ϕ_1, δ_2/ϕ_2, and ϕ_1/ϕ_2 are fixed;

(iii) $\phi_1\sim 0$ and $\phi_2\sim 0$, $|\delta_1|^{-1}\sim -\ln\phi_1$, $|\delta_2|^{-1}\sim -\ln\phi_2$.

These cases will allow us to extract important qualitative features of

$$\langle[(d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle]^2\rangle_{E_2} \quad\text{and}\quad \langle\ln S_\infty\rangle_{E_2}.$$

We may study cases (i) and (ii) together by means of a scale transformation. For convenience, first set

$$\phi_1=\phi\,,\quad \phi_2=\kappa\phi \tag{4.34}$$

and write (4.28) as

$$\frac{\partial^2}{\partial p_1^2}V(p_1,p_2)+\frac{\partial}{\partial p_1}\{\tfrac12(\delta_1+\delta_2)-\tfrac14\phi[e^{-p_1}(e^{-p_2}+\kappa e^{p_2})-e^{p_1}(e^{p_2}+\kappa e^{-p_2})]\}V(p_1,p_2)$$
$$+\frac{\partial}{\partial p_2}\{\tfrac12(\delta_1-\delta_2)-\tfrac14\phi[e^{-p_1}(e^{-p_2}-\kappa e^{p_2})-e^{p_1}(e^{p_2}-\kappa e^{-p_2})]\}V(p_1,p_2)=0. \tag{4.35}$$

When $\phi\to\infty$, we expect $V(p_1,p_2)$ to be a sharply peaked function of each of its variables. To be more precise, we define p_1^0 and p_2^0 to be the solutions of

$$\tfrac12(\delta_1+\delta_2)-\tfrac14\phi[e^{-p_1^0}(e^{-p_2^0}+\kappa e^{p_2^0})-e^{p_1^0}(e^{p_2^0}+\kappa e^{-p_2^0})]=0 \tag{4.36a}$$

and

$$\tfrac12(\delta_1-\delta_2)-\tfrac14\phi[e^{-p_1^0}(e^{-p_2^0}-\kappa e^{p_2^0})-e^{p_1^0}(e^{p_2^0}-\kappa e^{-p_2^0})]=0. \tag{4.36b}$$

Adding and subtracting these equations, we find

$$p_1^0+p_2^0=-\operatorname{arcsinh}(\delta_1/\phi) \tag{4.37a}$$

and

$$p_1^0-p_2^0=-\operatorname{arcsinh}(\delta_2/\kappa\phi). \tag{4.37b}$$

We then expand (4.35) about p_1^0 and p_2^0 as

$$\frac{\partial^2}{\partial p_1^2}V(p_1,p_2)+\frac12\frac{\partial}{\partial p_1}\{\phi(1+(\delta_1/\phi)^2)^{1/2}\times[p_1-p_1^0+p_2-p_2^0]+\kappa\phi(1+(\delta_2/\kappa\phi)^2)^{1/2}\times[p_1-p_1^0-p_2+p_2^0]\}V(p_1,p_2)$$
$$+\frac12\frac{\partial}{\partial p_2}\{\phi(1+(\delta_1/\phi)^2)^{1/2}[p_1-p_1^0+p_2-p_2^0]-\kappa\phi(1+(\delta_2/\kappa\phi)^2)^{1/2}[p_1-p_1^0-p_2+p_2^0]\}\times V(p_1,p_2)\sim 0. \tag{4.38}$$

We may now scale ϕ out of this equation by defining

$$p_j-p_j^0=\phi^{-1/2}\rho_j \tag{4.39}$$

and

$$\hat V(\rho_1,\rho_2)=\phi^{-1}V(p_1,p_2). \tag{4.40}$$

We then find

$$\frac{\partial^2}{\partial\rho_1^2}\hat V(\rho_1,\rho_2)+\frac12\frac{\partial}{\partial\rho_1}\{(1+(\delta_1/\phi)^2)^{1/2}(\rho_1+\rho_2)+\kappa(1+(\delta_2/\kappa\phi)^2)^{1/2}(\rho_1-\rho_2)\}\hat V(\rho_1,\rho_2)$$
$$+\frac12\frac{\partial}{\partial\rho_2}\{(1+(\delta_1/\phi)^2)^{1/2}(\rho_1+\rho_2)-\kappa(1+(\delta_2/\kappa\phi)^2)^{1/2}(\rho_1-\rho_2)\}\hat V(\rho_1,\rho_2)=0. \tag{4.41}$$

It is easily verified that a solution of (4.41) which is non-negative is

$$\hat V(\rho_1,\rho_2)=\exp(\alpha_{11}\rho_1^2+\alpha_{12}\rho_1\rho_2+\alpha_{22}\rho_2^2), \tag{4.42}$$

where

$$\alpha_{11}=-\tfrac12\{[1+(\delta_1/\phi)^2]^{1/2}+\kappa[1+(\delta_2/\kappa\phi)^2]^{1/2}\}, \tag{4.43a}$$

$$\alpha_{12}=-\frac{\{[1+(\delta_1/\phi)^2]^{1/2}+\kappa[1+(\delta_2/\kappa\phi)^2]^{1/2}\}^2}{[1+(\delta_1/\phi)^2]^{1/2}-\kappa[1+(\delta_2/\kappa\phi)^2]^{1/2}}, \tag{4.43b}$$

$$\alpha_{22}=-\{[1+(\delta_1/\phi)^2]^{1/2}+\kappa[1+(\delta_2/\kappa\phi)^2]^{1/2}\}\times\left\{\left(\frac{[1+(\delta_1/\phi)^2]^{1/2}+\kappa[1+(\delta_2/\kappa\phi)^2]^{1/2}}{[1+(\delta_1/\phi)^2]^{1/2}-\kappa[1+(\delta_2/\kappa\phi)^2]^{1/2}}\right)^2-\frac12\right\}. \tag{4.43c}$$

It is also clear that

$$\alpha_{11}<0,\quad \alpha_{22}<0, \tag{4.44a}$$

and

$$\alpha_{11}+\alpha_{22}-|\alpha_{12}|<0, \tag{4.44b}$$

so that the subsidiary condition (4.32) will hold. Thus, we conclude that as $\delta\to\infty$,

$$V(p_1,p_2)\sim C\exp\{\phi^{-1}[\alpha_{11}(p_1-p_1^0)^2+\alpha_{12}(p_1-p_1^0)(p_2-p_2^0)+\alpha_{22}(p_2-p_2^0)^2]\}, \tag{4.45a}$$

where C is an appropriate normalization constant. This approximation is valid as it stands for case (ii) and is appropriate for case (i) if we approximate $\operatorname{arcsinh}x\sim x$ in (4.37). A cruder approximation to (4.45a) is

$$V(p_1,p_2)\sim\delta(p_1-p_1^0)\delta(p_2-p_2^0). \tag{4.45b}$$

This approximation is precisely what is expected on the basis of (3.42).

The last case we consider is (iii), which is necessary for the study of analyticity properties near T_c. Because this case is more complicated than the previous cases, we will first present a heuristic analysis which yields a result accurate enough for the applications of this paper. Then we will give justification for this analysis by a more careful calculation that yields a more precise approximation much in the same sense that (4.45a) is a precise justification of (4.45b).

When $\phi\sim 0$, it is convenient to recognize that we are not interested in $V(p_1,p_2)$ for arbitrary $\delta_1-\delta_2$, but only for $\delta_1-\delta_2\sim 0$. To be precise, define

$$\delta=\tfrac{1}{2}(\delta_1+\delta_2) \tag{4.46a}$$

and

$$\epsilon=\tfrac{1}{2}(\delta_1-\delta_2). \tag{4.46b}$$

The only reason we need consider $\epsilon\neq 0$ is that in (2.52) there occurs $\partial^2/\partial\delta_1\partial\delta_2$. Therefore, since

$$\frac{\partial^2}{\partial\delta_1\partial\delta_2}\bigg|_{\delta_1=\delta_2}=\frac{1}{4}\left(\frac{\partial^2}{\partial\delta^2}-\frac{\partial^2}{\partial\epsilon^2}\right)\bigg|_{\epsilon=0},$$

we really need only consider

$$V|_{\epsilon=0},\quad \frac{\partial V}{\partial\epsilon}\bigg|_{\epsilon=0}\quad\text{and}\quad \frac{\partial^2 V}{\partial\epsilon^2}\bigg|_{\epsilon=0}.$$

We may derive differential equations for these derivatives by differentiating (4.35) with respect to ϵ. Therefore, instead of (4.35), we consider the three simpler equations

$$\frac{\partial^2 V_j(p_1,p_2)}{\partial p_1^2}+\frac{\partial}{\partial p_1}\{\delta-\tfrac{1}{4}\phi[e^{-p_1}(e^{-p_2}+\kappa e^{p_2}) - e^{p_1}(e^{p_2}+\kappa e^{-p_2})]\}V_j(p_1,p_2) -\tfrac{1}{4}\phi\frac{\partial}{\partial p_2}[e^{-p_1}(e^{-p_2}-\kappa e^{p_2})-e^{p_1}(e^{p_2}-\kappa e^{-p_2})]V_j(p_1,p_2) =-j\frac{\partial V_{j-1}(p_1,p_2)}{\partial p_2}\quad\text{for}\quad j=0,\,1,\,2,\,\cdots, \tag{4.47}$$

where we define

$$V_j(p_1,p_2)=\frac{\partial^j}{\partial\epsilon^j}V(p_1,p_2)|_{\epsilon=0},\quad j\geq 1 \tag{4.48a}$$

and

$$V_0(p_1,p_2)=V(p_1,p_2)|_{\epsilon=0}, \tag{4.48b}$$

and where there is no confusion, we will often omit the subscript 0. The subsidiary conditions for V_0 are (4.29) and (4.30) with $\epsilon=0$. There is no positivity requirement for $j\geq 1$, but subsidiary conditions similar to (4.30) are obtained by differentiation. Therefore,

$$\int_{-\infty}^{\infty}dq_1\,\hat{U}_j(q_1,q_2)=\tfrac{1}{2}(-1)^j\frac{\partial^j}{\partial\delta^j}\{[K_\delta(\phi_2)]^{-1} \times\exp[-\delta q_2-\tfrac{1}{2}\phi_2(e^{q_2}+e^{-q_2})]\} \tag{4.49a}$$

and

$$\int_{-\infty}^{\infty}dq_2\,\hat{U}_j(q_1,q_2)=\frac{1}{2}\frac{\partial^j}{\partial\delta^j}\{[K_\delta(\phi_1)]^{-1} \times\exp[-\delta q_1-\tfrac{1}{2}\phi_1(e^{q_1}+e^{-q_1})]\}. \tag{4.49b}$$

We begin our considerations of (4.47) for $j=0$ by noting that if we set $\phi=0$, the equation reduces to

$$\frac{\partial^2 V}{\partial p_1^2}+\delta\frac{\partial V}{\partial p_1}=0, \tag{4.50}$$

which has the trivial solutions

$$V(p_1,p_2)=[\text{const}\,e^{-\delta p_1}+\text{const}']f(p_2), \tag{4.51}$$

where $f(p_2)$ is some arbitrary function of p_2. This function cannot be determined merely by considering the region where $p_1=0(1)$ as $\phi\rightarrow 0$. To study it, we need to define

$$p_{1>}=p_1+\ln\tfrac{1}{4}\phi,\quad p_{1<}=p_1-\ln\tfrac{1}{4}\phi, \tag{4.52}$$

with

$$V_>(p_{1>},p_2)=V(p_1,p_2),\quad V_<(p_{1<},p_2)=V(p_1,p_2). \tag{4.53}$$

If we now consider the limit $p_{1>}$ fixed and $\phi\rightarrow 0$, we find

$$\frac{\partial^2 V_>(p_{1>},p_2)}{\partial p_{1>}{}^2}+\frac{\partial}{\partial p_{1>}}[\delta+e^{p_{1>}}(e^{p_2}+\kappa e^{-p_2})]V_>(p_{1>},p_2) +\frac{\partial}{\partial p_2}[e^{p_{1>}}(e^{p_2}-\kappa e^{-p_2})]V_>(p_{1>},p_2)=0. \tag{4.54}$$

Using the fact that

$$\frac{\partial}{\partial x}x\delta(x)=0, \tag{4.55}$$

we see that a solution to (4.54) is

$$V_>(p_{1>},p_2)=\text{const}\,\delta(p_2-\tfrac{1}{2}\ln\kappa) \times\exp[-\delta p_{1>}-2\kappa^{1/2}e^{p_{1>}}]. \tag{4.56a}$$

Similarly, we find a solution

$$V_<(p_{1<},p_2)=\text{const}\,\delta(p_2+\tfrac{1}{2}\ln\kappa) \times\exp[-\delta p_{1<}-2\kappa^{1/2}e^{-p_{1<}}]. \tag{4.56b}$$

Furthermore, it is easily verified that because in our approximation $\kappa^\delta\sim 1$, the constants in (4.56) may be chosen so that the subsidiary condition (4.30) is satisfied.

Loosely speaking, we now use (4.56) as a sort of boundary condition to determine the function $f(p_2)$. More precisely, we note that as $\phi\rightarrow 0$, we obtain (4.50) as an approximation to (4.47) with $j=0$ if

$$|p_1|+a\ln\phi=O(1)\quad\text{for}\quad 0\leq a<1. \tag{4.57}$$

For this range of p_1 we expect solutions of the form (4.51), except $f(p_2)$ may depend on a. We may obtain an equation from (4.47) that will determine this slow variation of $f(p_2)$ with p_1 by remarking that if we let $p_2=\phi x$ and then let $\phi\rightarrow 0$ with x fixed, we obtain an equation with a $\partial/\partial x$ term in addition of the other terms of (4.50). Such an equation describes a function that as a function of p_2 is in some sense localized with a width ϕ. In view of (4.56), it is natural to make such a scale transformation about $p_2=\pm\frac{1}{2}\ln\kappa$. Therefore, we define the scaled variables $p_\pm$:

$$p_2 \mp \tfrac{1}{2} \ln\kappa = \pm p_\pm \tfrac{1}{4}\phi\kappa^{-1/2}(1-\kappa^2), \tag{4.58}$$

where the big upper (or lower) signs go together and $p_\pm > 0$ if $-\frac{1}{2}|\ln\kappa| < p_2 < \frac{1}{2}|\ln\kappa|$. Correspondingly, we define

$$V_\pm(p_1,p_\pm) = \tfrac{1}{4}\phi\kappa^{-1/2}|1-\kappa^2|\,V(p_1,p_2). \tag{4.59}$$

Then from (4.47) we obtain

$$\frac{\partial^2 V_\pm}{\partial p_1{}^2} + \delta\frac{\partial}{\partial p_1}V_\pm - \frac{\partial}{\partial p_\pm}e^{\mp p_1}V_\pm = 0. \tag{4.60}$$

We may solve (4.60) by Laplace transforming on $p_\pm$ and reducing the resulting equation to Bessel's equation. This yields

$$V_\pm = \mathrm{const}_\pm(2\pi i)^{-1}\int_{-i\infty}^{i\infty} ds\, e^{sp_\pm}A_\pm(s)[2(se^{\mp p_1})^{1/2}]^{\pm\delta} \times K_\delta[2(se^{\mp p_1})^{1/2}], \tag{4.61}$$

where we must reject the corresponding solutions involving I_δ to get a bounded function as $\mp p_1 \to +\infty$. In order to determine the function $A(s)$ as we must study how the two functions $V_\pm$, which are valid approximations to V when $\phi \to 0$ only when $p_\pm = O(1)$, are to be connected together through the region where $p_\pm$ is large, and (4.61) is not a valid approximation. This will be done in Appendix A, and we will demonstrate that if

$$A_\pm(s) = s^{\mp\delta}, \tag{4.62}$$

we may choose the constants so that not only may V_+ and V_- be connected together, but also (4.30) may be satisfied. Indeed, if (4.62) holds, we may evaluate[11] (4.61) as

$$V_\pm(p_1,p) = C_\pm p_\pm{}^{-1\pm\delta}\exp(-p_\pm{}^{-1}e^{\mp p_1}). \tag{4.63}$$

To verify that $C_\pm$ may be chosen so that (4.30) holds, we note that when p_1 satisfies (4.57), the exponential factor in (4.63), which causes V to vanish when $p_2 = \pm\frac{1}{2}\ln\kappa$, approaches 1 and may be omitted when p_2 differs from $\pm\frac{1}{2}\ln\kappa$ by order 1 as $\phi \to 0$. Therefore, when

$$\delta^{-1} \sim O(\ln\phi)$$

the precise value of p_1 is irrelevant to order 1, in evaluating (4.30). We further remark that the form (4.63) is guaranteed to break down when $p_\pm \sim O(\phi^{-1})$. In particular, we cannot replace $p_\pm$ integration from zero to $O(\phi^{-1})$ by zero to ∞ because the integral of V_+ or V_- will diverge. One way to piece V_+ and V_- together is to restrict

$$0 < p_\pm < A^{-1},$$

where A is some arbitrary number of order 1 and then write an approximation

$$V(p_1,p_2) \sim 4\phi\kappa^{1/2}|1-\kappa^2|^{-1}[C_+p_+{}^{-1+\delta}\exp(-p_+{}^{-1}e^{-p_1}) + C_-p_-{}^{-1-\delta}\exp(-p_-{}^{-1}e^{p_1})]. \tag{4.64}$$

We then compute

$$\int_{-\infty}^{\infty} dq_1\,\hat{U}(q_1,q_2) = \int_{q_2-\ln\kappa}^{q_2+\ln\kappa} dq_1\,\hat{U}(q_1,q_2) \sim \int_{-\ln\kappa/2}^{\ln\kappa/2} dp_2 V(p_1,p_2)$$
$$= e^{-p_1}[C_+\Gamma(-\delta,\,\phi e^{-p_1}A^{-1}) + C_-\Gamma(\delta,\phi e^{p_1}A^{-1})], \tag{4.65}$$

where $\Gamma(a,x)$ is the incomplete Γ function.[12] Since δ and ϕ are small, we may expand this as

$$e^{-\delta p_1}\delta^{-1}\{C_+[-1+(\phi e^{-p_1}A^{-1})^{-\delta}] + C_-[1-(\phi e^{p_1}A^{-1})^{\delta}]\}. \tag{4.66}$$

Because $\delta \sim O[(\ln\phi)^{-1}]$, $A^\delta \sim 1$, so that the precise value of our order-1 cutoff A is immaterial, but we must keep the first two terms in the expansion of the incomplete Γ function. Then if we choose

$$C_+ = C\phi^\delta, \quad C_- = C\phi^{-\delta},$$

with

$$C = \delta^2(\phi^{-\delta}-\phi^\delta)^{-2}, \tag{4.67}$$

(4.30) is satisfied.

Approximation (4.64) is sharply peaked about $p_2 = \pm\frac{1}{2}\ln\kappa$. When p_1 satisfies (4.57), the width of the peak at $-\frac{1}{2}\ln\kappa$ is roughly ϕ^{-1+a} and thus spreads out as p_1 increases. Similarly, the peak at $\frac{1}{2}\ln\kappa$ has a width ϕ^{-1-a} and thus spreads out as p_1 decreases. When $p_1 \sim \pm\ln\phi$, the narrowing peak joins on to the δ function previously found. We can study this joining process in more detail by retaining a term in (4.47) that we dropped in obtaining (4.54) and consider

$$\frac{\partial^2 V_{>+}(p_{1>},\bar{p}_{2+})}{\partial^2 p_{1>}{}^2} + \frac{\partial}{\partial p_{1>}}(\delta + \tfrac{1}{2}\kappa^{1/2}e^{p_{1>}})V_{>+}(p_{1>},\bar{p}_{2+}) + \frac{\partial}{\partial \bar{p}_{2+}}(e^{p_1}2\kappa^{1/2}\bar{p}_{2+} - e^{-p_1})V_{>+}(p_{1>},\bar{p}_{2+}) = 0, \tag{4.68}$$

where $\bar{p}_{2+}$ is defined by

$$(\tfrac{1}{4}\phi)^2\kappa^{-1/2}(1-\kappa^2)\bar{p}_{2+} = p_2 - \tfrac{1}{2}\ln\kappa. \tag{4.69}$$

However, the details given by this equation are irrelevant for our purposes. It is also necessary to study how the spreading δ function disappears when $p_1 \pm \ln\phi = O(1)$. This will be done in Appendix A. We may summarize these considerations by the cruder approximation

$$V_0(p_1,p_2) = \tfrac{1}{2}\delta[\phi^{-\delta}-\phi^\delta]^{-1}e^{-\delta p_1} \times [\delta(p_2 - \tfrac{1}{2}\ln\kappa) + \delta(p_2 + \tfrac{1}{2}\ln\kappa)] \tag{4.70}$$

[11] *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. 1, p. 283.

[12] Reference 10, Vol. 2, p. 133.

when (4.57) holds and zero otherwise. This is the approximation that is used in the sequel.

To obtain an approximation for V_1 which is valid at the same level of accuracy as (4.70), we first remark that when $\kappa=1$, V_1 may be exactly computed. We know that if $\kappa=1$,

$$V_0=\tfrac{1}{2}[K_\delta(\phi)]^{-1}\delta(p_2)\exp[-\delta p_1-\tfrac{1}{2}\phi(e^{p_1}+e^{-p_1})], \quad (4.71)$$

so that when $j=1$, Eq. (4.47) becomes

$$\frac{\partial^2 V_1}{\partial p_1^2}+\frac{\partial}{\partial p_1}[\delta+\tfrac{1}{4}\phi(e^{p_1}-e^{-p_1})(e^{p_2}+e^{-p_2})]V_1-\tfrac{1}{4}\phi\frac{\partial}{\partial p_2}\times[(e^{p_1}+e^{-p_1})(e^{p_2}-e^{-p_2})]V_1=-\tfrac{1}{2}[K_\delta(\phi)]^{-1}\times\delta'(p_2)\exp[-\delta p_1-\tfrac{1}{2}\phi(e^{p_1}+e^{-p_1})]. \quad (4.72)$$

Using

$$\frac{\partial}{\partial x}x\delta'(x)=\delta'(x), \quad (4.73)$$

we find a particular solution to (4.72):

$$V_1(p_1,p_2)_{\kappa=1}=\tfrac{1}{2}[K_\delta(\phi)]^{-1}\delta'(p_2)f_1(p_1), \quad (4.74)$$

where

$$\frac{d^2f_1}{dp_1^2}+[\delta+\tfrac{1}{2}\phi(e^{p_1}-e^{-p_1})]\frac{df_1}{dp_1}=-\exp[-\delta p_1+\tfrac{1}{2}\phi(e^{p_1}+e^{-p_1})]. \quad (4.75)$$

To this particular solution may be added a solution of the homogeneous equation corresponding to (4.72). However, we note that in (4.35) the term proportional to ϵ may be neglected when $|\epsilon|\ll p_\pm$. Therefore, for all κ as $\phi\to 0$, V_j will be concentrated along the lines $p_2=\pm\frac{1}{2}\ln\kappa$ just as V_0 is. In the case $\kappa=1$ this means that the only solution of the homogeneous equation that is allowed which is less singular at $p_2=0$ than (4.47) is proportional to $V_0(p_1,p_2)_{\kappa=1}$. It is also clear that

$$\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\, V_j(p_1,p_2)=0, \quad j\geq 1. \quad (4.76)$$

Clearly, (4.74) satisfies this condition but $V_0(p_1,p_2)$ does not, and hence (4.74) is correct as it stands.

The general solution of (4.75) is

$$f_1=\int_{p_1}^{\infty}d\bar{p}_1\,\bar{p}_1\exp[-\delta\bar{p}_1-\tfrac{1}{2}\phi(e^{\bar{p}_1}+e^{-\bar{p}_1})]+C_1\exp[-\delta p_1-\tfrac{1}{2}\phi(e^{\bar{p}_1}+e^{-\bar{p}_1})]+C_2\int_{p_1}^{\infty}d\bar{p}_1\times\exp[-\delta\bar{p}_1-\tfrac{1}{2}\phi(e^{\bar{p}_1}+e^{-p_1})]. \quad (4.77)$$

The constants C_1 and C_2 are determined by subsidiary conditions (4.49), and we find

$$V_1(p_1,p_2)|_{\kappa=1}=-\delta'(p_2)\frac{\partial}{\partial\delta}\left\{\frac{1}{2}[K_\delta(\phi)]^{-1}\int_{p_1}^{\infty}d\bar{p}_1\times\exp[-\delta\bar{p}_1-\tfrac{1}{2}\phi(e^{\bar{p}_1}+e^{-\bar{p}_1})]\right\}. \quad (4.78)$$

In the general case when (4.57) holds (4.47) may be approximated as

$$\frac{\partial^2 V_1}{\partial p_1^2}+\delta\frac{\partial V_1}{\partial p_1}=-\delta(\phi^{-\delta}-\phi^{\delta})^{-1}e^{-\delta p_1}\times\tfrac{1}{2}[\delta'(p_2-\tfrac{1}{2}\ln\kappa)+\delta'(p_2+\tfrac{1}{2}\ln\kappa)]. \quad (4.79)$$

This equation has the particular solution

$$V_1(p_1,p_2)=\delta[\phi^{-\delta}-\phi^{\delta}]^{-1}f_1(p_1)\times\tfrac{1}{2}[\delta'(p_2-\tfrac{1}{2}\ln\kappa)+\delta'(p_2+\tfrac{1}{2}\ln\kappa)], \quad (4.80)$$

where $f_1(p_1)$ satisfies

$$\frac{d^2f_1}{dp_1^2}+\delta\frac{df_1}{dp_1}=-e^{-\delta p_1}. \quad (4.81)$$

As above, we reject any solutions of the homogeneous equation corresponding to (4.79). The general solution of (4.81) is

$$f_1(p_1)=\delta^{-1}p_1e^{-\delta p_1}+C_3e^{-\delta p_1}+C_4. \quad (4.82)$$

The constants C_3 and C_4 may be determined by (a) use of the subsidiary conditions (4.49) and (b) the requirement that $f_1(p_1)$ be finite when $\delta\to 0$. We then obtain

$$V_1(p_1,p_2)\sim-\frac{\partial}{\partial\delta}\{[e^{-\delta p_1}-\tfrac{1}{2}(\phi^{\delta}+\phi^{-\delta})](\phi^{-\delta}-\phi^{\delta})^{-1}\}\times\tfrac{1}{2}[\delta'(p_2-\tfrac{1}{2}\ln\kappa)+\delta'(p_2+\tfrac{1}{2}\ln\kappa)]. \quad (4.83)$$

This form makes manifest the symmetry property (4.33).

When $p_1+\ln\phi=O(1)$, approximation (4.83) breaks down. In this case (4.47) may be approximated as

$$\frac{\partial^2 V_1}{\partial p_1^2}+\frac{\partial}{\partial p_1}\{\delta+\tfrac{1}{4}\phi e^{p_1}(e^{p_2}+\kappa e^{-p_2})\}V_1+\tfrac{1}{4}\phi\frac{\partial}{\partial p_2}\times e^{p_1}(e^{p_2}-\kappa e^{-p_2})V_1=-\delta[\phi^{-\delta}-\phi^{\delta}]^{-1}\delta'(p_2-\tfrac{1}{2}\ln\kappa)\times\exp[-\delta p_1-\tfrac{1}{2}\phi\kappa^{1/2}e^{p_1}]. \quad (4.84)$$

Proceeding as before, we reject the solutions of the complementary equation that are proportional to $\delta(p_2-\frac{1}{2}\ln\kappa)$ and find that the unique solution of (4.84) which satisfies (4.49) is

$$V_1(p_1,p_2)\sim-\frac{\partial}{\partial\delta}\delta[\phi^{-\delta}-\phi^{\delta}]^{-1}\int_{p_1}^{\infty}d\bar{p}_1 \times\exp[-\delta\bar{p}_1-\tfrac{1}{2}\phi\kappa^{1/2}e^{\bar{p}_1}]\delta'(p_2-\tfrac{1}{2}\ln\kappa). \quad (4.85)$$

This solution vanishes when $p_1+\ln\phi\gg O(1)$. An analogous solution holds when $-p_1+\ln\phi=O(1)$. Therefore, we conclude that when (4.57) does not hold, V_1 may be approximated by zero. Finally, we may verify the consistency of our approximations by noting that when $\kappa=1$, (4.83) and (4.85) reduce to (4.78).

The functions $V_2(p_1,p_2)$ is more complicated than V_1. Indeed, we cannot explicitly solve (4.47) when $j=2$ even with $\kappa=1$. However, for the very limited purposes of Sec. 5, it will suffice to use approximation (4.83) for V_1 to find the corresponding approximation for V_2 when (4.57) holds. In this approximation (4.47) becomes

$$\frac{\partial^2 V_2}{\partial p_1^2}+\delta\frac{\partial V_2}{\partial p_1}=\frac{\partial}{\partial\delta}\{[e^{-\delta p_1}-\tfrac{1}{2}(\phi^{\delta}+\phi^{-\delta})](\phi^{-\delta}-\phi^{\delta})^{-1}\} \times[\delta''(p_2-\tfrac{1}{2}\ln\kappa)+\delta''(p_2+\tfrac{1}{2}\ln\kappa)]. \quad (4.86)$$

As before, we reject the solutions of the corresponding homogeneous equation that as a function of p_2 are less singular $\delta''(p_2)$ and write

$$V_2(p_1,p_2)\sim\tfrac{1}{2}[\delta''(p_2-\tfrac{1}{2}\ln\kappa)+\delta''(p_2+\tfrac{1}{2}\ln\kappa)]f_2(p_1), \quad (4.87)$$

where

$$\frac{d^2f_2}{dp_1^2}+\delta\frac{df_2}{dp_1}=\frac{\partial}{\partial\delta}\{[e^{-\delta p_1}-\tfrac{1}{2}(\phi^{2}+\phi^{-\delta})](\phi^{-\delta}-\phi^{\delta})^{-1}\}. \quad (4.88)$$

Requiring that (4.49) be satisfied and that f_2 be finite when $\delta\to 0$, we find

$$V_2(p_1,p_2)\sim\frac{\partial^2}{\partial\delta^2} \times\{\delta^{-1}[e^{-\delta p_1}-\tfrac{1}{2}(\phi^{\delta}+\phi^{-\delta})(1-\delta p_1)](\phi^{-\delta}-\phi^{\delta})^{-1}\} \times\tfrac{1}{2}[\delta''(p_2-\tfrac{1}{2}\ln\kappa)+\delta''(p_2+\tfrac{1}{2}\ln\kappa)]. \quad (4.89)$$

Finally, an analysis similar to that leading to (4.85) allows us to conclude that for our purposes $V_2(p_1,p_2)$ may be approximated by zero when (4.57) does not hold.

To summarize, approximations (4.70), (4.83), and (4.89) are the results of this analysis of case (iii) that will be used in Secs. 5 and 6.

5. VARIANCE OF $(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle$

After the preliminaries of the previous section we may now study the variance of $(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle$ when $T-T_c=O(N^{-2})$. From (2.52) and (2.31) we have

$$\langle[(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle]^2\rangle_{E_2}-\langle(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle\rangle_{E_2}{}^2$$
$$=(2\pi)^{-2}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2\int_0^{\infty}dE_2\,P(E_2)\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int_{-\infty}^{\infty}d\bar{x}_1\int_{-\infty}^{\infty}d\bar{x}_2\frac{\partial^2}{\partial T_1\partial T_2}$$
$$\times\{[\bar{\nu}(x_1,x_2)\bar{\nu}(\bar{x}_1,\bar{x}_2)-\nu(x_1;\,T_1)\nu(x_2;\,T_2)\nu(\bar{x}_1;\,T_1)\nu(\bar{x}_2;\,T_2)][(1-z_2^{(1)2})z_2^{(1)}(z_2^{(1)2}+x_1\bar{x}_1)^{-1}+z_2^{(1)}]$$
$$\times[(1-z_2^{(2)2})z_2^{(2)}(z_2^{(2)2}+x_2\bar{x}_2)^{-1}+z_2^{(2)}]\}. \quad (5.1)$$

If we define

$$W(\xi_j;\,T_j)d\xi_j=\hat{U}(q_j)dq_j, \quad (5.2)$$

where $W(\xi)$ is given by (3.11) and

$$\xi=e^{-q}, \quad (5.3)$$

and if we use (4.5), (4.14), (4.23), and (4.24), we find that (5.1) may be written, when $\delta=0(1)$, as

$$C_{10}{}^2\int_0^{\infty}d\phi_1\int_0^{\infty}d\phi_2\bigg(\frac{\partial^2}{\partial\delta_1\partial\delta_2}\int_{-\infty}^{\infty}dq_1\int_{-\infty}^{\infty}dq_2\int_{-\infty}^{\infty}d\bar{q}_1\int_{-\infty}^{\infty}d\bar{q}_2[\hat{U}(q_1,q_2)\hat{U}(\bar{q}_1,\bar{q}_2)-\hat{U}(q_1)\hat{U}(q_2)\hat{U}(\bar{q}_1)\hat{U}(\bar{q}_2)]$$
$$\times\tanh\tfrac{1}{2}(q_1+\bar{q}_1)\tanh\tfrac{1}{2}(q_2+\bar{q}_2)\bigg)_{\delta_1=\delta_2=\delta}+o(1). \quad (5.4)$$

In the upper limit in the ϕ_j integrations we have been allowed to replace N^2 by ∞ because of (3.42). Furthermore there is no additive constant in (5.4) because as $\delta\to\infty$ this variance must vanish and it is easily seen using (4.45) that this is indeed the case. An identical expression holds for the variance of $(d/dT)\langle\sigma_{0,0}\sigma_{0,1}\rangle$ if we replace C_{10} by C_{01}. These variances are of order 1, which is to be contrasted with the average values which are of order $\ln N^2$. We also see from (4.33) that these variances like, the average values, are even functions of δ.

To study the analyticity of (5.4) near $\delta=0$ it is useful to consider the average and the second moments of $(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle$ separately. The behavior of the average is easily obtained if we recall the behavior of $R(\delta)$ near

$\delta=0$ studied in Sec. 4 of I. We then see that

$$\langle (d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle\rangle_{E_2} \doteqdot C_{10}\frac{\partial^2}{\partial\delta^2}[\ln 2\delta-\psi(|2\delta|^{-1})]=4C_{10}\sum_{n=1}^{\infty}B_{2n}(2n-1)(2\delta)^{2(n-1)}, \tag{5.5}$$

where $\doteqdot$ is defined to mean that the most singular terms of both sides are the same as $\delta\to 0$, $\psi(z)=\Gamma'(z)/\Gamma(z)$, and B_{2n} are the Bernoulli numbers.

We obtain the corresponding approximation to the second moment of $(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle$ by remarking that the singularities in (5.4) occur only at $\delta=0$ and may be obtained by restricting the integration variables ϕ, ϕ_2 to the region

$$0\le\phi_1<\epsilon,\quad 0\le\phi_2<\epsilon, \tag{5.6}$$

where ϵ is some small positive constant. Therefore we study

$$\int_0^\epsilon d\phi_1\int_0^\epsilon d\phi_2\frac{\partial^2}{\partial\delta_1\partial\delta_2}\int_{-\infty}^{\infty}dq_1\int_{-\infty}^{\infty}dq_2\int_{-\infty}^{\infty}d\bar{q}_1\int_{-\infty}^{\infty}d\bar{q}_2\hat{U}(q_1,q_2)\hat{U}(\bar{q}_1,\bar{q}_2)\tanh\tfrac{1}{2}(q_1+\bar{q}_1)\tanh\tfrac{1}{2}(q_2+\bar{q}_2)|_{\delta_1=\delta_2=\delta}$$

$$=\frac{1}{4}\int_0^\epsilon d\phi_1\int_0^\epsilon d\phi_2\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{\infty}d\bar{p}_1\int_{-\infty}^{\infty}d\bar{p}_2\bigg(\frac{\partial^2}{\partial\delta_2}[V_0(p_1,p_2)V_0(\bar{p}_1,\bar{p}_2)]-2V_1(p_1,p_2)V_1(\bar{p}_1,\bar{p}_2)$$
$$-2V_0(p_1,p_2)V_2(\bar{p}_1,\bar{p}_2)\bigg)\tanh\tfrac{1}{2}(p_1+p_2+\bar{p}_1+\bar{p}_2)\tanh\tfrac{1}{2}(p_1-p_2+\bar{p}_1-\bar{p}_2). \tag{5.7}$$

To obtain the most singular part of this expression we use approximations (4.70), (4.83), and (4.87). Consider the first term in (5.7) separately and write

$$\frac{1}{4}\int_0^\epsilon d\phi_1\int_0^\epsilon d\phi_2\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{\infty}d\bar{p}_1\int_{-\infty}^{\infty}d\bar{p}_2\frac{\partial^2}{\partial\delta^2}$$
$$\times[V_0(p_1,p_2)V_0(\bar{p}_1,\bar{p}_2)]\tanh\tfrac{1}{2}(p_1+p_2+\bar{p}_1+\bar{p}_2)\tanh\tfrac{1}{2}(p_1-p_2+\bar{p}_1-\bar{p}_2)$$
$$\doteqdot\frac{1}{16}\frac{\partial^2}{\partial\delta^2}\int_0^\epsilon d\phi_1\int_0^\epsilon d\phi_2\int_{\ln\phi}^{-\ln\phi}dp_1\int_{\ln\phi}^{-\ln\phi}d\bar{p}_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{\infty}d\bar{p}_2e^{-\delta(p_1+\bar{p}_1)}\delta^2(\phi^{-\delta}-\phi^{\delta})^{-2}[\delta(p_2-\tfrac{1}{2}\ln\kappa)+\delta(p_2+\tfrac{1}{2}\ln\kappa)]$$
$$\times[\delta(\bar{p}_2-\tfrac{1}{2}\ln\kappa)+\delta(\bar{p}_2+\tfrac{1}{2}\ln\kappa)][\tanh\tfrac{1}{2}(p_1+p_2+\bar{p}_1+\bar{p}_2)\tanh\tfrac{1}{2}(p_1-p_2+\bar{p}_1-\bar{p}_2)-1], \tag{5.8}$$

where the -1 in the last bracket has been added to make the bracket vanish as $p_1+\bar{p}_1\to\pm\infty$ but makes no contribution to the final answer. It is now convenient to reexpress ϕ and κ in terms of r and α, where

$$\phi_1=r\cos\alpha,\quad \phi_2=r\sin\alpha, \tag{5.9}$$

and we recall $\kappa^\delta\sim 1$. We may also do the p_2 and $\bar{p}_2$ integrals to find that the most singular terms in (5.8) are given by

$$\frac{1}{8}\frac{\partial^2}{\partial\delta^2}\int_0^\epsilon dr\, r\delta^2(r^{-\delta}-r^\delta)^{-2}\int_0^{\pi/2}d\alpha\int_{\ln r}^{-\ln r}dp_1\int_{\ln r}^{-\ln r}d\bar{p}_1$$
$$\times e^{-\delta(p_1+\bar{p}_1)}[\tanh\tfrac{1}{2}(p_1+\bar{p}_1+\ln\tan\alpha)\tanh\tfrac{1}{2}(p_1+\bar{p}_1-\ln\tan\alpha)+\tanh^2\tfrac{1}{2}(p_1+\bar{p}_1)-2]. \tag{5.10}$$

Let

$$t_+=\tfrac{1}{2}(p_1+\bar{p}_1)$$

and

$$t_-=\tfrac{1}{2}(p_1-\bar{p}_1) \tag{5.11}$$

and do the t_- integration. In the remaining t_+ integral the integrand vanishes exponentially as $t_+\to+\infty$, so we may replace $e^{-2\delta t_+}$ by one and the limit $-\ln r$ by ∞ to obtain

$$Q_0\frac{\partial^2}{\partial\delta^2}\int_0^\epsilon dr\delta^2\, r(r^{-\delta}-r^\delta)^{-2}\ln r+Q_1\frac{\partial^2}{\partial\delta^2}\int_0^\epsilon dr\delta^2\, r(r^{-\delta}-r^\delta)^{-2}, \tag{5.12}$$

where

$$Q_j = -\frac{1}{2}\int_0^{\pi/2} d\alpha \int_0^{\infty} dt\, t^j[\tanh(t+\tfrac{1}{2}\ln\tan\alpha)\tanh(t-\tfrac{1}{2}\ln\tan\alpha)+\tanh^2 t-2]. \tag{5.13}$$

The first r integral in (5.12) may be written as

$$\int_0^{\epsilon} dr\, r\ln r(r^{-\delta}-r^{\delta})^{-2} = -\frac{1}{8}\frac{\partial}{\partial|\delta|}\int_0^{\epsilon} dr (r^{-\delta/2}+r^{\delta/2})(r^{-\delta/2}-r^{\delta/2})^{-1}. \tag{5.14}$$

This integral is the same as the one studied in Sec. 4 of I and we may use that analysis to find

$$\int_0^{\epsilon} dr\, r\ln r(r^{-\delta}-r^{\delta})^{-2} \doteq -\frac{1}{4}\frac{\partial}{\partial|\delta|}|\delta|^{-1}[\ln|\delta|^{-1}-\psi(|\delta|^{-1})]. \tag{5.15}$$

Furthermore, if we integrate the second term in (5.12) by parts,

$$\delta^2\int_0^{\epsilon} dr\, r(r^{-\delta}-r^{\delta})^{-2} \doteq -\tfrac{1}{2}\delta\int_0^{\epsilon} dr\, r(r^{-\delta}+r^{\delta})(r^{-\delta}-r^{\delta}) \doteq -\tfrac{1}{2}[\ln|\delta|^{-1}-\tfrac{1}{2}|\delta|-4(|\delta|^{-1})]. \tag{5.16}$$

This is clearly less singular than (5.15), so the most singular term in (5.12) is

$$Q_0\frac{1}{4}\frac{\partial^2}{\partial\delta^2}\delta^2\frac{\partial}{\partial|\delta|}|\delta|^{-1}[\ln|\delta|^{-1}-\psi(|\delta|^{-1})]. \tag{5.17}$$

The analyticity of the remaining two terms of (5.7) may be investigated together using the relation (which is easily demonstrated by integrating by parts twice)

$$\begin{aligned}\int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} d\bar{p}_2[\delta'(p_2-\tfrac{1}{2}\ln\kappa)+\delta'(p_2+\tfrac{1}{2}\ln\kappa)]&\\ \times[\delta'(\bar{p}_2-\tfrac{1}{2}\ln\kappa)+\delta'(\bar{p}_2+\tfrac{1}{2}\ln\kappa)]\tanh\tfrac{1}{2}(p_1+\bar{p}_1+p_2+\bar{p}_2)\tanh\tfrac{1}{2}(p_1+\bar{p}_1-\bar{p}_2-\bar{p}_2)&\\ =\int_{-\infty}^{\infty} dp_2\int_{-\infty}^{\infty} d\bar{p}_2[\delta(p_2-\tfrac{1}{2}\ln\kappa)+\delta(p_2+\tfrac{1}{2}\ln\kappa)][\delta''(\bar{p}_2-\tfrac{1}{2}\ln\kappa)+\delta''(\bar{p}_2+\tfrac{1}{2}\ln\kappa)]&\\ \times\tanh\tfrac{1}{2}(p_1+\bar{p}_1+p_2+\bar{p}_2)\tanh\tfrac{1}{2}(p_1+p_1-\bar{p}_2-\bar{p}_2).&\end{aligned} \tag{5.18}$$

Therefore we define I as

$$\begin{aligned}I=&-\frac{1}{2}\int_0^{\epsilon} d\phi_1\int_0^{\epsilon} d\phi_2\int_{-\infty}^{\infty} dp_1\int_{-\infty}^{\infty} dp_2\int_{-\infty}^{\infty} d\bar{p}_1\int_{-\infty}^{\infty} d\bar{p}_2[V_1(p_1,p_2)V_1(\bar{p}_1,\bar{p}_2)+V_0(p_1,p_2)V_2(\bar{p}_1,\bar{p}_2)]\\ &\times\tanh\tfrac{1}{2}(p_1+\bar{p}_1+p_2+\bar{p}_2)\tanh\tfrac{1}{2}(p_1+\bar{p}_1-p_2-\bar{p}_2)\\ =&-\frac{1}{2}\int_0^{\epsilon} d\phi_1\int_0^{\epsilon} d\phi_2\int_{\ln\phi}^{-\ln\phi} dp_1\int_{\ln\phi}^{-\ln\phi} d\bar{p}_1\bigg(\frac{\partial}{\partial\delta}\{[e^{-\delta p_1}-\tfrac{1}{2}(\phi^{\delta}+\phi^{-\delta})](\phi^{-\delta}-\phi^{\delta})^{-1}\}\\ &\times\frac{\partial}{\partial\delta}\{[e^{-\delta\bar{p}_1}-\tfrac{1}{2}(\phi^{\delta}+\phi^{-\delta})](\phi^{-\delta}-\phi^{\delta})^{-1}\}+\delta e^{-\delta p_1}(\phi^{-\delta}-\phi^{\delta})^{-1}\frac{\partial^2}{\partial\delta^2}\{\delta^{-1}[e^{-\delta\bar{p}_1}-\tfrac{1}{2}(\phi^{\delta}+\phi^{-\delta})(1-\delta\bar{p}_1)](\phi^{-\delta}-\phi^{\delta})^{-1}\}\bigg)\\ &\times\int_{-\infty}^{\infty} dp_2\int_{-\infty}^{\infty} d\bar{p}_2\,\tfrac{1}{2}[\delta'(p_2-\tfrac{1}{2}\ln\kappa)+\delta'(p_2+\tfrac{1}{2}\ln\kappa)]\tfrac{1}{2}[\delta'(\bar{p}_2-\tfrac{1}{2}\ln\kappa)+\delta'(\bar{p}_2+\tfrac{1}{2}\ln\kappa)]\\ &\times\tanh\tfrac{1}{2}(p_1+\bar{p}_1+p_2+\bar{p}_2)\tanh\tfrac{1}{2}(p_1+\bar{p}_1-p_2-\bar{p}_2).\end{aligned} \tag{5.19}$$

We now rewrite the second term in (5.19) by writing

$$\delta e^{-\delta p_1}(\phi^{-\delta}-\phi^{\delta})^{-1} = -\frac{\partial}{\partial p_1}\{[e^{-\delta p_1}-\tfrac{1}{2}(\phi^{\delta}+\phi^{-\delta})](\phi^{-\delta}-\phi^{\delta})^{-1}\}, \tag{5.20}$$

integrating by parts on p_1, transferring the derivative in the resulting expression to $\bar{p}_1$, and integrating by parts again. This yields

$$I \doteq I_1+I_2+I_3, \tag{5.21}$$

with

$$I_1=-\frac{1}{4}\frac{\partial^2}{\partial\delta^2}\int_0^\epsilon dr\, r\int_0^{\pi/2} d\alpha\int_{\ln r}^{-\ln r} dp_1\int_{\ln r}^{-\ln r} d\bar{p}_1[e^{-\delta p_1}-\tfrac{1}{2}(r^\delta+r^{-\delta})][e^{-\delta\bar{p}_1}-\tfrac{1}{2}(r^\delta+r^{-\delta})](r^{-\delta}-r^\delta)^{-2}$$
$$\times\int_{-\infty}^{\infty} dp_2\int_{-\infty}^{\infty} d\bar{p}_2\, \tfrac{1}{2}[\delta'(p_2-\tfrac{1}{2}\ln\tan\alpha)+\delta'(p_2+\tfrac{1}{2}\ln\tan\alpha)]\tfrac{1}{2}[\delta'(\bar{p}_2-\tfrac{1}{2}\ln\tan\alpha)+\delta'(\bar{p}_2+\tfrac{1}{2}\ln\tan\alpha)]$$
$$\times\tanh\tfrac{1}{2}(p_1+\bar{p}_1+p_2+\bar{p}_2)\tanh\tfrac{1}{2}(p_1+\bar{p}_1-p_2-\bar{p}_2), \tag{5.22a}$$

$$I_2=\frac{1}{4}\int_0^\epsilon dr\, r\int_0^{\pi/2} d\alpha\int_{\ln r}^{-\ln r} d\bar{p}_1\frac{\partial^2}{\partial\delta^2}\{\delta^{-1}[e^{-\delta\bar{p}_1}-\tfrac{1}{2}(r^\delta+r^{-\delta})(1-\delta\bar{p}_1)](r^{-\delta}-r^\delta)^{-1}\}$$
$$\times\int_{-\infty}^{\infty} dp_2\int_{-\infty}^{\infty} d\bar{p}_2\, \tfrac{1}{2}[\delta'(p_2-\tfrac{1}{2}\ln\tan\alpha)+\delta'(p_2+\tfrac{1}{2}\ln\tan\alpha)]$$
$$\times\tfrac{1}{2}[\delta'(\bar{p}_2-\tfrac{1}{2}\ln\tan\alpha)+\delta'(\bar{p}_2+\tfrac{1}{2}\ln\tan\alpha)][\tanh\tfrac{1}{2}(-\ln r+p_2+\bar{p}_1+\bar{p}_2)\tanh\tfrac{1}{2}(-\ln r-p_2+\bar{p}_1-\bar{p}_2)$$
$$+\tanh\tfrac{1}{2}(\ln r+p_2+\bar{p}_1+\bar{p}_1)\tanh\tfrac{1}{2}(\ln r-p_2+\bar{p}_1-\bar{p}_2)], \tag{5.22b}$$

and

$$I_3=-\frac{1}{4}\int_0^\epsilon dr\, r\int_0^{\pi/2} d\alpha\int_{\ln r}^{-\ln r} dp_1\frac{\partial^2}{\partial\delta^2}\{\delta^{-1}[1+\delta(r^{-\delta}+r^\delta)(r^{-\delta}-r^\delta)^{-1}\ln r]\}[e^{-\delta p_1}-\tfrac{1}{2}(r^\delta+r^{-\delta})](r^{-\delta}-r^\delta)^{-1}$$
$$\times\int_{-\infty}^{\infty} dp_2\int_{-\infty}^{\infty} d\bar{p}_2\, \tfrac{1}{2}[\delta'(p_2-\tfrac{1}{2}\ln\tan\alpha)+\delta'(p_2+\tfrac{1}{2}\ln\tan\alpha)]\tfrac{1}{2}[\delta'(\bar{p}_2-\tfrac{1}{2}\ln\tan\alpha)+\delta'(\bar{p}_2+\tfrac{1}{2}\ln\tan\alpha)]$$
$$\times[\tanh\tfrac{1}{2}(p_1+p_2-\ln r+\bar{p}_2)\tanh\tfrac{1}{2}(p_1+p_2-\ln r-\bar{p}_2)$$
$$+\tanh\tfrac{1}{2}(p_1+p_2+\ln r+\bar{p}_2)\tanh\tfrac{1}{2}(p_1-p_2+\ln r-\bar{p}_2)]. \tag{5.22c}$$

It is now straightforward to analyze (5.22). I_1 is studied exactly as (5.8) was. I_2 and I_3 are studied by making the substitutions

$$\tfrac{1}{2}(\bar{p}_1-\ln r)=x \tag{5.23a}$$

in their first terms and

$$\tfrac{1}{2}(\bar{p}_1+\ln r)=-x \tag{5.23b}$$

in their second terms, then combining these terms together and expanding $e^{\delta x}$ about $x=0$. The results are

$$I_1\doteq\tilde{Q}_0\frac{1}{2}\frac{\partial^2}{\partial\delta^2}\delta^{-1}\frac{\partial}{\partial\delta}\{-\tfrac{1}{2}|\delta|+\ln|\delta|^{-1}-\psi(|\delta|^{-1})\}, \tag{5.24a}$$

$$I_2\doteq\tilde{Q}_2\frac{\partial^2}{\partial|\delta|^2}[\ln|\delta|^{-1}-\psi(|\delta|^{-1})], \tag{5.24b}$$

$$I_3\doteq-\tilde{Q}_1\tfrac{1}{2}\delta\frac{\partial^3}{\partial\delta^3}[-\tfrac{1}{2}|\delta|+\ln|\delta|^{-1}-\psi(|\delta|^{-1})], \tag{5.24c}$$

where

$$\tilde{Q}_j=\frac{1}{4}\int_0^\infty dx\int_0^{\pi/2} d\alpha\int_{-\infty}^{\infty} dp_2\int_{-\infty}^{\infty} d\bar{p}_2$$
$$\times[\delta'(p_2-\tfrac{1}{2}\ln\tan\alpha)+\delta'(p_2+\tfrac{1}{2}\ln\tan\alpha)]$$
$$\times[\delta'(\bar{p}_2-\tfrac{1}{2}\ln\tan\alpha)+\delta'(\bar{p}_2+\tfrac{1}{2}\ln\tan\alpha)]$$
$$\times\tanh[x+\tfrac{1}{2}(p_2+\bar{p}_2)]\tanh[x-\tfrac{1}{2}(p_2-p_2)]. \tag{5.25}$$

I_1 is more singular than I_2 or I_3; therefore we combine (5.24) with (5.17) to find

$$\langle[(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle]^2\rangle_{E_2}\doteq C_{10}{}^2\left\{\frac{1}{4}Q_0\frac{\partial^2}{\partial\delta^2}\delta^2\frac{\partial}{\partial|\delta|}|\delta|^{-1}\right.$$
$$\times[\ln|\delta|^{-1}-\psi(|\delta|^{-1})]+\tfrac{1}{2}\tilde{Q}_0\frac{\partial^2}{\partial\delta^2}\delta^{-1}\frac{\partial}{\partial\delta}$$
$$\left.\times[-\tfrac{1}{2}|\delta|+\ln|\delta|^{-1}-\psi(|\delta|^{-1})]\right\}. \tag{5.26}$$

Asymptotically[13] as $\delta \to 0$

$$\ln|\delta|^{-1}-\psi(|\delta|^{-1})=\tfrac{1}{2}|\delta|+\sum_{n=0}^{\infty} B_{2n}\delta^{2n}/(2n). \quad (5.27)$$

We therefore see that the second moment (5.26) is infinitely differentiable but not analytic at $\delta=0$. Furthermore, by comparing (5.5) and (5.26) we see that the singularities of $\langle(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle\rangle_{E_2}{}^2$ and $\langle[(d/dT)\langle\sigma_{0,0}\sigma_{1,0}\rangle]^2\rangle_{E_2}$ are different. Therefore we conclude that the variance (5.1) possesses an infinitely differentiable essential singularity at $\delta=0$.

Finally, we remark that just as in Sec. 3 we were able to study $\langle(d/dT)\langle\sigma_{0,0}\sigma_{l,m}\rangle\rangle_{E_2}$ near $\delta=0$ when $l^2+m^2\ll N^4$ using $\nu(x)$ alone, we are able to study the corresponding variance in terms of $\tilde{\nu}(x_1,x_2)$. An analogous argument allows us to conclude that the variance of $(d/dT)\times\langle\sigma_{0,0}\sigma_{l,m}\rangle$ at least when $l^2+m^2<N^4$ is infinitely differentiable but not analytic at $\delta=0$.

6. BEHAVIOR OF SPONTANEOUS MAGNETIZATION

The magnetization of an Ising model with $2\mathfrak{M}+1$ rows and $2\mathfrak{N}$ columns in the presence of a magnetic field H is defined in the thermodynamic limit as

$$M(H)=\lim_{\mathfrak{M}\to\infty,\ \mathfrak{N}\to\infty}\langle[2\mathfrak{N}(2\mathfrak{M}+1)]^{-1}\sum_{j,k}\sigma_{j,k}\rangle. \quad (6.1)$$

The spontaneous magnetization is then defined as

$$\lim_{H\to 0^+} M(H)=M. \quad (6.2)$$

However, since it has proven impossible to compute the partition function of a two-dimensional Ising model in the presence of a finite magnetic field, it has not been possible to evaluate M by direct application of (6.2). Instead, one way of evaluating M for Onsager's lattice is to use the relation

$$M^2=\lim_{l^2+m^2\to\infty}\langle\sigma_{0,0}\sigma_{l,m}\rangle. \quad (6.3)$$

From this formula the spontaneous magnetization M_O for the Onsager lattice of Sec. 3 with interaction energies E_1 and $\bar{E}_2$ may be shown to be[14]

$$\begin{aligned} M_O&=[1-(\sinh 2E_1\beta\,\sinh 2\bar{E}_2\beta)^{-2}]^{1/8} && \text{if } T\le T_c \\ &=0 && \text{if } T>T_c. \end{aligned} \quad (6.4)$$

It is important to realize that for most lattices other than Onsager's, (6.3) cannot be taken as a definition of the spontaneous magnetization because in general the limit will not exist. For example, consider one particular member of the class of lattices we have been considering where the variation of E_2 is $E_2{}^0$, $E_2{}^0$, $E_2{}^1$, $E_2{}^1$ and then repeats periodically. In Appendix B we generalize the formalism of Sec. 2 to compute for this lattice

$$S_m(l)=\langle\sigma_{l,0}\sigma_{l,m}\rangle \quad (6.5)$$

and to show that $S_m(l)$ is *not* independent of l. In particular, for any given m, $S_m(l)$ may take on one of three values depending on whether l is between $E_2{}^0$ and $E_2{}^0$, $E_2{}^1$ and $E_2{}^1$, or $E_2{}^0$ and $E_2{}^1$. This dependence on l does not vanish as $m\to\infty$ and $S_\infty(l)$ can take on three distinct values. In fact, because $S_\infty(l)^{1/2}$ depends on l, a limit will not exist if in (6.5) l and m go to infinity together. In this case, we expect the analog of (6.3) to be

$$M=\tfrac{1}{4}[S_\infty(1)^{1/2}+S_\infty(2)^{1/2}+S_\infty(3)^{1/2}+S_\infty(4)^{1/2}] \quad (6.6)$$

and we may think of $S_\infty(l)^{1/2}$ as a measure of the local magnetization of the lth row.

This dependence of $S_\infty(l)$ on l implies that $\langle\sigma_{0,0}\sigma_{0,m}\rangle$ is not a probability-1 object even as $m\to\infty$. The spontaneous magnetization defined by (6.2) must be a probability-1 object because it is a property of the entire lattice and not just of a particular row, and we may compute M in terms of spin-spin correlations as

$$M=\langle S_\infty{}^{1/2}\rangle_{E_2}. \quad (6.7)$$

A direct application of the formalism of Sec. 2 to evaluate (6.7) is extremely difficult because of the necessity of considering ν functions with an arbitrarily large number of variables. We will therefore confine ourselves to the simpler problem of studying the geometric rather than the arithmetic mean of $S_\infty{}^{1/2}(l)$ and consider $\langle\ln S_\infty(l)\rangle_{E_2}$. We have been able to study this average only by restricting ourselves to sets $\{E_2\}$ which are symmetric about the lth row. We will combine the formalism of Sec. 2 with Szegö's theorem[15] to express $\langle\ln S_\infty(l)\rangle_{E_2}$ in terms of $\nu(x_1,x_2)$ and use the results of Sec. 4 to establish (1.5).

The formalism of Sec. 2 is easily applied to an arbitrary set $\{E_2\}$ to show that

$$\langle\sigma_{0,0}\sigma_{0,m}\rangle^2=\begin{vmatrix} a^{(0)} & \cdots & a^{(m-1)} \\ a^{(-m+1)} & \cdots & a^{(0)} \end{vmatrix}, \quad (6.8)$$

where $a^{(m)}$ is the 2×2 matrix whose elements are given by

$$a_{11}{}^{(m)}=a_{22}{}^{(m)}=(1-z_1{}^2)A^{-1}(0,0;0,m)_{RR}, \quad (6.9a)$$

$$a_{12}{}^{(m)}=-a_{21}{}^{(-m)}=(1-z_1{}^2)A^{-1}(0,0;0,m+1)_{RL}-z_1\delta_{m,0}, \quad (6.9b)$$

and the matrix elements of A^{-1} are given by (2.36). This determinant is a block Toeplitz determinant of

[13] Reference 10, Vol. 1, p. 47.

[14] This famous result was first obtained by C. N. Yang [Phys. Rev. **85**, 808 (1952)] from a somewhat different point of view. The point of view adopted here is that of Ref. 7.

[15] G. Szegö, in Communications du séminaire mathématique de l'université de Lund; tôme supplémentaire dédié à Macel Reisz, 1952, p. 228 (unpublished). Szegö only proves the theorem for $a_{12}(\theta)$ real. For the case that $a_{12}(\theta)$ is complex see I. I. Hirschmann, J. D'Analyse Math. **14**, 225 (1965); A. Devinatz, Illinois J. Math. **11**, 160 (1967).

2×2 matrices. To the authors' knowledge there is no explicit formula known for the limit of such a determinant as $l\to\infty$. However, if we make the restriction

$$E_2(j)=E_2(-1-j)\,, \tag{6.10}$$

then

$$\bar{x}(0,\mathfrak{M})=x(-\mathfrak{M},0)\,, \tag{6.11}$$

$$a_{11}{}^{(m)}=a_{22}{}^{(m)}=0\,, \tag{6.12a}$$

and

$$\begin{aligned}a_{12}{}^{(m)}=-a_{21}{}^{(m)}=&-(2\pi)^{-1}\\&\times\int_{-\pi}^{\pi}d\theta\;e^{i(m+1)\theta}(1+z_1e^{-i\theta})(1+z_1e^{i\theta})^{-1}\\&\times[b+i(\bar{x}(0,\mathfrak{M})-a)][b-i(\bar{x}(0,\mathfrak{M})-a)]^{-1}\\&=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\;e^{im\theta}a_{12}(\theta)\,,\end{aligned} \tag{6.12b}$$

which defines $a_{12}(\theta)$. Thus (6.8) may be reexpressed as

$$S_m=\langle\sigma_{0,0}\sigma_{0,m}\rangle=\pm\begin{vmatrix}a_{12}{}^{(0)} & \cdots & a_{12}{}^{(m-1)}\\ a_{12}{}^{(-m+1)} & \cdots & a_{12}{}^{(0)}\end{vmatrix} \tag{6.13}$$

(the $\pm$ sign is chosen to make $\langle\sigma_{0,0}\sigma_{0,m}\rangle$ positive), which is now a Toeplitz determinant whose entries are scalars rather than matrices. Since we have

$$\bar{x}(0,\mathfrak{M};\,-\theta)=\bar{x}(0,\mathfrak{M};\,\theta)\,, \tag{6.14}$$

$$a_{12}(-\theta)=[a_{12}(\theta)]^{-1}. \tag{6.15}$$

If, in addition, $\ln a_{12}(\theta)$ is continuous and periodic with period 2π, we may apply Szegö's theorem[15] to (6.13) in the form

$$\lim_{m\to\infty}\ln S_m=\frac{1}{8}\int_{-\pi}^{\pi}d\theta_1\,(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta_2\,(2\pi)^{-1}\times\left[\frac{\ln a_{12}(\theta_1)-\ln a_{12}(\theta_2)}{\sin\frac{1}{2}(\theta_1-\theta_2)}\right]^2. \tag{6.16}$$

[It is easily seen by direct substitution that (6.16) is equivalent to the more familiar form of Szegö's theorem

$$\lim_{m\to\infty}\ln S_m=\sum_{n=-\infty}^{\infty}nk_nk_{-n}\,, \tag{6.17}$$

where

$$k_n=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\;e^{in\theta}\ln a_{12}(\theta)\,.] \tag{6.18}$$

Because (6.16) holds (at least for sufficiently low temperature) for all sets $\{E_2\}$ consistent with the symmetry requirement (6.10), we may average $\ln S_\infty$ over these sets by use of the two-variable function $\nu(x_1,x_2)$ as

$$\begin{aligned}\langle\ln S_\infty\rangle_{E_2}=&\frac{1}{4}\int_{-\pi}^{0}d\theta_1\,(2\pi)^{-1}\int_{-\pi}^{0}d\theta_2\,(2\pi)^{-1}\int_0^\infty dx_1\int_0^\infty dx_2\;\nu(x_1,x_2;\,\theta_1,\theta_2)\\&\times\left\{\left[\left(\ln\frac{1+z_1e^{-i\theta_1}}{1+z_1e^{i\theta_1}}\frac{b_1+i(x_1-a_1)}{b_1-i(x_1-a_1)}-\ln\frac{1+z_1e^{-i\theta_2}}{1+z_1e^{i\theta_2}}\frac{b_2+i(x_2-a_2)}{b_2-i(x_2-a_2)}\right)\Big/\sin\tfrac{1}{2}(\theta_1-\theta_2)\right]^2\right.\\&\left.+\left[\left(\ln\frac{1+z_1e^{-i\theta_1}}{1+z_1e^{i\theta_1}}\frac{b_1+i(x_1-a_1)}{b_1-i(x_1-a_1)}+\ln\frac{1+z_1e^{-i\theta_2}}{1+z_1e^{i\theta_2}}\frac{b_2+i(x_2-a_2)}{b_2-i(x_2-a_2)}\right)\Big/\sin\tfrac{1}{2}(\theta_1+\theta_2)\right]^2\right\}.\end{aligned} \tag{6.19}$$

When $\delta=O(1)$, we use (4.5), (4.14), (4.15), (4.20), (4.23), (4.24), and (4.26) to approximate (6.19) as

$$\begin{aligned}\langle\ln S_\infty\rangle_{E_2}=&-\pi^{-2}\int_0^{N^2}d\phi_1\int_0^{N^2}d\phi_2\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\;V(p_1,p_2)\\&\times\left[\left(\frac{\arctan e^{-(p_1+p_2)}-\arctan e^{-(p_1-p_2)}}{\phi_1-\phi_2}\right)^2+\left(\frac{\arctan e^{-(p_1+p_2)}+\arctan e^{-(p_1-p_2)}}{\phi_1+\phi_2}\right)^2\right]+\bar{C}_M+o(1).\end{aligned} \tag{6.20}$$

The dependence of the right-hand side of (6.20) on N^2 may be made more explicit if we use approximation (4.45) for $V(p_1,p_2)$ to show that for ϕ_1 and ϕ_2 large

$$\begin{aligned}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\;V(p_1,p_2)&\left[\left(\frac{\arctan e^{-(p_1+p_2)}-\arctan e^{-(p_1-p_2)}}{\phi_1-\phi_2}\right)^2\right.\\&\left.+\left(\frac{\arctan e^{-(p_1+p_2)}+\arctan e^{-(p_1-p_2)}}{\phi_1+\phi_2}\right)^2\right]+\tfrac{1}{4}\pi^2(\phi_1+\phi_2)^{-2}\sim(\tfrac{1}{2}\delta\phi_1{}^{-1}\phi_2{}^{-1})^2,\end{aligned} \tag{6.21}$$

so that the N^2 dependence of (6.20) is given as

$$\langle \ln S_\infty \rangle_{E_2} \sim -\frac{1}{4}\int^{N^2} dr\, r^{-1} \int_0^{\pi/4} d\alpha (\sin\alpha+\cos\alpha)^{-2} = -\tfrac{1}{4}\ln N^2 + \text{const}. \quad (6.22)$$

We exhibit this dependence on N^2 explicitly by subtracting out the asymptotic behavior (6.21) from the ϕ_1, ϕ_2 integrand of (6.20) and obtain

$$\langle \ln S_\infty \rangle_{E_2} = -\pi^{-2}\int_0^\infty d\phi_1 \int_0^\infty d\phi_2 \left\{ \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2\, V(p_1,p_2) \left[\left(\frac{\arctan e^{-(p_1+p_2)} - \arctan e^{-(p_1-p_2)}}{\phi_1-\phi_2} \right)^2 + \left(\frac{\arctan e^{-(p_1+p_2)} + \arctan e^{-(p_1-p_2)}}{\phi_1+\phi_2} \right)^2 \right] - \tfrac{1}{4}\pi^2(\phi_1+\phi_2+1)^{-2} \right\} - \tfrac{1}{4}\ln N^2 + C_M + o(1). \quad (6.23)$$

The constant C_M may be computed from the requirement that the $\delta \to -\infty$ (6.23) and the $T \to T_c$ behavior of $\ln M_O{}^2$ agree to order $O(1)$. From (6.4) we see, correct to leading order in N^{-1}, that as $T \to T_c-$

$$\ln M_O \sim \tfrac{1}{8}\{\ln(1-T/T_c) + \ln 2\beta_c(z_{1c}{}^{-1}+z_{1c})(1-z_{1c})^{-1}[E_1(1-z_{1c})+E_2{}^0(1-z_{2c}{}^0)]\}. \quad (6.24)$$

Here we have replaced E_2 by $E_2{}^0$ since $\bar{E}_2 - E_2{}^0 = O(N^{-1})$ and have used identity (3.41b). Furthermore, we use approximation (4.45) to show that as $\delta \to -\infty$

$$\langle \ln S_\infty \rangle_{E_2} \sim -\pi^{-2}\int_0^\infty d\phi_1 \int_0^\infty d\phi_2 \left[\left(\frac{\arctan[\delta\phi_1{}^{-1}+(\delta^2\phi_1{}^{-2}-1)^{1/2}] - \arctan[\delta\phi_2{}^{-1}+(\delta^2\phi_2{}^{-2}+1)^{1/2}]}{\phi_1-\phi_2} \right)^2 + \left(\frac{\arctan[\delta\phi_1{}^{-1}+(\delta^2\phi_1{}^{-2}+1)^{1/2}] + \arctan[\delta\phi_2{}^{-1}+(\delta^2\phi_2{}^{-2}+1)^{1/2}]}{\phi_1+\phi_2} \right)^2 \right] - \tfrac{1}{4}\pi^2(\phi_1+\phi_2+1)^{-2} - \tfrac{1}{4}\ln N^2 + C_M = C_M{}' + \tfrac{1}{4}\ln(-\delta/N^2) + C_M, \quad (6.25)$$

where

$$C_M{}' = -\pi^{-2}\int_0^\infty dy_1 \int_0^\infty dy_2 \left[\left(\frac{\arctan[y_1{}^{-1}-(y_1{}^{-2}+1)^{1/2}] - \arctan[y_2{}^{-1}-(y_2{}^{-2}+1)^{1/2}]}{y_1-y_2} \right)^2 + \left(\frac{\arctan[y_1{}^{-1}-(y_1{}^{-2}+1)^{1/2}] + \arctan[y_2{}^{-1}-(y_2{}^{-2}+1)^{1/2}]}{y_1+y_2} \right)^2 - \tfrac{1}{4}\pi^2(y_1+y_2+1)^{-2} \right]. \quad (6.26)$$

Comparing (6.24) and (6.25) and using (3.1), we find

$$C_M = \tfrac{1}{4}\ln[\tfrac{1}{4}(z_{1c}+z_{1c}{}^{-1})] - C_M{}'. \quad (6.27)$$

It is now a simple matter to study the $\delta \to -0$ behavior of (6.23) by using approximation (4.70) for $V(p_1,p_2)$. We find

$$\langle \ln S_\infty \rangle_{E_2} \sim -\delta\pi^{-2}\int_0^\epsilon dr\, r^{-1}(r^{-\delta}-r^{\delta})^{-1} \int_0^{\pi/2} d\alpha \int_{\ln r}^{-\ln r} dp_1\, e^{-\delta p_1} \left[\left(\frac{\arctan[e^{-p_1}(\tan\alpha)^{1/2}] - \arctan[e^{-p_1}(\cot\alpha)^{1/2}]}{\sin\alpha-\cos\alpha} \right)^2 + \left(\frac{\arctan[e^{-p_1}(\tan\alpha)^{1/2}] + \arctan[e^{-p_1}(\cot\alpha)^{1/2}]}{\sin\alpha+\cos\alpha} \right)^2 \right] - \tfrac{1}{4}\ln N^2. \quad (6.28)$$

We may determine the r dependence of the p_1 integral accurately enough to compute the terms in (6.28) that diverge as $\delta \to 0^-$ if we use

$$\arctan[e^{-p_1}(\tan\alpha)^{1/2}] - \arctan[e^{-p_1}(\cot\alpha)^{1/2}] = \arctan\left(\frac{(\tan\alpha)^{1/2}-(\cot\alpha)^{1/2}}{e^{p_1}+e^{-p_1}} \right) \quad (6.29\text{a})$$

and

$$\arctan[e^{-p_1}(\tan\alpha)^{1/2}] + \arctan[e^{-p_1}(\cot\alpha)^{1/2}] = \pi\Theta(-p_1) + \arctan\left[\frac{(\tan\alpha)^{1/2}+(\cot\alpha)^{1/2}}{e^{p_1}-e^{-p_1}} \right], \quad (6.29\text{b})$$

where

$$\Theta(x)=1,\quad x>0$$
$$\quad=0,\quad x<0 \tag{6.30}$$

and the branch of the arctangent is defined by

$$\arctan 0=0. \tag{6.31}$$

Using (6.29), we find

$$\pi^{-2}\int_0^{\pi/2}d\alpha\int_{\ln r}^{-\ln r}dp_1\, e^{-\delta p_1}\Bigg[\left(\frac{\arctan[e^{-p_1}(\tan\alpha)^{1/2}]-\arctan[e^{-p_1}(\cot\alpha)^{1/2}]}{\sin\alpha-\cos\alpha}\right)^2$$
$$+\left(\frac{\arctan[e^{-p_1}(\tan\alpha)^{1/2}]+\arctan[e^{-p_1}(\cot\alpha)^{1/2}]}{\sin\alpha+\cos\alpha}\right)^2\Bigg]\sim-\delta^{-1}(1-r^{-\delta})+2\pi^{-1}\int_0^{\pi/2}d\alpha\int_{-\infty}^{\infty}dp_1\,(\sin\alpha+\cos\alpha)^{-2}$$
$$\times\arctan\left(\frac{(\tan\alpha)^{1/2}+(\cot\alpha)^{1/2}}{e^{p_1}-e^{-p_1}}\right)+\pi^{-2}\int_0^{\pi/2}d\alpha\int_{-\infty}^{\infty}dp_1\left\{(\sin\alpha-\cos\alpha)^{-2}\left[\arctan\left(\frac{(\tan\alpha)^{1/2}-(\cot\alpha)^{1/2}}{e^{p_1}+e^{-p_1}}\right)\right]^2\right.$$
$$\left.+(\sin\alpha+\cos\alpha)^{-2}\left[\arctan\left(\frac{(\tan\alpha)^{1/2}+(\cos\alpha)^{1/2}}{e^{p_1}-e^{-p_1}}\right)\right]^2\right\}=-\delta^{-1}(1-r^{-\delta})+2C_2, \tag{6.32}$$

where the last equation defines C_2. Putting this approximation in (6.28), we obtain

$$\langle\ln S_\infty\rangle_{E_2}\sim-\int_0^\epsilon dr\, r^{-1}(r^\delta+1)^{-1}$$
$$-2C_2\delta\int_0^\epsilon dr\, r^{-1}(r^{-\delta}-r^\delta)^{-1}-\tfrac{1}{4}\ln N^2. \tag{6.33}$$

Now

$$-\int_0^\epsilon dr\, r^{-1}(r^\delta+1)^{-1}=\delta^{-1}\ln(1+\epsilon^{-\delta})$$
$$=\delta^{-1}\ln 2-\tfrac{1}{2}\ln\epsilon+O(\delta) \tag{6.34a}$$

and

$$\delta\int_0^\epsilon dr\, r^{-1}(r^{-\delta}-r^\delta)^{-1}=-\tfrac{1}{2}\ln(\epsilon^\delta+1)/(\epsilon^\delta-1)$$
$$=-\tfrac{1}{2}\ln(-\delta)+\tfrac{1}{2}\ln(-\tfrac{1}{2}\ln\epsilon)+O(\delta). \tag{6.34b}$$

The coefficients of the terms in (6.34) that diverge as $\delta\to0^-$ do not depend on ϵ and thus are correctly given by our approximations of $\nu(x_1,x_2)$. On the other hand, the constant terms do depend on ϵ and consequently are not computable from approximation (4.70) for $\nu(x_1,x_2)$. We thus may combine (6.33) and (6.34) to find, as $\delta\to0^-$,

$$\langle\ln S_\infty\rangle_{E_2}=-\tfrac{1}{4}\ln N^2+\delta^{-1}\ln 2$$
$$+C_2\ln(-\delta)+O(1). \tag{6.35}$$

If we recall the definition of δ(3.1), we obtain (1.5). Furthermore, since S_∞ lies between zero and 1, we infer that as $\delta\to0^-$ the geometric mean of S_∞ is

$$\text{const}N^{-1/2}2^{1/\delta}(-\delta)^{C_2}. \tag{6.36}$$

In Onsager's lattice S_∞ is not a probabilistic quantity and the analog of (6.36) is the square of (6.4), which, $\delta\to0^-$, behaves as

$$\text{const}N^{-1/2}(-\delta)^{1/4}. \tag{6.37}$$

Clearly, as $\delta\to0^-$, the ratio of (6.36) to (6.37) may be made arbitrarily small. We speculate that the further increase in randomness that comes from totally destroying the symmetry requirements (6.10) can only serve to further decrease this ratio so that the geometric mean of S_∞ in an arbitrary lattice is bounded above by (6.36) as $\delta\to0^-$.

In the following paper,[5] by considering boundary effects in our random Ising lattice, we demonstrate that

$$\langle S_\infty^{1/2}\rangle_{E_2}=M\geq\text{const}(T_c-T). \tag{6.38}$$

The only way that as $\delta\to0^-$ the geometric mean of $S_\infty^{1/2}$ [(6.36)] can be so much smaller than the arithmetic mean is for $P(S_\infty^{1/2})$, the probability distribution function for $S_\infty^{1/2}$, to be very concentrated near $S_\infty^{1/2}=0$ while still having a long tail that is appreciable in the region where $S_\infty^{1/2}N^{1/2}=O(1)$ as $N\to\infty$. One such probability function is discussed in Sec. 4 of the following paper. We will only remark here that such a spread-out probability distribution is not surprising. The fact that there is or is not a spontaneous magnetization is a property of the lattice as a whole. However, since the distribution of bonds $E_2(j)$ is by no means uniform, we do not expect the local magnetization $S_\infty(l)^{1/2}$ to be uniform. When $T>T_c$, each $S_\infty(l)^{1/2}$ is expected to be zero. However, when $T<T_c$, even though the arithmetic mean of S_∞ will be different from zero, there will exist large strips in our lattice where the bonds $E_2(j)$ are so much weaker than the average that if all bonds had the strength of these bonds, the critical temperature would be less than T. In these strips the value of $S_\infty^{1/2}(l)$ is expected to be much smaller than the arithmetic mean. As the strips get larger, the value of

$S_\infty^{1/2}(l)$ should get smaller. Therefore, as $\delta \to 0^-$ the local magnetization should break up into strips such that the strips in which $S_\infty^{1/2}(l)$ is comparable to the arithmetic mean become more and more separated by strips where $S_\infty^{1/2}(l)$ is extremely small.

The foregoing description can be, of course, no more precise than the foregoing calculations. In particular, it is not clear how dependent this description is on the narrowness of $P(E_2)$. Nevertheless, it does show that on a microscopic scale even our lattice with a narrow distribution function can be dramatically different from Onsager's lattice. It also suggests that there is a great deal of structure in the asymptotic behavior of the correlation functions and their joint probability functions when the separations and the correlation lengths are larger than or comparable to N^2. It is the intent of the next paper of this series to make more precise what some of this structure may be.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix, we justify the analysis of case (iii) of Sec. 4 by demonstrating that in this approximation $V(p_1,p_2)$ is nonvanishing only when

$$-\tfrac{1}{2}|\ln\kappa| < p_2 < \tfrac{1}{2}|\ln\kappa| \tag{A1}$$

and by studying $V(p_1,p_2)$ when (A1) holds but $p_\pm$ defined by (4.58) are not of order 1. The analysis is complicated and roughly proceeds as follows: (i) When $|p_1|+\ln\phi = O(1)$, we find an approximation for $V_>$ and $V_<$ that is more precise than (4.56). These two solutions are expressed as an integral of some appropriate Green's function times the function $f(p_2)$ of (4.51). (ii) We then obtain a one-dimensional homogeneous integral equation for $f(p_2)$ by requiring that when p_1 is of order 1, $V_>$ and $V_<$ and their p_1 derivatives are equal. Such a condition may be imposed and, indeed, must be imposed because $V_>$ and $V_<$ must join together as smoothly as possible since the exact $V(p_1,p_2)$ is analytic in p_1 for p_1 of order 1. Only if (A1) holds will this equation have a nontrivial solution. (iii) We then approximately solve this integral equation for p_2 away from $\pm\frac{1}{2}|\ln\kappa|$. (iv) This approximation to $V(p_1,p_2)$ is then used to restrict the possible solutions for $V_\pm$ found in Sec. 4 by demanding that the $p_2 \to \pm\frac{1}{2}|\ln\kappa|$ asymptotic behavior of $V(p_1,p_2)$ agree with the $p_\pm \to \infty$ asymptotic behavior of $V_\pm$.

When $p_{1>}$ is of order 1, we found in Sec. 4 that $V(p_1,p_2)$ is sharply peaked about $-\frac{1}{2}\ln\kappa$ with a width proportional to ϕ^2. In this region V was approximated by $V_{>+}$ satisfying (4.68). When $p_{1>}$ is still of order 1 but p_2 is away from $-\frac{1}{2}\ln\kappa$, V is approximated by $V_>$,

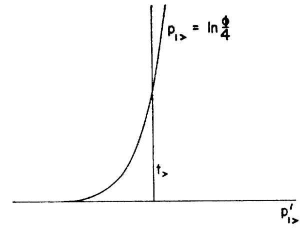

FIG. 2. Boundary curves in the $t_>$, $p_{1>}'$ plane along which $\bar{V}_>$ and $\bar{V}_<$ are to be joined.

which satisfies the approximate equation (4.54). One solution to (4.54) was considered in (4.56). However, this approximation is not accurate enough for the present purpose. It merely says that to leading order in ϕ the region away from $-\frac{1}{2}\ln\kappa$ does not contribute to the subsidiary condition integral over q_1 and q_2.

To study (4.54) (and the analogous equation for $V_<$) in more detail let

$$\begin{aligned} p_{2>(<)} &= p_2 \mp \tfrac{1}{2}\ln\kappa\,, \\ p_{1>(<)}' &= p_{1>(<)} \mp \ln\tfrac{1}{2}|\sinh p_{2>(<)}| \pm \tfrac{1}{2}\ln\kappa\,, \end{aligned} \tag{A2}$$

and

$$\hat{V}_{>(<)}(p_{1>(<)}',p_{2>(<)}) = V_{>(<)}(p_{1>(<)},p_2)\,, \tag{A3}$$

where the lower signs and subscripts in parentheses go together. Then (4.54) becomes

$$\frac{\partial^2\hat{V}_{>(<)}}{\partial p_{1>(<)}'^2} + \delta\frac{\partial\hat{V}_{>(<)}}{\partial p_{1>(<)}'} + \operatorname{sgn}p_{2>(<)}\frac{\partial}{p_{2>(<)}} \times e^{\pm p_{1>(<)}'}\sinh^2 p_{2>(<)}\hat{V}_{>(<)} = 0\,. \tag{A4}$$

Furthermore, set

$$t_{>(<)} = \coth|p_{2>(<)}| - 1 \tag{A5}$$

and

$$\bar{V}_{>(<)}(p_{1>(<)}',t_{>(<)}) = \sinh^2 p_{2>(<)}\hat{V}_{>(<)}(p_{1>(<)}',p_{2>(<)}) \tag{A6}$$

and obtain

$$\frac{\partial^2\bar{V}_{>(<)}}{\partial p_{1>(<)}'^2} + \delta\frac{\partial\bar{V}_{>(<)}}{\partial p_{1>(<)}'} - e^{\pm p_{1>(<)}'}\frac{\partial\bar{V}_{>(<)}}{\partial t_{>(<)}} = 0\,. \tag{A7}$$

From (4.32) we see that $\bar{V}_{>(<)}$ satisfies the boundary condition

$$\bar{V}_{>(<)}(p_{1>(<)}',0) = 0\,. \tag{A8}$$

This boundary condition is not by itself sufficient to specify a unique solution to (A7). To obtain further boundary conditions we recall from (4.51) that if $p_{1>(<)}' \sim \pm\ln\frac{1}{4}\phi$, $V(p_1,p_2)$ is given by

$$[\mathrm{const}\,e^{-\delta p_1} + \mathrm{const}']f(p_2)\,. \tag{A9}$$

This approximation may also be obtained from (A7) by omitting the last term. Therefore, we should be able to recover the form (A9) from either $\bar{V}_>$ or $\bar{V}_<$, and we will impose the restriction expressing the continuity *of* $V(p_1,p_2)$ in p_1 at $p_1=0$ of

$$\hat{V}_>(\ln\tfrac{1}{4}\phi,\, p_{2>}+\tfrac{1}{2}\ln\kappa) = \hat{V}_<(-\ln\tfrac{1}{4}\phi,\, p_{2<}-\tfrac{1}{2}\ln\kappa) = f(p_2)\,, \quad \text{(A10)}$$

where in the last equation we have redefined $f(p_2)$ by a constant factor. Indeed, $V(p_1,p_2)$ is not only continuous but even differentiable in p_1 at $p_1=0$ and so we would like to also impose

$$\frac{\partial}{\partial p_{1>}}\hat{V}_>(p_{1>},\, p_{2>}+\tfrac{1}{2}\ln\kappa)\Big|_{p_{1>}=\ln\frac{1}{4}\phi} = \frac{\partial}{\partial p_{1<}}\hat{V}_<(p_{1<},\, p_{2<}-\tfrac{1}{2}\ln\kappa)\Big|_{p_{1<}=\ln\frac{1}{4}\phi}\,. \quad \text{(A11)}$$

It is slightly unfortunate that the partial differential equation (A7) is for $\bar{V}$ instead of $\hat{V}$ because the boundary condition (A11) for $\hat{V}$ at the fixed point $p_{1>(<)} = \pm\ln\frac{1}{4}\phi$ implies a boundary condition for $\hat{V}$ along a curve in the $p_{1>(<)}',t$ plane (Fig. 2). If $\bar{V}_{>(<)}(p_{1>(<)}',t)$ were to be matched along a line $p_{1>(<)}'=$ const, then (A7) could be interpreted as a heat equation for a semi-infinite rod with a conductivity that depends exponentially on position. The boundary condition (A8) specifies that at time $t_{>(<)}=0$ this semi-infinite rod has zero temperature with the exception of the end where the temperature is prescribed [by a single function like $f(p_2)$]. The lines $p_{1>(<)}'=$ const correspond to the curves in the p_1,p_2 plane:

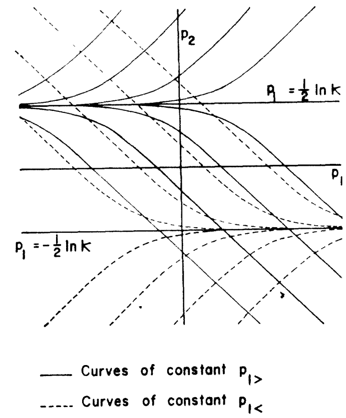

FIG. 3. Curves in the p_1, p_2 plane along which $p_{1>}$ and $p_{1<}$ are constant.

$$\text{const} = p_1 \pm [\ln\tfrac{1}{4}\phi + \tfrac{1}{2}\ln\kappa - \ln\tfrac{1}{2}|\sinh(p_2 \mp \tfrac{1}{2}\ln\kappa)|\,]\,, \quad \text{(A12)}$$

which are shown in Fig. 3. These two sets of curves clearly do not coincide. However, from (A9) we see that when $\delta^{-1}\sim -\ln\phi$, $(\partial^n/\partial p_1{}^n)V(p_1,p_2)\sim O[(\ln\phi)^{-n}]$ when p_1 is of order 1. Therefore, $V(p_1,p_2)$ and $\partial V(p_1,p_2)/\partial p_1$, while not exactly independent of p_1 along the curves $p_{1>(<)}'=\pm(\ln\frac{1}{4}\phi-\frac{1}{2}\ln\kappa)$, will be independent of p_1 to leading order in ϕ as long as p_2 is away from $\pm\frac{1}{2}\ln\kappa$. On this basis we replace the requirements (A10) and (A11) by

$$\hat{V}_>[\ln\tfrac{1}{4}\phi - \tfrac{1}{2}\ln\kappa + \ln\tfrac{1}{2}|\sinh(p_{2>}+\tfrac{1}{2}\ln\kappa)|,\, p_{2>}+\tfrac{1}{2}\ln\kappa] \doteqdot \hat{V}_>[-\ln\tfrac{1}{4}\phi + \tfrac{1}{2}\ln\kappa - \ln\tfrac{1}{2}|\sinh(p_{2<}-\tfrac{1}{2}\ln\kappa)|,\, p_{2<}-\tfrac{1}{2}\ln\kappa] = f(p_2)\,, \quad \text{(A10}')$$

$$\frac{\partial}{\partial p_{1>}}\hat{V}_>(p_>,\, p_{2>}+\tfrac{1}{2}\ln\kappa)\Big|_{p_{1>}=\ln\frac{1}{4}\phi-\frac{1}{2}\ln\kappa+\ln\frac{1}{2}|\sinh(p_{2>}+\frac{1}{2}\ln\kappa)} \doteqdot \frac{\partial}{\partial p_{1<}}\hat{V}_<(p_{1<},\, p_{2<}-\tfrac{1}{2}\ln\kappa)\Big|_{p_{1<}=-\ln\frac{1}{2}\phi-\frac{1}{2}\ln\kappa-\ln\frac{1}{2}|\sinh(p_{2<}-\frac{1}{2}\ln\kappa)\frac{1}{2}} \quad \text{(A11}')$$

where $\doteqdot$ means equality of the leading term as $\phi\to 0$.

Let

$$g_{>(<)}(t_{>(<)}) = f(p_2)\frac{dp_2}{dt_{>(<)}}\,. \quad \text{(A13)}$$

Then (A10′) gives the additional boundary condition

$$\bar{V}_{>(<)}(\pm\ln\tfrac{1}{4}\phi, t_{>(<)}) = g_{>(<)}(t_{>(<)})\,. \quad \text{(A14)}$$

We may now follow the procedure indicated above and express $\bar{V}_{>(<)}(p_{1>}',t_{>(<)})$ in terms of $f(p_2)$. Define the Green's functions $G_{>(<)}(p_{1>(<)}',y_{>(<)}-t')$ to satisfy (A7) with the boundary conditions

$$G_{>(<)}(p_{1>(<)}',t_{>(<)}-t')=0 \quad \text{for } t_{>(<)}\le t',\quad |p_{1>(<)}'| > -\ln\tfrac{1}{4}\phi \quad \text{(A15)}$$

and

$$G_{>(<)}(\pm\ln\tfrac{1}{4}\phi,\, t_{>(<)}-t') = \delta(t_{>(<)}-t')\,. \quad \text{(A16)}$$

Then when $t_\pm>0$,

$$\bar{V}_{>(<)}(p_{1>(<)}',t_{>(<)}) = \int_0^\infty dt'\, G_{>(<)}(p_{1>(<)}',\, t_{>(<)}-t')g_{>(<)}(t')\,. \quad \text{(A17)}$$

Even without the explicit form for $G_{>(<)}$ we may now see why $V(p_1,p_2)$ vanishes outside the strip (A1). From (A2) and (A5) we see that

$$\text{if } p_2=\tfrac{1}{2}\ln\kappa,\quad \text{then } t_>=\infty \quad \text{and} \quad \text{if } p_2=-\tfrac{1}{2}\ln\kappa,\quad \text{then } t_<=\infty\,. \quad \text{(A18)}$$

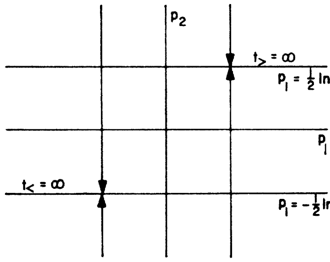

FIG. 4. Plot in p_1, p_2 space showing the directions in which the timelike variables $t_>$ and $t_<$ increase. Arrows point in the direction of increasing "time."

From our interpretation of (A7) as a heat equation [or, equivalently, from (A14) and (A17)], $\bar{V}_{>(<)}(p_{1>(<)}',t_{>(<)})$ and $\partial\bar{V}_{>(<)}/\partial p_{1>(<)}$ depend on the values of $\bar{V}_{>(<)}(\pm\ln\frac{1}{4}\phi,t_{>(<)})$ only for times $t_{>(<)}'$ that are smaller than $t_{>(<)}$. Outside the strip (A1) the timelike variables $t_>$ and $t_<$ are increasing in the same direction (Fig. 4). Therefore, the values of $(\partial V_>/\partial p_1)(p_1,p_2)$ and $(\partial V/\partial p_1)(p_1,p_2)$ which we are trying to equate by (A11′) are both computed from (A17) in terms of an integral over $f(p_2')$ where p_2' is greater (or lesser) than p_2 if p_2 is greater (or lesser) than $\frac{1}{2}|\ln\kappa|$ (or $-\frac{1}{2}|\ln\kappa|$). However, the Green's functions for $V_>$ and $V_<$ will be different. Therefore, we conclude that if $\partial V_>/\partial p_1$ and $\partial V_</\partial p_1$ agree to leading order in $(\ln\phi)^{-1}$ for some value of p_2 they cannot, in general, agree for some other value of p_2 unless they both vanish identically. However, when (A1) holds, the timelike variables are increasing in opposite directions, so that if $\partial V_>/\partial p_1$ is determined by values of $f(p_2')$ for $p_2'>p_2$, then $\partial V_</\partial p_1$ is determined by values of $f(p_2')$ for $p_2'<p_2$. Therefore, we conclude that $V(p_1,p_2)$ can only be different from zero (to our approximation) when (A1) holds.

To obtain further information on $f(p_2)$ in the strip (A1) we need the explicit form for $G_{>(<)}$. To obtain this, Laplace-transform (A7). We define

$$\tilde{G}_{>(<)}(p_{1>(<)}',s)=\int_0^\infty dt\, G(p_{1>(<)}',t)e^{-ts} \quad \text{(A19)}$$

to find

$$\frac{\partial^2\tilde{G}_{>(<)}}{\partial p_{1>(<)}'^2}+\delta\frac{\partial\tilde{G}_{>(<)}}{\partial P_{1>(<)}'}-e^{\pm p_{1>(<)}'}s\tilde{G}_{>(<)}=0 \quad \text{(A20)}$$

with the boundary condition

$$\tilde{G}_{>(<)}(\pm\ln\tfrac{1}{4}\phi,\, s)=e^{-st}. \quad \text{(A21)}$$

Clearly, we obtain $\tilde{G}_<$ from $\tilde{G}_>$ if we replace $p_{1>}'$ by $-p_{1<}'$ and δ by $-\delta$. This equation may be solved and we find

$$G_{>(<)}(p_{1>(<)}',\, t_{>(<)}-t')=(2\pi i)\int_{-i\infty}^{+i\infty} ds\, e^{s(t_{>(<)}-t')} \times\frac{[2(se^{\pm p_{>(<)}'})^{1/2}]^{\mp\delta}K_\delta[2(se^{\pm p_{>(<)}'})^{1/2}]}{(s\phi)^{\mp\delta/2}K_\delta[(s\phi)^{1/2}]}, \quad \text{(A22)}$$

where the numerical coefficient of ϕ is arbitrary, reflecting the arbitrariness of the path along which $V_>$ and $V_<$ are joined. We now use (A22) and (A17) in (A11′) to obtain the approximate one-dimensional integral equation for $\kappa>1$ and p_2 away from $\pm\frac{1}{2}\ln\kappa$

$$[\sinh(p_2-\tfrac{1}{2}\ln\kappa)]^{-2}\phi^{1/2}\int_{-\frac{1}{2}\ln\kappa}^{p_2} d\bar{p}_2\, f(\bar{p}_2) \times\bar{G}_>[\coth(\tfrac{1}{2}\ln\kappa-p_2)-\coth(\tfrac{1}{2}\ln\kappa-\bar{p}_2)]$$

$$\doteqdot -[\sinh(p_2+\tfrac{1}{2}\ln\kappa)]^{-2}\phi^{1/2}\int_{p_2}^{\frac{1}{2}\ln\kappa} d\bar{p}_2\, f(\bar{p}_2) \times\bar{G}_<[\coth(p_2+\tfrac{1}{2}\ln\kappa)-\coth(\bar{p}_2+\tfrac{1}{2}\ln\kappa)], \quad \text{(A23)}$$

where

$$\bar{G}_{>(<)}(t)=\mp4\phi^{-1/2}\frac{\partial}{\partial p_{1>(<)}'}G_{>(<)}(p_{1>(<)}',t)\Big|_{p_{1>(<)}=\pm\ln\frac{1}{4}\phi}$$

$$=\phi(2\pi i)^{-1}\int_{-i\infty}^{i\infty} ds e^{st}s^{1/2}\frac{K_{\pm\delta+1}[(s\phi)^{1/2}]}{K_\delta[(s\phi)^{1/2}]}. \quad \text{(A24)}$$

To analyze (A23) further it is useful to define the variable ξ by

$$e^\xi=(\kappa e^{2p_2}-1)/(\kappa-e^{2p_2}) \quad \text{(A25)}$$

so that as p_2 goes from $-\frac{1}{2}\ln\kappa$ to $+\frac{1}{2}\ln\kappa$, ξ goes from $-\infty$ to $+\infty$,

$$\coth[p_2+\tfrac{1}{2}\ln\kappa]-1=2(\kappa^2-1)^{-1}(\kappa e^{-\xi}+1) \quad \text{(A26)}$$

and

$$\sinh^2(p_2+\tfrac{1}{2}\ln\kappa)=\tfrac{1}{4}(\kappa-\kappa^{-1})^2 \times(e^{-\xi}+\kappa^{-1})^{-1}(e^{-\xi}+\kappa)^{-1}. \quad \text{(A27)}$$

Define also $g(\xi)$ by

$$f(p_2)=g(\xi)\frac{d\xi}{dp_2}=g(\xi)2(\kappa^2-1) \times(\kappa e^{2p_2}-1)^{-1}(\kappa e^{-2p_2}-1)^{-1}. \quad \text{(A28)}$$

Then (A23) becomes

$$e^\xi\int_{-\infty}^{\xi} d\bar{\xi}\, g(\bar{\xi})\bar{G}_>[2(\kappa-\kappa^{-1})^{-1}(e^\xi-e^{\bar{\xi}})]$$

$$\doteqdot -e^{-\xi}\int_{\xi}^{\infty} d\bar{\xi}\, g(\bar{\xi})\bar{G}_<[2(\kappa-\kappa^{-1})^{-1}(e^{-\xi}-e^{-\bar{\xi}})]. \quad \text{(A29)}$$

This integral equation is somewhat complicated by the fact that when $t=0$, $\bar{G}_{>(<)}(t)$ as given by (A24) does not exist. This must be interpreted as meaning that $\bar{G}_{>(<)}(t)$ is a distribution at $t=0$ instead of a function. This may be made more precise if we note from (A22) that

$$\lim_{t\to\infty} G_{>(<)}(p_{1>(<)}',t)=0. \quad \text{(A30)}$$

Then if we recall (A8), we may integrate the partial differential equation for $G_{>(<)}$ (A7) and find

$$\left(\frac{\partial^2}{\partial p_{1>(<)}'^2}+\delta\frac{\partial}{\partial p_{1>(<)}'}\right)\times\int_{-\infty}^{\infty} dt\, G_{>(<)}(p_{1>(<)}',t)=0. \quad \text{(A31)}$$

Therefore

$$\int_{-\infty}^{\infty} dt\, G_{>(<)}(p_{1>(<)}',t)=\text{const}e^{-\delta p_{1>(<)}'}+\text{const}' \quad \text{(A32)}$$

and

$$\int_{-\infty}^{\infty} dt\, \bar{G}_{>(<)}(t)=-\delta\,\text{const}. \quad \text{(A33)}$$

However, this constant can be no larger than $O(\phi^{1/2})$ because $V(p_1,p_2)$ is normalized to 1. Therefore we conclude that

$$\int_{-\infty}^{\infty} dt\, \bar{G}_{>(<)}(t)=O[\phi^{1/2}\delta]=O[\phi^{1/2}(\ln\phi)^{-1}]. \quad \text{(A34)}$$

If $\phi\ll t$, we may approximate $\bar{G}_{>(<)}(t)$ by

$$\bar{G}_{>(<)}(t)\sim-\phi^{1/2}(2\pi i)^{-1}\times\int_{-i\infty}^{i\infty} ds\, e^{st}s^{1/2}\tfrac{1}{2}(s\phi)^{-1/2(1\pm\delta)}[\Gamma(\mp\delta)]^{-1}\operatorname{sgn}\delta\times\{(s\phi)^{-\delta/2}[\Gamma(1-\delta)]^{-1}-(s\phi)^{\delta/2}[\Gamma(1+\delta)]^{-1}\}^{-1}. \quad \text{(A35)}$$

The integrand of (A35) is analytic in s except for a cut which may be taken on the negative real s axis. We may deform the contour to go around this cut and obtain

$$\bar{G}_{>(<)}(t)\sim\tfrac{1}{2}\delta^2\int_0^{\infty} ds\, e^{-st}\{(\phi s)^{-\delta/2}-(\phi s)^{\delta/2}\}^{-2}. \quad \text{(A36)}$$

If, in addition,

$$t^{|\delta|}\sim 1 \quad (t\ll\phi^{-1}), \quad \text{(A37)}$$

we have

$$\bar{G}_{>(<)}(t)\sim\tfrac{1}{2}\delta^2(\phi^{-\delta/2}-\phi^{\delta/2})^{-2}t^{-1}. \quad \text{(A38)}$$

If, instead of (A37), we have

$$t^{|\delta|}\gg 1 \quad (t\gg\phi^{-1}), \quad \text{(A39)}$$

then

$$\bar{G}_{>(<)}(t)\sim\delta^2\phi^{|\delta|}t^{-1-|\delta|}. \quad \text{(A40)}$$

Since $\delta^{-1}=O(\ln\phi)$, approximations (A36) and (A40) show that $\bar{G}_{>(<)}(t)$ is much larger than its integral as given by (A34). Therefore we may use (A34) to rewrite (A29) as

$$\int_{-\infty}^{\xi} d\bar{\xi}\, \bar{G}_>[2(\kappa-\kappa^{-1})^{-1}(e^{\xi}-e^{\bar{\xi}})][e^{\xi}g(\bar{\xi})-e^{\bar{\xi}}g(\xi)]+\int_{\xi}^{\infty} d\bar{\xi}\, \bar{G}_<[2(\kappa-\kappa^{-1})^{-1}(e^{-\xi}-e^{-\bar{\xi}})][e^{-\xi}g(\bar{\xi})-e^{-\bar{\xi}}g(\xi)]$$

$$\doteqdot -g(\xi)\left\{\int_0^{e^{\xi}} d\bar{t}\, \bar{G}_>[2(\kappa-\kappa^{-1})^{-1}(e^{\xi}-\bar{t})]+\int_0^{e^{-\xi}} d\bar{t}\, \bar{G}_<[2(\kappa-\kappa^{-1})^{-1}(e^{-\xi}-\bar{t})]\right\}$$

$$=g(\xi)\left\{\int_{-\infty}^{0} d\bar{t}\, \bar{G}_>[2(\kappa-\kappa^{-1})^{-1}(e^{\xi}-\bar{t})]+\int_{-\infty}^{0} d\bar{t}\, \bar{G}_<[2(\kappa-\kappa^{-1})(e^{-\xi}-\bar{t})]\right\}. \quad \text{(A41)}$$

In the two integrals on the right-hand side, if ξ is of order 1, we may repalce $\bar{G}_{>(<)}$ by approximation (A38) because (a) the argument is never larger than order 1 and (b) the distribution at $\bar{\xi}=\xi$ does not contribute to leading order because the rest of the integrand vanishes there. We may approximate $\bar{G}_{>(<)}$ in the integrals of the right-hand side by (A38) as long as $|\bar{t}|\lesssim\phi^{-1}$. When $|\bar{t}|\gtrsim\phi^{-1}$, the extra factor of $t^{-|\delta|}$ in approximation (A40) acts as a cutoff and assures us that the integrals converge. We therefore obtain the leading term correctly if we replace $-\infty$ by $-\phi^{-1}$. Therefore (A41) is, when $\xi=O(1)$, correctly approximated as

$$\tfrac{1}{4}(\kappa-\kappa^{-1})\{\phi^{-\delta/2}-\phi^{\delta/2}\}^{-2}\delta^2\left\{\int_{-\infty}^{\xi} d\bar{\xi}\, (e^{\xi}-e^{\bar{\xi}})^{-1}[e^{\xi}g(\bar{\xi})-e^{\bar{\xi}}g(\xi)]+\int_{\xi}^{\infty} d\bar{\xi}\, (e^{-\xi}-e^{-\bar{\xi}})^{-1}[e^{-\xi}g(\bar{\xi})-e^{-\bar{\xi}}g(\xi)]\right\}$$

$$\doteqdot g(\xi)\tfrac{1}{4}(\kappa-\kappa^{-1})\{\phi^{-\delta/2}-\phi^{\delta/2}\}^{-2}\delta^2\left\{\int_{-\phi^{-1}}^{0} d\bar{t}\, [(e^{\xi}-\bar{t})^{-1}+(e^{-\xi}-\bar{t})^{-1}]\right\}. \quad \text{(A42)}$$

We therefore obtain

$$\int_{-\infty}^{\infty} d\bar{\xi}\, g(\bar{\xi})+\int_{-\infty}^{\infty} d\bar{\xi}\, (e^{\xi-\bar{\xi}}-1)^{-1}[g(\bar{\xi})-g(\xi)]+\int_{\xi}^{\infty} d\bar{\xi}\, (e^{-\xi+\bar{\xi}}-1)^{-1}[g(\bar{\xi})-g(\xi)]$$

$$\doteqdot g(\xi)[\ln(e^{\xi}+\phi^{-1})+\ln(e^{-\xi}+\phi^{-1})]\doteqdot -g(\xi)2\ln\phi. \quad \text{(A43)}$$

This expression is valid for ξ of order 1 as $\phi \to 0$. It is clear that the only possible solution is

$$g(\xi) = \text{const} \quad \text{for } \ln\phi + |\xi| \ll 1$$
and
$$g(\xi) = 0 \quad \text{for } \ln\phi + |\xi| \gg 1. \tag{A44}$$

When $\ln\phi + |\xi| = O(1)$, $g(\xi)$ approaches zero in some manner which is not explicitly needed for our purposes. We finally make use of (A28) to obtain the desired result

$$f(p_2) = \text{const}(\kappa^2-1)(\kappa e^{2p_2}-1)^{-1}(\kappa e^{-2p_2}-1)^{-1} \quad \text{for } |p_2 \pm \tfrac{1}{2}\ln\kappa| \gg \phi^{-1}. \tag{A45}$$

It now remains to use (A45) to demonstrate that the choice of the functions $V_\pm$ of Sec. 4 is correct. To do this we first notice that $f(p_2)$ possesses simple poles at $p_2 = \pm\frac{1}{2}\ln\kappa$ with equal residues. This implies that the δ functions at $p_2 = \pm\frac{1}{2}\ln\kappa$ that we obtained in approximation (4.70) must have equal coefficients. It is this equality of coefficients that is produced by the choice $A_\pm(s) = s^{\mp\delta}$. If $A_\pm(s) = s^{\alpha_\pm}$ with $\alpha_\pm \neq \mp\delta$, then, if we satisfy the boundary condition (4.30), the corresponding δ functions of (4.70) will not have equal coefficients.

To see in slightly more detail how $V_\pm$ of (4.63) joins to (A45) we note that as $p_\pm \to O(\phi^{-1})$

$$V_\pm(p_1,p_\pm) = \tfrac{1}{4}\phi\kappa^{-1/2}|1-\kappa^2|\,V(p_1,p_2) \to \phi[\pm(p_2 \mp \tfrac{1}{2}\ln\kappa)4\kappa^{1/2}(1-\kappa^2)]^{-1\pm\delta}. \tag{A46}$$

All quantities in the square brackets are of order 1, so δ may be set equal to zero. However, (A46) with $\delta \to 0$ is exactly the form (A45) takes when $p_2 \to \pm\frac{1}{2}\ln\kappa$, so we conclude that $V_\pm(p_1,p)$ given by (4.63) is asymptotically equal to $V(p_1,p_2)$ obtained from (A45).

Finally, it may be argued that we have only fixed the form of $A_\pm(s)$ when $s \to 0$. This is indeed correct; however, a more detailed specification of $A_\pm(s)$ merely alters the detailed nature of the sharp peaks at $p_2 = \pm\frac{1}{2}\ln\kappa$ and can alter neither the qualitative picture of $V(p_1,p_2)$ given in Sec. 4 for $\phi \sim 0$, $|\delta|^{-1} \sim -\ln\phi$ nor the final approximation (4.70). Indeed, we may use our function $f(p_2)$ in conjunction with $V_\pm$ and the Green's-function representation (A17) to study in detail how the one peak of $V(p_1,p_2)$ spreads out and vanishes when $|p_1| + \ln\phi = O(1)$. However, since these refined details do not affect the results of Secs. 5 and 6, we will pursue them no further.

APPENDIX B

We may study the $m \to \infty$ limits of the correlation functions $S_m^{(4)}(l)$ for the example of Sec. 6 where the variation of E_2 is E_2^0, E_2^0, E_2^1, E_2^1 repeated periodically by a straightforward application of Szegö's theorem. [The superscript (4) indicates the number of bonds in the fundamental cell which is periodically repeated.] For concreteness consider a lattice with the rows labeled as in Fig. 5. Then in any row we can always apply (6.8) and write $[S_m^{(4)}(l)]^2$ as a $2m \times 2m$ block Toeplitz determinant. It is clear from Fig. 5 that in the thermodynamic limit if

$$E_2^0 \leftrightarrow E_2^1, \tag{B1}$$

then

$$S_m^{(4)}(0) \leftrightarrow S_m^{(4)}(2). \tag{B2}$$

Furthermore, the lattice is symmetric about row zero (in fact, about any even row) in the sense of (6.10). Therefore $S_m^{(4)}(0)$ is given by the $m \times m$ Toeplitz determinant (6.13). We will demonstrate that this expression is not invariant under the substitution (B1) so that $S_m^{(4)}(0) \neq S_m^{(4)}(2)$.

To evaluate (6.13) we need an expression for $\bar{x}^{(4)}(0,\mathfrak{M};\theta)$ that appears in (6.12b) when $\mathfrak{M} \to \infty$. Since

$$\lim_{\mathfrak{M}\to\infty} \bar{x}^{(4)}(0,\mathfrak{M};\theta)$$

exists, we may let $\mathfrak{M}$ tend the ∞ through multiples of 4 without altering the limit. For this particular lattice we have

$$\bar{D}^{(4)}(0,4\mathfrak{M}') = -D^{(4)}(0,4\mathfrak{M}'), \tag{B3}$$

so we have

$$\bar{x}^{(4)}(0,4\mathfrak{M}') = x^{(4)}(0,4\mathfrak{M}'), \tag{B4}$$

where we may evaluate the right-hand side in the $\mathfrak{M} \to \infty$ limit by use of the recursion relation (2.23), which in this case becomes

$$\begin{pmatrix} C^{(4)}(0,4\mathfrak{M}'+1) \\ D^{(4)}(0,4\mathfrak{M}'+1) \end{pmatrix} = \begin{pmatrix} a^2+b^2 & a \\ a & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & z_2^2(4\mathfrak{M}') \end{pmatrix} \begin{pmatrix} C^{(4)}(0,4\mathfrak{M}') \\ D^{(4)}(0,4\mathfrak{M}') \end{pmatrix}. \tag{B5}$$

To compute (B4) we need a recursion relation connecting $4\mathfrak{M}'$ and $4(\mathfrak{M}'+1)$. Therefore we iterate (B5) to find

$$\begin{pmatrix} C^{(4)}(0,4(\mathfrak{M}'+1)) \\ D^{(4)}(0,4(\mathfrak{M}'+1)) \end{pmatrix} = \begin{pmatrix} a^2+b^2 & az_2^2(4\mathfrak{M}'+3) \\ a & z_2^2(4\mathfrak{M}'+3) \end{pmatrix} \begin{pmatrix} a^2+b^2 & az_2^2(4\mathfrak{M}'+2) \\ a & z_2^2(4\mathfrak{M}'+2) \end{pmatrix} \begin{pmatrix} a^2+b^2 & az_2^2(4\mathfrak{M}'+1) \\ a & z_2^2(4\mathfrak{M}'+1) \end{pmatrix}$$
$$\times \begin{pmatrix} a^2+b^2 & az_2^2(4\mathfrak{M}') \\ a & z_2^2(4\mathfrak{M}') \end{pmatrix} \begin{pmatrix} C^{(4)}(0,4\mathfrak{M}') \\ D^{(4)}(0,4\mathfrak{M}') \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} C^{(4)}(0,4\mathfrak{M}') \\ D^{(4)}(0,4\mathfrak{M}') \end{pmatrix}, \tag{B6}$$

where, if we use

$$z_2^2(4\mathfrak{M}') = z_2^2(4\mathfrak{M}'+3) = \lambda_0 \tag{B7a}$$

and

$$z_2^2(4\mathfrak{M}'+1) = z_2^2(4\mathfrak{M}'+2) = \lambda_1, \tag{B7b}$$

we have

$$y_{11}=[(a^2+b^2)^2+a^2\lambda_0][(a^2+b^2)^2+a^2\lambda_1]+a^2\lambda_1[a^2+b^2+\lambda_0][a^2+b^2+\lambda_1], \quad \text{(B8a)}$$

$$y_{12}=a\lambda_0\{[(a^2+b^2)^2+a^2\lambda_0][a^2+b^2+\lambda_1]+\lambda_1[a^2+b^2+\lambda_0](a^2+\lambda_1)\}, \quad \text{(B8b)}$$

$$y_{21}=a\{[(a^2+b^2)^2+a^2\lambda_1][a^2+b^2+\lambda_0]+\lambda_1[a^2+b^2+\lambda_1](a^2+\lambda_0)\}, \quad \text{(B8c)}$$

and

$$y_{22}=\lambda_0[a^2(a^2+b^2+\lambda_0)(a^2+b^2+\lambda_1)+\lambda_1(a^2+\lambda_0)(a^2+\lambda_1)]. \quad \text{(B8d)}$$

The matrix in (B6) is independent of $\mathfrak{M}'$ because of the underlying periodicity of the lattice, so the determination of $x^{(4)}(0,4\mathfrak{M}')$ as $\mathfrak{M}'\to\infty$ is reduced to determining the eigenvector of (B6) with the largest eigenvalue. From (B6)

$$x^{(4)}[0,4(\mathfrak{M}'+1)]=[y_{11}x^{(4)}(0,4\mathfrak{M}')+y_{12}]+[y_{21}x^{(4)}(0,4\mathfrak{M}')+y_{22}]^{-1}, \quad \text{(B9)}$$

so, letting

$$x^{(4)}=x^{(4)}[0,4(\mathfrak{M}'+1)]=x^{(4)}(0,4\mathfrak{M}'), \quad \text{(B10)}$$

we find

$$x^{(4)}=(2y_{21})^{-1}\{y_{11}-y_{22}+[(y_{22}-y_{11})^2+4y_{12}y_{21}]^{1/2}\}, \quad \text{(B11)}$$

where the plus sign is chosen for the square root in order that $x^{(4)}=x_0$ [defined by (3.5)] when $\lambda_1=\lambda_0$.

If we now use (B11) in (6.12b), we find

$$S_m^{(4)}(0)=\pm\det\begin{bmatrix} a_{12}^{(0)} & \cdots & a_{12}^{(m-1)} \\ \vdots & & \vdots \\ a_{12}^{(-m+1)} & \cdots & a_{12}^{(0)} \end{bmatrix}, \quad \text{(B12)}$$

with

$$a_{12}^{(m)}=(2\pi)^{-1}\int_{-\pi}^{\pi} d\theta\, e^{im\theta}a_{12}^{(4)}(\theta) \quad \text{(B13)}$$

and

$$a_{12}^{(4)}(\theta)=\{b-ia+i(2y_{21})^{-1}(y_{11}-y_{22}+[(y_{22}-y_{11})^2+4y_{12}y_{21}]^{1/2})\}\{b+ia-i(2y_{21})^{-1}(y_{11}-y_{22}+[y_{22}-y_{11})^2+4y_{12}y_{21}]^{1/2})\}^{-1}. \quad \text{(B14)}$$

From inspection of (B14) we see that $a_{12}^{(4)}(\theta)$ is not invariant under the interchange $\lambda_0\leftrightarrow\lambda_1$. To verify that this implies that $S_m^{(4)}(0)\neq S_m^{(4)}(2)$ we note that (B14) has the required property (6.15) that

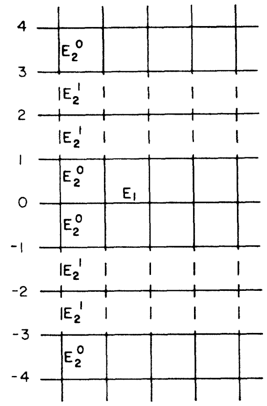

FIG. 5. Lattice used to demonstrate that $S_\infty(l)$ may depend upon l.

$$a_{12}^{(4)}(-\theta)=[a_{12}^{(4)}(\theta)]^{-1} \quad \text{(B15)}$$

and, furthermore, that it may be put in the form

$$a_{12}^{(4)}(\theta)=i[(y_{22}-y_{11})^2+4y_{21}y_{12}]^{1/2}\times[y_{21}(b+ia)^2+y_{12}+(b+ia)(y_{22}-y_{11})]^{-1}=\left(\frac{(1-A_1e^{i\theta})(1-A_2e^{-i\theta})(1-A_3e^{i\theta})(1-A_4e^{-i\theta})}{(1-A_1e^{-i\theta})(1-A_2e^{i\theta})(1-A_3e^{-i\theta})(1-A_4e^{i\theta})}\right)^{1/2}. \quad \text{(B16)}$$

The A's are the roots of a polynomial of eighth degree and will not be explicitly written out. However, if the A's are suitably chosen, we can easily see that if $\lambda_0\to\lambda_1$,

$$A_1\to\alpha_1,\quad A_2\to\alpha_2,\quad\text{and}\quad A_3\to A_4. \quad \text{(B17)}$$

Furthermore, since $(y_{22}-y_{11})^2+4y_{21}y_{12}$ is invariant under $\lambda_0\leftrightarrow\lambda_1$, this substitution merely permutes the A's and, in particular, A_1 and A_2 are left invariant while $A_3\leftrightarrow A_4$. We may now apply Szegö's theorem[15] [in the form (6.17)] to (B12) and find

$$S_\infty^{(4)}(0)=[(1-A_1^2)(1-A_2^2)(1-A_3^2)(1-A_4^2)]^{1/4}\times[(1-A_1A_3)(1-A_2A_4)]^{1/2}[(1-A_1A_2)(1-A_1A_4)\times(1-A_3A_2)(1-A_3A_4)]^{-1/2}. \quad \text{(B18)}$$

This is clearly not invariant if $A_3\leftrightarrow A_4$, so we conclude that $S_\infty^{(4)}(0)\neq S_\infty^{(4)}(2)$ if $\lambda_0\neq\lambda_1$.