that report suggests that the data itself is not inconsistent with the results of the present experiment. The fluctuation in f, which they interpret as a maximum connected with the ferroelectric transition, is of the same order and shape as several unexplained fluctuations in f at other temperatures. In the published data, the fluctuation in f near T_c is not clearly established as reproducible. Finally, the fluctuation near T_c is of the same order as the general scatter of the data points and could actually be interpreted as a minimum rather than a maximum if a slightly different choice is made as to which of the scattered data points are considered to be more reliable. In light of all these features of the data published by Hazony *et al.*, it is reasonable to say that the previous data are consistent with a negative result, and that the present data, with smaller scatter of the data points and better temperature resolution, further establish that negative result.

VI. CONCLUSIONS

The temperature dependence of f for FAS is consistent with the anomalous lattice-mode theory of ferroelectric transitions and clearly indicates the displacive nature of the FAS transition. The linewidth variation for FAS indicates a temperature dependence for the structure of the ferroelectric alums, which deserves further study by appropriate experimental and theoretical techniques. The negative results for KFCT suggest an order-disorder model for the KFCT transition but do not exclude one particular displacive mechanism, as was discussed in Sec. V.

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Hydrodynamic Theory of Spin Waves

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A hydrodynamic theory of spin waves is developed for certain magnetic systems in analogy with the derivation of two-fluid hydrodynamics for liquid helium. The systems considered are "isotropic" and "planar" ferromagnets and antiferromagnets. In each system, low-frequency spin waves are predicted to exist at long wavelengths for any temperature below the transition to the paramagnetic phase. The real part of the frequency is given exactly in terms of thermodynamic quantities. The damping rate is proportional to the square of the real part of the frequency in each case, and hence is negligible in the long-wavelength limit, compared to the real part. These results for the damping rates are new, and disagree with previous microscopic calculations for the Heisenberg ferromagnet and antiferromagnet. An experiment using neutron diffraction is proposed to test the hydrodynamic theory in the almost isotropic antiferromagnet $RbMnF_3$. The assumptions necessary to derive the hydrodynamic theory are discussed in detail, as are the limits of validity of the theory, and the applicability of the results to real systems.

1. INTRODUCTION

A NUMBER of models of magnetic systems are similar to superfluid helium in that ground states of their ordered phases exhibit broken symmetry with respect to a continuous symmetry of the Hamiltonian.[1,2] It is therefore reasonable to attempt to deduce dynamic properties of magnetic systems from macroscopic hydrodynamic considerations, just as one can derive the theory of two-fluid hydrodynamics for the properties of superfluid helium.[3,4] By hydrodynamic considerations we mean a theory utilizing only the conservation laws for the Hamiltonian, the symmetry properties of the ground state, and assumptions that one can expand certain quantities in powers of the gradients and magnitudes of the deviations from equilibrium, when these derivations are sufficiently small and slowly varying.

The motivation for carrying out a hydrodynamic investigation of magnetic systems is twofold. In the first place, we hope to increase our understanding of magnetic systems; in particular, we may make predictions about the spectrum of spin fluctuations as measured by inelastic diffraction of neutrons in certain real systems. In the second place, one can gain insight into the foundations of two-fluid hydrodynamics for

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[1] T. Matsubara and H. Matsuda, Progr. Theoret. Phys. (Kyoto) **16**, 416 (1956); **16**, 569 (1956); **17**, 19 (1957); P. R. Zilsel, Phys. Rev. Letters **15**, 476 (1965); D. D. Betts and M. H. Lee, *ibid.*, **20**, 1507 (1968).

[2] See also M. E. Fisher, Rept. Progr. Phys., **30**, 615 (1967).

[3] L. D. Landau, J. Phys. (USSR) **5**, 71 (1941). (Reprinted in Ref. 4 below.) See also L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959), Chap. 16.

[4] I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965).

helium, by careful examination of the underlying assumptions in the magnetic systems, which are, in fact, somewhat simpler systems than the superfluid. The principal results of this paper have been stated previously by the authors,[5–7] and applied to a discussion of critical phenomena.

The theory of two-fluid hydrodynamics[3,4] describes the nonequilibrium behavior of superfluid helium for situations in which physical quantities vary slowly in space and time. This macroscopic theory contains, in addition to the conserved densities of ordinary fluid mechanics, a new velocity field pertaining to the superfluid, whose variations lead to an additional propagating normal mode in the system, second sound. Although this theory was introduced on purely phenomenological grounds by Landau, it was later realized that its microscopic basis lay in the phenomenon of Bose condensation.[8–12] At low temperatures, the equilibrium states of a Bose liquid possess an additional thermodynamic variable, the Bose field density $\langle\psi(\mathbf{r})\rangle$, whose nonzero value corresponds to the breaking of gauge symmetry. For uniform systems in equilibrium, the magnitude of the complex function $\langle\psi(\mathbf{r})\rangle$ is determined by the temperature and pressure of the system, but the phase φ is an arbitrary constant. For a nonuniform state, spatial variations of the phase of $\langle\psi(\mathbf{r})\rangle$ lead to superfluid flow, with a velocity field $\mathbf{v}_s=(\hbar/m)\nabla\varphi$. In order to obtain two-fluid hydrodynamics, one assumes that the *magnitude* of $\langle\psi(\mathbf{r})\rangle$ is entirely determined by the local temperature and pressure.

The ideal magnetic systems considered in this paper are the planar ferromagnet, the planar antiferromagnet, the isotropic antiferromagnet, and the isotropic ferromagnet. The first two of these are systems with an easy plane of spin alignment, with Hamiltonians that are invariant under any uniform rotation of the spins in this plane.[13] The isotropic systems have Hamiltonians which are invariant under an *arbitrary* uniform rotation of the spins. An equivalence of the planar ferromagnet to a lattice-gas model for Bose condensation in liquid helium has been noted by other authors,[1,2] who have been primarily interested in static phenomena.

For the magnetic systems, the hydrodynamic variables analogous to $\mathbf{v}_s$ are the gradients of the direction of orientation of the order parameter. The other variables which enter the hydrodynamics are the energy density, and the densities of the conserved components of the total magnetization. The magnitude of the order parameter is a conserved quantity only for the isotropic ferromagnet.

A central assumption of our analysis is the existence of a microscopic relaxation time τ, characterizing the approach to local equilibrium. Specifically, we assume that if a system initially disturbed from equilibrium is left to evolve undisturbed for a time long compared to τ, then it will reach a state of local equilibrium entirely determined by the local values of the conserved densities and their gradients. Moreover, for the antiferromagnets and the planar ferromagnet, just as for helium,[10,11] the magnitude of the order parameter is not treated as a separate variable but is assumed to relax in a time τ to a value determined by the local values of the other conserved densities.

Given these assumptions, the main result of this paper is, in each case, the prediction of propagating spin-wave modes, whose damping rate is much smaller than the frequency, in the limit of long wavelengths, at any temperature below the transition to the paramagnetic phase. For the isotropic antiferromagnet, as well as for the planar ferromagnet and antiferromagnet, we predict a spin-wave frequency linear in the wave vector $\mathbf{k}$,

$$\omega(\mathbf{k})=ck, \tag{1.1}$$

where the spin-wave velocity c is exactly related to a stiffness constant ρ_s and a susceptibility χ by

$$c=(\rho_s/\chi)^{1/2}. \tag{1.2}$$

Both ρ_s and χ are temperature-dependent thermodynamic quantities that can be obtained from static measurements: ρ_s, can, in principle, be obtained from the quasielastic scattering of neutrons, while χ is the ordinary magnetic susceptibility in a direction perpendicular to the direction of sublattice magnetization for the isotropic antiferromagnet, and parallel to the symmetry axis for the planar systems. In these systems, the damping rate for spin waves is proportional to k^2, which is indeed negligible compared to the real part of the frequency for small k.

In the isotropic ferromagnet, the real part of the spin-wave frequency is found to be proportional to k^2, for small k. The constant of proportionality is again related to thermodynamic quantities,

$$\omega/k^2=\rho_s/M_0, \tag{1.3}$$

where M_0 is the equilibrium magnetization. This is just the well-known formula of Landau and Lifshitz[14] with the use of temperature-dependent values of ρ_s and M_0. The spin-wave damping rate is predicted to

[5] B. I. Halperin and P. C. Hohenberg, Phys. Rev. Letters **19**, 700 (1967); **19**, 940E (1967).

[6] B. I. Halperin and P. C. Hohenberg, Phys. Rev. **177**, 952 (1969).

[7] B. I. Halperin and P. C. Hohenberg, J. Appl. Phys. **40**, 1554 (1969).

[8] N. N. Bogoliubov, J. Phys. (USSR) **11**, 23 (1947).

[9] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).

[10] N. N. Bogoliubov, Dubna report, 1963 (unpublished).

[11] P. C. Hohenberg and P. C. Martin, Ann. Phys. (N. Y.) **34**, 291 (1965).

[12] P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).

[13] We do *not* mean by planar that the lattice is two dimensional, or that the axial components of the spins are forced to lie in a plane for nonequilibrium states of the system. Both the lattice and the spins are three dimensional.

[14] L. D. Landau and E. M. Lifshitz, Physik Z. Sowjetunion **8**, 153 (1935).

be proportional to k^4, for small k, which is again small compared to the real part.

The spin waves derived in this work bear the same relation to the spin wave elementary excitations found at low temperatures by Bloch,[15] Dyson[16] and others,[17] as hydrodynamic first sound bears to the phonons in helium[11] or a crystal.[18] A sufficiently accurate calculation of renormalized spin waves in terms of elementary excitations should at long wavelengths reproduce the hydrodynamic results. In addition, a microscopic calculation can give approximate numerical values for the parameters of the hydrodynamic theory.

Equation (1.2) for the spin-wave frequency was derived microscopically for the isotropic Heisenberg antiferromagnet at low temperatures by Tani.[19] Published microscopic calculations of the damping in this system,[19–21] on the other hand, have not found a damping proportional to k^2 for small k, so that this result represents a new prediction of hydrodynamics, which we believe to be correct. A more recent perturbation theory calculation at low temperatures by Harris, Kumar, and the authors[22] has in fact succeeded in reproducing microscopically the k^2 damping at long wavelengths.

For the isotropic ferromagnet, the spin-wave frequency Eq. (1.3) has also been derived by various authors.[17] Microscopic calculations have predicted a spin-wave damping proportional to $k^4 \ln k$,[23] for small k and low temperatures, rather than the k^4 predicted by hydrodynamics.

For both normal fluids[24] and superfluids,[11] it is possible to deduce the form of the correlation functions of the conserved densities for small k and ω, once the hydrodynamic equations are known. Similarly, for the magnetic systems considered here, we may find the long-wavelength low-frequency form of certain correlation functions which are directly related to inelastic neutron scattering experiments.

In Sec. 2 we discuss in detail the derivation of the hydrodynamic equations for the planar ferromagnet. In Sec. 3 we use these results to obtain dynamic linear response functions and correlation functions for the system. In Sec. 4 we interpret the planar ferromagnet as a lattice gas model for superfluid helium, and we discuss the similarities and differences between the hydrodynamics of the magnetic and helium systems. The planar antiferromagnet, the isotropic antiferromagnet and the isotropic ferromagnet are considered in Sec. 5, 6, and 7, respectively.

In Sec. 8, we discuss the probable limits of validity of the long-wavelength hydrodynamic theory in the ideal magnetic models under consideration. The applicability of the hydrodynamic theory to a real antiferromagnet or ferromagnet with small anisotropy is discussed in Sec. 9.

In Sec. 10, we reexamine one of the assumptions used in this paper and, implicitly, in derivations of two-fluid hydrodynamics,[3,4,10,11] namely, the assumption that the magnitude of the order parameter relaxes to local equilibrium at a nonzero "microscopic" rate. Although this assumption is indeed questionable, we argue that it is almost certainly not necessary for the validity of hydrodynamics in all of the magnetic models in the absence of an external magnetic field. However, a possible breakdown of this assumption could affect the hydrodyanmic predictions for the intrinsic linewidth of Brillouin scattering in superfluid helium and in ordinary crystalline solids.

In Sec. 11 we discuss the breakdown of linear hydrodynamics in the ideal magnetic systems, for *finite* deviations from equilibrium. Since the only direct measurements of the spin-wave spectrum, via inelastic diffraction of neutrons, yield the response of the system to *infinitesimal* perturbing fields, this section is not of direct experimental relevance. It is useful, however, in revealing differences between the dynamic properties of the various magnetic systems and of helium. In the Appendix we derive the relation between the stiffness constant ρ_s and the linear response to a static inhomogeneous magnetic field.

The possibility of testing the hydrodynamic theory of spin waves by neutron experiments on a nearly isotropic antiferromagnet is discussed in Sec. 9.

2. PLANAR FERROMAGNET

A. Equilibrium States

We shall first consider an anisotropic ferromagnet with an easy plane of magnetization, which we call the planar ferromagnet.[13] Let us consider a lattice of spins whose Hamiltonian is

$$\mathcal{H}=\mathcal{H}_0-H_z\sum_i S_i^z, \tag{2.1}$$

$$\mathcal{H}_0=-\sum_{i,j}\left[J_{ij}^z S_i^z S_j^z+J_{ij}^1(S_i^x S_j^x+S_i^y S_j^y)\right], \tag{2.2}$$

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[15] F. Bloch, Z. Physik **61**, 206 (1930).
[16] F. J. Dyson, Phys. Rev. **102**, 1217 (1956); **102**, 1230 (1956).
[17] For a comprehensive review see F. Keffer, in *Handbuch der Physik* (Springer-Verlag, New York, 1966), Vol. XVIII/2, p. 1.
[18] P. C. Kwok and P. C. Martin, Phys. Rev. **142**, 495 (1966).
[19] K. Tani, Progr. Theoret. Phys. (Kyoto) **31**, 335 (1964).
[20] M. I. Kaganov, V. M. Tsukernik, and I. Ye. Chupis, Fiz. Metal. i Metalloved. **10**, 797 (1960) [English transl.: Phys. Metals Metallog. (USSR) **10**, 154 (1960)].
[21] J. Sólyom, Zh. Eksperim. i Teor. Fiz. **55**, 2355 (1968) [English transl.: Soviet Phys.—JETP **28**, 1251 (1969).]
[22] A. B. Harris, D. Kumar, B. I. Halperin, and P. C. Hohenberg, 15th Annual Conference on Magnetism and Magnetic Materials, Philadelphia, Pa., Nov. 1969, Paper 7F4. Proceedings to be published J. Appl. Phys., March 1970.
[23] V. N. Kascheev and M. A. Krivoglaz, Fiz. Tverd. Tela **3**, 1541 (1961) [English transl.: Soviet Phys.—Solid State **3**, 1117 (1961)]; A. B. Harris, Phys. Rev. **175**, 674 (1968); V. G. Vaks, A. I. Larkin, and S. A. Pikin, Zh. Eksperim. i Teor. Fiz. **53**, 1089 (1967) [English transl.: Soviet Phys.—JETP **26**, 674 (1968)].
[24] L. P. Kadanoff and P. C. Martin, Ann. Phys. (N. Y.) **24**, 419 (1963). See also, L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959), Chap. 17.

where $\mathbf{S}_i$ is the value of the spin angular momentum on the ith lattice site, $\mathcal{H}_0$ is the intrinsic Hamiltonian of the spin system, and H_z is an applied magnetic field in the z direction. (We have chosen our units of magnetic field strength so that the gyromagnetic ratio is unity). The coupling constants J_{ij}^z and $J_{ij}^\perp$ are assumed to depend only on the coordinate difference between lattice sites i and j. Note that the Hamiltonian $\mathcal{H}$ is invariant if all the spins are rotated simultaneously about the z axis. The z component M_z of the total magnetization, as well as the internal Hamiltonian $\mathcal{H}_0$, commute with the total Hamiltonian $\mathcal{H}$, and are constants of the motion.

With proper choice of the coupling constants J, the ground state of the system without an external field will have a ferromagnetic alignment of the spins, with $\langle \mathbf{S}_i \rangle$ in an arbitrary direction of the x-y plane,

$$\langle S_i^x \rangle + i\langle S_i^y \rangle = M_\perp e^{i\varphi}/\mathfrak{N}, \tag{2.3}$$

$$\langle S_i^z \rangle = 0, \tag{2.4}$$

where $\mathfrak{N}$ is the number of lattice sites in the system, $M_\perp$ is positive, and the angle φ is arbitrary. (For a model with nearest-neighbor interactions only, this planar alignment will occur if $J^\perp$ is positive and greater than $|J^z|$.) If a field H_z is applied which is less than some critical-field strength, the ground state will still have a ferromagnetic alignment in which $M_\perp \neq 0$, but it will now also have

$$\langle S_i^z \rangle = \mathfrak{N}^{-1} M_z \neq 0. \tag{2.5}$$

If the temperature is raised, one will eventually reach a critical temperature T_c, above which $M_\perp$ will vanish. We shall refer to the system in the paramagnetic phase above T_c as a planar paramagnet.

The equilibrium states of the planar paramagnet are determined by the values of the two conserved quantities: M_z and the internal energy

$$E = \langle \mathcal{H}_0 \rangle. \tag{2.6}$$

The entropy of the equilibrium system is a function of E and M_z:

$$S = S_0(E, M_z). \tag{2.7}$$

If the paramagnet is in true thermal equilibrium at some temperature T, then the value of M_z is determined by the external field H_z and the entropy S; in particular, M_z must be chosen to minimize $E - H_z M_z$ for the given entropy. Thus, in true thermal equilibrium we have

$$h(E, M_z) = H_z, \tag{2.8}$$

where h is defined in terms of Eq. (2.7) by

$$h \equiv \partial E/\partial M_z|_S. \tag{2.9}$$

Because M_z is a constant of the motion, however, we may also have an equilibrium state in which M_z is not equal to the thermodynamic value determined by (2.8) for the given field H_z. In fact, we expect that any state of the system in the paramagnetic region will eventually relax to an *equilibrium* state completely determined by the initial values of E and M_z, and independent of the field H_z.[25] [In particular, the entropy is given by (2.7), independent of the value of H_z.] Only, if the system is in *true thermal equilibrium* defined by (2.8), however, will the state be stable in the presence of any small perturbation which does not conserve M_z. It will be convenient, throughout this section, to retain the distinction between equilibrium, where M_z is independent of H_z, and true thermal equilibrium where M_z and H_z are related by Eq. (2.8).

Below the transition temperature, the equilibrium states of the ferromagnet are not adequately described by specifying E and M_z; one must also specify the direction φ of alignment of the perpendicular component of the magnetization. Of course, the entropy S will be independent of φ, and Eq. (2.8) will still describe the condition for the equilibrium state to be the state of true thermal equilibrium in a field H_z.

In order to specify an equilibrium state of the ferromagnet it is *not* necessary to give the value of $M_\perp$. The x and y components of the total magnetization do not commute with the Hamiltonian $\mathcal{H}$ and are not constants of the motion. Although the component $M_\perp$ can be changed from its equilibrium value $M_\perp = M_0(E, M_z)$ by the application of an external perturbation (e.g., by a magnetic field in the x direction), the value of $M_\perp$ will presumably relax, after the perturbation is turned off, towards the value of $M_\perp$ determined by the values of E and M_z at the instant the perturbation is turned off. The exact nature of this relaxation is not known. For the bulk of this paper, we shall assume that this relaxation is characterized by a nonzero *microscopic* rate, even in the limit where the deviation from equilibrium is small. As noted in Sec. 1, a similar assumption about the relaxation of fluctuations in the magnitude of the order parameter is implicit in the Landau-Khalatnikov[3,4] derivation of two-fluid hydrodynamics for helium. There is some reason for questioning this assumption, however, and we shall discuss this point in greater detail in Sec. 10.

B. Hydrodynamic Description

Let us define a magnetization density $\mathbf{m}$ and an (internal) energy density ϵ according to

$$\mathbf{m}(\mathbf{r}) = \langle \sum_i \mathbf{S}_i \delta(\mathbf{r} - \mathbf{r}_i) \rangle, \tag{2.10a}$$

$$\epsilon(\mathbf{r}) = -\langle \sum_{ij} [J_{ij}^z S_i^z S_j^z + J_{ij}^\perp (S_i^x S_j^x + S_i^y S_j^y)] \delta(\mathbf{r} - \mathbf{r}_i) \rangle, \tag{2.10b}$$

where $\mathbf{r}_i$ is the position of the ith spin. The expectation

[25] The total effect of the applied field on the time evolution of the system is equivalent to a transformation to a spin coordinate system which is rotating about the z axis, as we shall see below.

values denoted by the angular brackets in (2.10) may be evaluated if a density matrix for the system is known. In the equilibrium states, the functions $\mathbf{m}$ and ϵ are uniform and are equal to $\mathbf{M}$ and E, respectively. (We take the volume of the system to be unity.) We shall also wish to consider systems in which $\epsilon(\mathbf{r})$ and/or $\mathbf{m}(\mathbf{r})$ are not uniform, but exhibit small deviations from uniformity which are slowly varying in space. [We are not concerned with short-wavelength variations in these quantities, and we may, if we wish, choose the δ functions in (2.10) to be broad and give an average over some region of space. In any hydrodynamic theory one is concerned with states in which the long-wavelength variations in densities such as $\mathbf{m}(\mathbf{r})$ and $\epsilon(\mathbf{r})$ are known, but where the short-wavelength variations are unspecified.]

If a state in the ferromagnetic region has a magnetization $\mathbf{m}(\mathbf{r})$ which varies slowly in space, then we may define a phase function $\varphi(\mathbf{r})$ according to

$$m_x(\mathbf{r})+im_y(\mathbf{r})=m_\perp(\mathbf{r})e^{i\varphi(\mathbf{r})} \tag{2.11}$$

If a state with an initially nonuniform value for $\mathbf{m}(\mathbf{r})$ or $\epsilon(\mathbf{r})$ is allowed to evolve undisturbed, it will eventually relax to an equilibrium state in which $\mathbf{m}$ and ϵ are uniform. Because the total M_z and E are conserved, however, $m_z(\mathbf{r})$ and $\epsilon(\mathbf{r})$ cannot relax to uniform values in a microscopic time τ if the spatial scale of variation is very large; energy and/or magnetization must be transported from one place to another, and this transport cannot occur rapidly over arbitrarily large distances. Similarly, in the ferromagnetic phase, a long-wavelength variation in $\varphi(\mathbf{r})$ cannot relax rapidly to uniformity; knowledge of a perturbation in the phase at one point in space cannot be transmitted instantaneously to another point a large distance away.

If one wishes to describe the state, after some elapsed time t, of a system which was initially nonuniform, it is at least necessary to specify the long-wavelength variation of $\epsilon(\mathbf{r})$, $m_z(\mathbf{r})$, and $\varphi(\mathbf{r})$. We assume that after some microscopic time τ, the state of the system is *completely* determined by the long-wavelength variation of ϵ, m_z, and φ, and that the details of its past history are unimportant. For example, in a system with slowly varying $m_z(\mathbf{r})$ and/or $\epsilon(\mathbf{r})$, the magnitude of the order parameter $m_\perp(\mathbf{r})$ will also be nonuniform. After a time τ, however, the value of $m_\perp(\mathbf{r})$ will be determined by the functions m_z, ϵ, and φ. In particular, to lowest order in the variation of these fields, the value of $m_\perp(\mathbf{r})$ will be related to the local values of $\epsilon(\mathbf{r})$ and of $m_z(\mathbf{r})$ in the same way that $M_\perp$ is determined by E and M_z in equilibrium[26]:

$$m_\perp(\mathbf{r})=M_0[\epsilon(r),m_z(r)]. \tag{2.12}$$

[26] If m_z, ϵ, and φ are uniform, of course, the system must be in the equilibrium state for the given M_z, E, and φ.

If the state of the system is determined by the functions ϵ, m_z, and φ, then we must be able to express $\partial\epsilon/\partial t$, $\partial m_z/\partial t$, and $\partial\varphi/\partial t$ as functionals of ϵ, m_z, and φ. *We shall assume that it is possible to express the time derivatives of ϵ, m_z, and φ as an expansion in the spatial derivatives of these quantities.* It is convenient to introduce a new variable, the vector function

$$\mathbf{v}(\mathbf{r})\equiv\nabla\varphi(\mathbf{r}). \tag{2.13}$$

Fluctuations in $\nabla\varphi$ are of the same order as fluctuations in m_z in the long-wavelength limit because physical properties such as the energy or entropy are independent of the over-all magnitude of φ.

Let us write

$$\frac{\partial\varphi(\mathbf{r})}{\partial t}=-\psi(\mathbf{r}), \tag{2.14}$$

$$\frac{\partial\mathbf{v}(\mathbf{r})}{\partial t}=-\nabla\psi(\mathbf{r}), \tag{2.15}$$

$$\frac{\partial m_z(\mathbf{r})}{\partial t}=-\nabla\cdot\mathbf{j}^{m_z}(\mathbf{r}), \tag{2.16}$$

$$\frac{\partial\epsilon(\mathbf{r})}{\partial t}=-\nabla\cdot\mathbf{j}^{\epsilon}(\mathbf{r}), \tag{2.17}$$

where ψ, $\mathbf{j}^{m_z}$ and $\mathbf{j}^{\epsilon}$ are functionals of m_z, ϵ, and φ. The right-hand sides of (2.16) and (2.17) can be written as the divergences of vector fields because the volume integrals of the left-hand sides vanish as a consequence of the conservation laws. Microscopic definitions of the currents $\mathbf{j}^{m_z}$ and $\mathbf{j}^{\epsilon}$, in terms of the spin operators $\mathbf{S}_i$, may be written down but are unnecessary for our purposes. In view of (2.15), we may consider that $\mathbf{v}(\mathbf{r})$ is a conserved quantity, whose current is given by the product of $\psi(\mathbf{r})$ and the unit tensor. In fact, from the definition of $\mathbf{v}(\mathbf{r})$, it is clear that if we impose periodic boundary conditions on the ferromagnet, then $\int\mathbf{v}(\mathbf{r})d\mathbf{r}$ must be constant in time, as long as $\varphi(r)$ is continuous. If there are only small deviations from equilibrium ($|\varphi(\mathbf{r})-\varphi(\mathbf{r}')|<\pi$), then we must have

$$\int\mathbf{v}(\mathbf{r})d\mathbf{r}=0.$$

The vector field $\mathbf{v}(\mathbf{r})$ is completely analogous to the superfluid velocity $\mathbf{v}_s$ in liquid helium.

Let us first evaluate the function ψ under conditions of equilibrium in a uniform system. If the system is in *true* thermal equilibrium [Eq. (2.8)], we assert that $d\varphi/dt=0$. This must be true, physically, because the true thermal equilibrium state is a state which minimizes the total energy for a given value of the entropy; this state must be stable in the presence of a small dissipative term, such as coupling to a radiation field, which does not conserve M_z. If there were a macro-

scopic $M_\perp$ rotating at a nonzero rate, the system would radiate and would lose energy to the radiation field, which is impossible if the system is already in the lowest possible energy state for the given entropy. On the other hand, if the system is initially in equilibrium with $h(M_z,E)$ not equal to H_z, the system could radiate energy by letting M_z relax to its true thermal equilibrium value, determined by Eq. (2.8); in fact, we shall see that $d\varphi/dt$ is not zero in this case. Let us write the Hamiltonian of the system as $\mathcal{H}=\mathcal{H}_0-H_zS_z$, where $S_z=\sum_i S_i^z$. Since $\mathcal{H}_0$ commutes with H_zS_z, we may write

$$\exp(-i\mathcal{H}t/\hbar)=\exp(iH_zS_zt/\hbar)\exp(-i\mathcal{H}_0t/\hbar), \quad (2.18)$$

where the first factor is equivalent to a rotation of the spin system about the z axis by an angle $(-H_zt)$,[27] and the second factor is the time-evolution operator in the absence of the external field H_z. The equilibrium state of the system characterized by a given value of M_z and of E will have $d\varphi/dt=0$ when it is placed in an external field $\bar{H}_z=h(E,M_z)$, corresponding to *true* thermal equilibrium. The same state in the absence of the field $\bar{H}_z$ will have

$$d\varphi/dt=\bar{H}_z=h(E,M_z),$$

as can be seen from Eq. (2.18) with $H_z=\bar{H}_z$. More generally, if the state is placed in an arbitrary field H_z, it will precess at a rate

$$d\varphi/dt=h(E,M_z)-H_z. \quad (2.19)$$

[For example, if $M_z=0$, we have $h(E,M_z)=0$ and the system will precess at the Larmor frequency $(-H_z)$.] Thus, we have

$$\psi(\mathbf{r})=H_z-h[\epsilon(\mathbf{r}),m_z(\mathbf{r})]-h'(\mathbf{r}), \quad (2.20)$$

where h' involves the various space derivatives (first derivatives and higher) of ϵ, m, and $\mathbf{v}$.[28] There will also be terms in ψ proportional to v^2, etc., but we may neglect these for small deviations from equilibrium. We shall assume for convenience that the lattice is invariant under reflection; it is then clear by symmetry that there can be no term in ψ linear in $\mathbf{v}$ itself.

Let us now consider $\mathbf{j}^\epsilon(\mathbf{r})$. We require that the energy current vanish if ϵ, m_z, and φ are constant. The energy current may therefore be written

$$\mathbf{j}^\epsilon(\mathbf{r})=A(\mathbf{r})\mathbf{v}(\mathbf{r})+j^{\epsilon'}(\mathbf{r}), \quad (2.21)$$

where A is a function of $\epsilon(\mathbf{r})$ and $m_z(\mathbf{r})$, to be determined, and $j^{\epsilon'}$ is proportional to various space derivatives of ϵ, m_z, and $\mathbf{v}$. We neglect terms in j^ϵ of order v^3 and higher. In general, the coefficient A will be a second-rank tensor. We shall assume, however, that the lattice has cubic symmetry, so that A can be treated as a scalar.[29] For $\mathbf{j}^{m_z}$ we may write the following, corresponding to (2.21):

$$\mathbf{j}^{m_z}(\mathbf{r})=B(\mathbf{r})\mathbf{v}(r)+\mathbf{j}^{m_z'}(\mathbf{r}). \quad (2.22)$$

Much of the content of the hydrodynamic theory is obvious at this point. Let us define a column vector

$$\mathbf{U}(\mathbf{r})=\begin{bmatrix} m_z(\mathbf{r}) \\ \epsilon(\mathbf{r}) \\ \mathbf{v}(\mathbf{r}) \end{bmatrix}. \quad (2.23)$$

We shall write linearized equations of motion expressing $\partial\mathbf{U}/\partial t$ as an expansion in gradients of $\mathbf{U}$. If we take the Fourier transform of these equations of motion, we find an equation of the form

$$-i\omega\mathbf{U}=ik\mathbf{R}_1\mathbf{U}-k^2\mathbf{R}_2\mathbf{U}+\cdots, \quad (2.24)$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are real matrices. If we consider only the first term on the right-hand side of (2.24), the equation will have solutions at

$$-\omega=r_ik, \quad i=1,2,3, \quad (2.25)$$

where r_i are the eigenvalues of $\mathbf{R}_1$. Because $\mathbf{R}_1$ is real, it follows that the complex conjugate of any eigenvalue is also an eigenvalue. If the equilibrium state of the system is to be stable against the growth of inhomogeneities, then all the eigenvalues of $\mathbf{R}_1$ must be real. The second term on the right-hand side of (2.24) will lead to a damping of order k^2.

In Sec. 2 C we shall find an explicit expression for $\mathbf{R}_1$ in terms of thermodynamic quantities. The three eigenvalues will have the form $r=0$ and $r=\pm c$, corresponding to heat fluctuation and spin-wave modes, respectively. In Sec. 2 D, we shall find an expression for the damping rate $\mathbf{R}_2$, in terms of transport coefficients for the spin system. These coefficients cannot be measured directly, however, except for the thermal conductivity, and the purpose of Sec. 2 D is primarily to complete the analogy with Khalatnikov's treatment[4] of the helium system.

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[27] Since S_z is the generator for rotations of the spins about the z axis, the unitary operator $\exp[iH_zS_zt/\hbar]$ will change the value of φ by the amount $(-H_zt)$. This may be expressed by the statement that φ is canonically conjugate to M_z.

[28] For superfluid helium, the equation equivalent to (2.19) is $\partial\mathbf{v}_s/\partial t=-\nabla\mu$, where μ is the chemical potential. In the derivation of two-fluid hydrodynamics in Ref. 4, Khalatnikov never introduces the microscopic definition of $\mathbf{v}_s$ as the gradient of the phase of the Bose field, and therefore does not use the fact that the phase is conjugate to the particle number (see Refs 12 and 27). Khalatnikov claims to derive the equation for $\partial\mathbf{v}_s/\partial t$ directly from a differential equation comparable to Eq. (2.32) of this paper. We believe that his derivation is not complete, however, and that his differential equations are compatible with the relation $\partial\mathbf{v}_s/\partial t=-\nabla(\mu+A)$, where A is an arbitrary function of temperature, independent of the density. Khalatnikov's equation for the entropy current would be simultaneously modified to include the additional term $\rho_s(\mathbf{v}_s-\mathbf{v}_n)dA/dT$. In Landau's original phenomenological derivation of two-fluid hydrodynamics (Ref. 3), he *assumed* that the superfluid carries no entropy and hence A is a constant which may be set equal to zero.

[29] By cubic symmetry we mean that the constants J_{ij}^z and $J_{ij}^\perp$ have cubic symmetry separately.

C. Expression for Spin-Wave Velocity

In the hydrodynamic theory, we assume that the functions $\epsilon(\mathbf{r})$, $m_z(\mathbf{r})$, and $\mathbf{v}(\mathbf{r})$ determine the entropy of the system. In fact, there are many possible density matrices that one may associate with a given nonequilibrium state of the system, and the entropy will be different for the different cases. (The entropy, of course, is not an observable quantity.) We assume, however, that the entropy is at least uniquely determined to lowest order in the gradients. Thus we may write

$$S=\int\{s[\epsilon(\mathbf{r}),m_z(\mathbf{r}),\mathbf{v}(\mathbf{r})]+s'(\mathbf{r})\}d\mathbf{r}, \tag{2.26}$$

where s' involves various space derivatives of ϵ, m_z, and $\mathbf{v}$, and may not be uniquely determined. If $\mathbf{v}(\mathbf{r})$ is zero, we must have

$$s(\mathbf{r})=S_0[\epsilon(\mathbf{r}),m_z(\mathbf{r})], \tag{2.27}$$

where S_0 is the equilibrium function defined in (2.7). More generally, we may write

$$s=S_0(\epsilon,m_z)-\frac{v^2}{2}\frac{\rho_s(\epsilon,m_z)}{T(E,m_z)}+\cdots. \tag{2.28}$$

In (2.28) we have neglected terms of order v^4 and higher; the temperature $T(\epsilon,m_z)$ is

$$T^{-1}\equiv\partial S_0/\partial\epsilon, \tag{2.29}$$

and the coefficient ρ_s is defined by (2.28). The coefficient ρ_s, which is analogous to the superfluid density in helium,[4] may be identified as a stiffness constant for variations in the direction of spin orientation; ρ_s must be nonnegative if the ferromagnetic system is stable. We show in the Appendix that ρ_s is related to the linear response of the system to a static inhomogeneous magnetic field in the x-y plane by

$$\chi^{\perp}(k,\omega=0)=M_0^2/\rho_s k^2. \tag{2.30}$$

Using (2.28), (2.29), and (2.9), we find that

$$T\frac{\partial s}{\partial t}+h\frac{\partial m_z}{\partial t}+\rho_s\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t}-\frac{\partial\epsilon}{\partial t}=-T\frac{v^2}{2}\frac{\partial(\rho_s T^{-1})}{\partial t}\approx 0. \tag{2.31}$$

The right-hand side of (2.31) is negligible, for our purposes, because it is cubic in the deviations from equilibrium.

Let us now substitute Eqs. (2.15)–(2.17) and (2.20)–(2.22) in the left-hand side of (2.31). After some rearrangement we find

$$\frac{\partial s}{\partial t}=T^{-1}[h\nabla\cdot(B\mathbf{v}+\mathbf{j}^{m_z\prime})-\rho_s\mathbf{v}\cdot\nabla(h+h')-\nabla\cdot(A\mathbf{v}+\mathbf{j}^{\epsilon\prime})]. \tag{2.32}$$

It is convenient to let h, T, and $(\rho_s\mathbf{v})$ be the independent variables, rather than m_z, ϵ, and $\mathbf{v}$. In order to determine the function B, we suppose that at a certain instant of time we have

$$\begin{aligned}\rho_s\mathbf{v}&=a\hat{k}\cos\mathbf{k}\cdot\mathbf{r},\\ h&=h_0+b\sin\mathbf{k}\cdot\mathbf{r},\\ T&=\text{const},\end{aligned} \tag{2.33}$$

where $|a|$, $|b|$, and $\mathbf{k}$ are small, and $\hat{k}$ is a unit vector in the direction of $\mathbf{k}$. Let us integrate (2.32) over the volume of the system, and let us keep only terms first-order in k, and lowest-order (quadratic) in a and b, on the right-hand side of (2.32). We find (after integration by parts)

$$\frac{\partial S}{\partial t}-\frac{\partial}{\partial t}\left(\int s'd\mathbf{r}\right)=-\frac{1}{2}\frac{abk}{T}\left[1+\frac{B}{\rho_s}\right]+O(k^2), \tag{2.34}$$

where B and ρ_s are evaluated at $h=h_0$. The quantity $(\int s'd\mathbf{r})$ is at least of order k, and its time derivative will be at least or order k^2. Since ab can be positive or negative, and since the total entropy cannot decrease with time, we must have

$$B=-\rho_s. \tag{2.35}$$

In a similar fashion, consider a system where h is constant, but where $\rho_s\mathbf{v}$ and T have a sinusoidal variation. We find that

$$A=-\rho_s h. \tag{2.36}$$

We may now use these results to find the linearized equations of motion, up to terms first-order in the gradients:

$$\partial\mathbf{v}/\partial t=\nabla h, \tag{2.37a}$$

$$\partial m_z/\partial t=\rho_s\nabla\cdot\mathbf{v}, \tag{2.37b}$$

$$\frac{\partial\epsilon}{\partial t}-h\frac{\partial m_z}{\partial t}=0, \tag{2.37c}$$

$$\partial s/\partial t=0. \tag{2.37d}$$

Because the entropy does not vary in time, Eq. (2.37a) may be written

$$\frac{\partial^2\mathbf{v}}{\partial t^2}=\frac{\partial}{\partial t}(-\chi_s^{-1}\nabla m_z), \tag{2.38}$$

where χ_s is the isentropic susceptibility, defined by

$$\chi_s^{-1}=\partial h/\partial M_z|_S. \tag{2.39}$$

Equations (2.37b) and (2.38) have a spin-wave solution at frequency

$$\omega=\pm ck, \tag{2.40}$$

with

$$c=(\rho_s/\chi_s)^{1/2}. \tag{2.41}$$

Equation (2.37c) may be put in a simple form for the case of small fluctuations about true thermal equi-

librium ($h \approx H_z$):

$$\partial[\epsilon(r) - H_z m_z(r)]/\partial t = 0. \tag{2.42}$$

In the case where the average magnetization M_z equals zero, we need not distinguish between the isentropic and isothermal susceptibilities.

If the external field H_z is zero, then χ^{-1} is a measure of the preference of the spins to line up in the x-y plane. In the limit $J^z \to J^1$, the susceptibility χ will tend towards infinity, and the spin-wave velocity c will tend towards zero. This is consistent with the fact that the spin-wave velocity in an *isotropic* ferromagnet is zero in the limit of long wavelengths. We shall discuss this system further in Sec. 7.

D. Dissipative Terms

Let us now return to the full equation (2.32). Substituting (2.35) and (2.36) we find

$$T\left[\frac{\partial s}{\partial t} + \nabla \cdot (\mathbf{q}T^{-1})\right] = -\mathbf{j}^{m_z'} \cdot \nabla h + h' \nabla \cdot (\rho_s \mathbf{v}) - T^{-1}\mathbf{q} \cdot \nabla T, \tag{2.43}$$

where the "heat flow" $\mathbf{q}$ is defined by

$$\mathbf{q} = \mathbf{j}^{\epsilon'} - h\mathbf{j}^{m_z'} + h'\rho_s \mathbf{v}. \tag{2.44}$$

The right-hand side of (2.43) is called the dissipative function, and is generally positive. To lowest order in the deviations from equilibrium and lowest order in ∇, we may write

$$\mathbf{q} = -K_{11}\nabla T - K_{12}\nabla h, \tag{2.45a}$$

$$\mathbf{j}^{m_z'} = -K_{21}\nabla T - K_{22}\nabla h, \tag{2.45b}$$

$$h' = K_{33}\rho_s \nabla \cdot \mathbf{v}, \tag{2.45c}$$

where the transport coefficients K_{ij} depend on E and M_z. By symmetry, the vectors $\mathbf{q}$ and $\mathbf{j}^{m_z}$ cannot have a term proportional to $\nabla \mathbf{v}$. Similarly, the scalar h' cannot have terms proportional to ∇T or ∇h.

We shall now restrict ourselves to the case $M_z = 0$, which is appropriate for fluctuations about true thermal equilibrium in zero external field. Here one can show by symmetry that $K_{12} = K_{21} = 0$. The constant $K_{11} = \kappa$ has the interpretation of a thermal conductivity, while K_{22} is a spin transport coefficient. The constant K_{33} is analogous to the second viscosity coefficient in the theory of liquid helium.[4] When $M_z = 0$, we may write

$$dT = C^{-1} d\epsilon, \tag{2.46}$$

where C is the specific heat, and we need not distinguish between the specific heats at constant magnetization and constant field. Similarly, we may drop the subscript s from the susceptibility χ. We then find the following linearized equations of motion for m_z, ϵ, and $\mathbf{v}$:

$$\frac{\partial \epsilon}{\partial t} = \kappa C^{-1} \nabla^2 \epsilon, \tag{2.47a}$$

$$\frac{\partial m_z}{\partial t} = \rho_s \nabla \cdot \mathbf{v} + \chi^{-1} K_{22} \nabla^2 m_z, \tag{2.47b}$$

$$\frac{\partial \mathbf{v}}{\partial t} = \chi^{-1} \nabla m_z + K_{33}\rho_s \nabla (\nabla \cdot \mathbf{v}). \tag{2.47c}$$

The solution of Eqs. (2.47) yields a spin-wave mode with frequency

$$\omega_{\pm}(\mathbf{k}) = \pm ck - \tfrac{1}{2} i D k^2, \tag{2.48}$$

and a heat diffusion mode with frequency

$$\omega_{\epsilon}(\mathbf{k}) = -i\kappa C^{-1} k^2 \equiv -i D_T k^2. \tag{2.49}$$

In Eq. (2.48), c is given by Eq. (2.41), and D is given by

$$D = \chi^{-1} K_{22} + \rho_s K_{33}, \tag{2.50}$$

and we have neglected terms of order k^3. Equations (2.48) and (2.49) with the definitions (2.28) and (2.41), constitute the central result of this paper. They are valid at any temperature $T < T_c$, for sufficiently long wavelengths, i.e., for $k \ll k_c(T)$, where the cutoff k_c depends on T but is presumably finite for any fixed temperature in the range $0 < T < T_c$.

E. Paramagnetic State

In the paramagnetic state, the phase φ is not well defined, and there can be no terms involving $\mathbf{v}$ in the equations of motion for m_z and ϵ. The correct expressions for the paramagnetic state may be obtained from those of the ferromagnetic state simply by setting $\rho_s = 0$ in the latter. We now find two diffusive modes for the time dependence of m_z and ϵ. For small deviations from $M_z = 0$, the two modes are uncoupled; we then find a thermal diffusion mode whose frequency is given by Eq. (2.49) and a diffusion mode for M_z whose frequency is given by

$$\omega = -i\chi^{-1} K_{22} k^2 \equiv -i D k^2. \tag{2.51}$$

3. LINEAR RESPONSE AND CORRELATION FUNCTIONS FOR PLANAR FERROMAGNET

A. Linear Response Functions

Let us consider the linear response of the planar ferromagnet to an external perturbation

$$\mathcal{H}'(t) = -\int b(\mathbf{r},t) B(\mathbf{r}) d\mathbf{r}, \tag{3.1}$$

where the operator $B(\mathbf{r})$ is an arbitrary linear combination of the operators $\epsilon(\mathbf{r})$ and $\mathbf{m}(\mathbf{r})$. Let us assume that

for times t less than an initial time t_0 the coefficient $b(\mathbf{r},t)$ is zero, the system is at thermal equilibrium in zero external field H_z, and the system is in a ferromagnetic state with $\mathbf{M}$ aligned in the x-direction.

Let $\langle A(\mathbf{r},t)\rangle$ denote the expectation value of an operator $A(\mathbf{r})$ at time t in the perturbed system. The linear response function $\hat{\chi}_{AB}(\mathbf{r},t)$ may be defined by

$$\langle A(\mathbf{r},t)\rangle = \langle A\rangle_0 + \int \hat{\chi}_{AB}(\mathbf{r}-\mathbf{r}', t-t')b(\mathbf{r}',t')d\mathbf{r}'dt' + O(b^2), \quad (3.2)$$

where $\langle A\rangle_0$ denotes the expectation value of A in the unperturbed system. Let us now suppose that $b(\mathbf{r},t)$ vanishes for $t>t_1$. If the disturbance is of sufficiently long wavelength so that hydrodynamics applies, then for $t>t_1+\tau$, the quantities $\langle\epsilon(\mathbf{r},t)\rangle$, $\langle m_z(\mathbf{r},t)\rangle$, and $\langle m_y(\mathbf{r},t)\rangle$ must obey[24] the hydrodynamic linear equations of motion derived in the previous sections. (Since we keep only terms linear in b, we have $\langle m_y(\mathbf{r},t)\rangle = M_0(E,M_z)\varphi(\mathbf{r},t)$.] It follows that the Fourier-transformed response function $\chi_{AB}(k,\omega)$ will have poles at the frequencies $\omega_\pm(\mathbf{k})$ and $\omega_c(\mathbf{k})$, given by (2.48) and (2.49). Consider first the case where $A=\epsilon$. Here the residues at $\omega_\pm(k)$ vanish and we may write

$$\chi_{\epsilon B}(\mathbf{k},\omega) = r_{\epsilon B}(\mathbf{k})/(\omega+iD_T k^2) + R_{\epsilon B}(\mathbf{k},\omega), \quad (3.3)$$

where the poles of $R_{\epsilon B}$ are at least a distance $1/\tau$ from the real axis. (More accurately, since the singularities in $R_{\epsilon B}$ are probably not a discrete set of poles, we assert that all variations of $R_{\epsilon B}$ take place over a frequency range equal to or larger than $1/\tau$.) The correction term $R_{\epsilon B}$ arises from the short time behavior of $\hat{\chi}_{\epsilon B}$ where the hydrodynamic equations need not be valid. Because the energy is a conserved quantity, the energy density $\langle\epsilon(\mathbf{r},t)\rangle$ can only change very slightly between times t_1 and $t_1+\tau$, if the perturbation H' is slowly varying in space. Consequently, the correction term $R_{\epsilon B}$ will be negligible in the long-wavelength limit. The residue $r_{\epsilon B}$ can be determined if the static susceptibility $\chi_{\epsilon B}(\mathbf{k},0)$ is known. For the case $B=\epsilon$, we have

$$\lim_{k\to 0}\chi_{\epsilon\epsilon}(\mathbf{k},0) = TC, \quad (3.4)$$

where C is the specific heat per unit volume. We therefore have, in the long-wavelength limit,

$$\chi_{\epsilon\epsilon}(\mathbf{k},\omega) \approx iD_T CTk^2/(\omega+iD_T k^2). \quad (3.5)$$

Let us now consider the case where A is m_z or m_y. We may now write

$$\chi_{AB}(\mathbf{k},\omega) = \frac{r_{AB}}{\omega-\omega_+} - \frac{r_{AB}^*}{\omega-\omega_-} + R_{AB}(\omega). \quad (3.6)$$

In (3.6) we have suppressed the k dependence on the right-hand side, and we have made use of the general relation (for A and B Hermitian)

$$\chi_{AB}(\mathbf{k},\omega) = \chi_{AB}(\mathbf{k},-\omega)^* \quad (3.7)$$

for ω on the real axis. Once again we may argue that the function R_{AB} is negligible in the long-wavelength limit, if A is the conserved quantity m_z. We may also argue that R_{AB} is negligible compared to the first two terms in χ_{AB} when A is m_y, because the angle φ has no preferred direction (cf. the Appendix). As a consequence of Eqs. (2.47), the residues r_{AB} must obey the relations

$$(-i\omega_+ + \chi^{-1}K_{22}k^2)r_{m_z B} = -\rho_s M_0^{-1}k^2 r_{m_y B}. \quad (3.8)$$

For the static response functions we have

$$\chi_{m_z m_z}(\mathbf{k},0) = \chi + O(k^2), \quad (3.9a)$$

$$\chi_{m_y m_y}(\mathbf{k},0) = M_0^2/\rho_s k^2 + O(k^0), \quad (3.9b)$$

$$\chi_{m_z m_y}(\mathbf{k},0) = \chi_{m_y m_z}(\mathbf{k},0) = 0. \quad (3.9c)$$

Equation (3.9b) is established in the Appendix, while (3.9c) is an exact relation that follows from simple symmetry considerations.

We can now determine the residues in the long-wavelength limit.[30]

$$r_{m_z m_z} = (-\rho_s k/2c)[1-(ik/c)\chi^{-1}K_{22}+O(k^2)], \quad (3.10a)$$

$$r_{m_y m_z} = -r_{m_z m_y} = -\tfrac{1}{2}iM_0[1-(ik/2c)D+O(k^2)], \quad (3.10b)$$

$$r_{m_y m_y} = -(M_0^2 c/2\rho_s k) \times [1-(ik/c)\rho_s K_{33}+O(k^2)]. \quad (3.10c)$$

We may readily verify that the approximations to $\chi_{AB}(k,\omega)$ obtained from (3.6) and (3.10), neglecting R_{AB}, are consistent to lowest order in k, with the exact sum rules derived from the equal-time commutation relations

$$\lim_{L\to\infty}\int_{-L}^{L}\chi_{m_y m_z}(\mathbf{k},\omega)d\omega = -\lim_{L\to\infty}\int_{-L}^{L}\chi_{m_z m_y}(\mathbf{k},\omega)d\omega = \pi M_0, \quad (3.11a)$$

$$\lim_{L\to\infty}\int_{-L}^{L}\chi_{AA}(\mathbf{k},\omega)d\omega = 0. \quad (3.11b)$$

Alternatively, we could have derived (3.10) using (3.11), instead of (3.9). Note however, that the approximation (3.5) for $\chi_{\epsilon\epsilon}$ does not satisfy the sum rule (3.11b), because of contributions to the integral from the region of large ω where the approximation is not

[30] The assumption that we can neglect the remainder term R_{AB} in (3.6) when computing $\chi_{m_y m_z}(\mathbf{k},0)$, is somewhat more difficult to justify than the other assumptions we have used in this section, because the contributions from the two poles involve the difference between two large numbers. If we were to relax this assumption, we would have to multiply each residue in (3.10) by the phase factor $(1+i\alpha k)$, where α is an unknown constant. Such a phase factor would have a negligible effect on the quantities which are observable in practice, i.e., the position, area, and width of the spin-wave peaks in neutron diffraction.

valid. A third method of arriving at the leading term in the three equations (3.10) is to assume initially response functions of the form (3.6) with the spin-wave frequency $\omega_+ = -\omega_-^*$ as an undetermined *real* quantity. If we neglect the contributions of R_{AB} to the sum rules (3.9) and (3.11), then these equations suffice to determine both the positions and residues of the poles. This method does not take dissipative terms into account, nor does it show that they may be neglected at long wavelengths.

Let us now consider the remaining linear response functions for the variables $\mathbf{m}$ and $\boldsymbol{\epsilon}$. By exploiting the invariance of the ferromagnet under a spin rotation of 180° about the x axis, we can show that $\chi_{AB}(\mathbf{k},\omega) = \chi_{BA}(\mathbf{k},\omega) = 0$, if A is m_y or m_z and B is ϵ or m_x. On the other hand, the response function $\chi_{m_x\epsilon}$ will not vanish. For times $t > t_1 + \tau$, we know that $\langle m_x(\mathbf{r},t)\rangle$ is related to $\langle \epsilon(\mathbf{r},t)\rangle$ and $\langle m_z(\mathbf{r},t)\rangle$ by (2.12). It follows that

$$\chi_{m_x\epsilon}(\mathbf{k},\omega) = \left(\frac{\partial M_0}{\partial E}\right)\chi_{\epsilon\epsilon}(\mathbf{k},\omega) + R_{m_x\epsilon}(\mathbf{k},\omega). \quad (3.12)$$

Using the invariance of the ferromagnetic state under the combined operations of time reversal and rotation of the spins by 180° about the z axis, one can show that

$$\chi_{\epsilon m_x}(\mathbf{k},\omega) = \chi_{m_x\epsilon}(\mathbf{k},\omega). \quad (3.13)$$

Finally, using (2.12) again we get

$$\chi_{m_xm_x}(\mathbf{k},\omega) = \left(\frac{\partial M_0}{\partial E}\right)^2\chi_{\epsilon\epsilon}(\mathbf{k},\omega) + R_{m_xm_x}(\mathbf{k},\omega). \quad (3.14)$$

As in the previous equations, the remainder terms R of (3.12)–(3.14) have no poles closer than $1/\tau$ to the real axis. Unlike the previous cases, however, the term $R_{m_xm_x}$ is definitely *not* small compared to the first term; in fact, the remainder term in (3.14) is probably larger by a factor $1/k$ in the long-wavelength limit at zero frequency.[23,31] It is not clear how large the remainder terms will be in (3.12) and (3.13).

B. Correlation Functions

Let us define the equilibrium correlation function

$$\hat{C}_{AB}(\mathbf{r},t) \equiv \tfrac{1}{2}\langle\{A(\mathbf{r},t),B(0,0)\}\rangle_0 - \langle A\rangle_0\langle B\rangle_0,$$

where the curly brackets denote the anticommutator. As is well known, the susceptibilities $\hat{\chi}$ are determined by the correlation functions $\hat{C}$ and vice versa.[24] In particular, when $A = B$, the Fourier transformed correlation function C is related to χ by

$$C_{AA}(\mathbf{k},\omega) = \hbar\coth(\hbar\omega/2k_BT)\,\mathrm{Im}\chi_{AA}(\mathbf{k},\omega). \quad (3.15)$$

In the range where hydrodynamics is valid, we have $\hbar\omega/k_BT \ll 1$ and

$$C_{AA}(\mathbf{k},\omega) = 2k_BT\,\mathrm{Im}\chi_{AA}(\mathbf{k},\omega)/\omega. \quad (3.16)$$

Thus we find

$$C_{\epsilon\epsilon}(\mathbf{k},\omega) = 2Ck_BT^2\frac{D_Tk^2}{\omega^2 + (D_Tk^2)^2}, \quad (3.17)$$

$$C_{m_zm_z}(\mathbf{k},\omega) = \frac{2\chi k_BT[c^2Dk^4 + \chi^{-1}K_{22}k^2(\omega^2 - c^2k^2)]}{[(\omega - ck)^2 + (\tfrac{1}{2}Dk^2)^2][(\omega + ck)^2 + (\tfrac{1}{2}Dk^2)^2]}, \quad (3.18)$$

$$C_{m_ym_y}(\mathbf{k},\omega) = \frac{2M_0^2k_BT[c^2Dk^4 + \rho_sK_{33}k^2(\omega^2 - c^2k^2)]}{\rho_sk^2[(\omega - ck)^2 + (\tfrac{1}{2}Dk^2)^2][(\omega + ck)^2 + (\tfrac{1}{2}Dk^2)^2]}. \quad (3.19)$$

The above correlation functions consist of one or two narrow peaks, with widths of order k^2 and Lorentzian line shapes, in the long-wavelength limit. Corrections to the above formulas should be smaller by a factor k^2, as measured by the areas under the frequency distributions at fixed k.

The correlation function $C_{m_xm_x}(\mathbf{k},\omega)$ should have a peak of large area, and width $1/\tau$, arising from the remainder term in (3.14), in addition to a narrow peak proportional to $C_{\epsilon\epsilon}(\mathbf{k},\omega)$.

C. Paramagnetic State

We may find the correlation and response functions in the paramagnetic state by the same methods we have used for the ferromagnetic state. We again consider the case where the system is initially in true thermal equilibrium in zero external field H_z. We find

$$\chi_{\epsilon\epsilon}(\mathbf{k},\omega) \approx iD_TTCk^2/(\omega + iD_Tk^2), \quad (3.20)$$

$$\chi_{m_zm_z}(\mathbf{k},\omega) \approx iD\chi k^2/(\omega + iDk^2), \quad (3.21)$$

so that the corresponding correlations functions have narrow diffusive peaks

$$C_{\epsilon\epsilon}(\mathbf{k},\omega) = 2Ck_BT^2D_Tk^2/[\omega^2 + (D_Tk^2)^2], \quad (3.22)$$

$$C_{m_zm_z}(\mathbf{k},\omega) = 2\chi k_BTDk^2/[\omega^2 + (Dk^2)^2]. \quad (3.23)$$

[31] K. Kawasaki and H. Mori, Progr. Theoret. Phys. (Kyoto) **25**, 1043 (1961); **38**, 1052 (1967); K. Tani and H. Tanaka, Phys. Letters **26A**, 68 (1967); V. G. Vaks *et al.*, Ref. 23.

On the other hand, the response function $\chi_{m_y m_y}(\mathbf{k},\omega)$, which is now equal to $\chi_{m_x m_x}(\mathbf{k},\omega)$, has no poles nearer the real axis than $1/\tau$, and the corresponding correlation function will have a broad peak with width of order $1/\tau$. The cross-correlation functions and response functions all vanish for A and B equal to ϵ, m_x, m_y, or m_z.

4. EQUIVALENCE OF PLANAR FERROMAGNET AND BOSE LATTICE GAS

The planar ferromagnet with spin $\frac{1}{2}$ is equivalent to a lattice model for a system of interacting bosons,[1,2] whose Hamiltonian is

$$\mathcal{H}=\sum_{\mathbf{k}} \epsilon(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k})+\tfrac{1}{2}\sum_{i,i'} V(\mathbf{r}_i-\mathbf{r}_{i'})a_i^\dagger a_{i'}^\dagger a_{i'}a_i, \quad (4.1)$$

where $a_i^\dagger$ is the creation operator for an electron on lattice site i, $a^\dagger(\mathbf{k})$ is the Fourier transform

$$a^\dagger(\mathbf{k})=N^{-1/2}\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i}a_i^\dagger, \quad (4.2)$$

and $V(\mathbf{r}_i-\mathbf{r}_{i'})$ is an interaction potential which is infinite if $i=i'$. The equivalence between the spin and boson systems is established by setting

$$\begin{aligned} S_i^+&=\hbar a_i^\dagger,\\ S_i^-&=\hbar a_i,\\ S_i^z&=\hbar(a_i^\dagger a_i-\tfrac{1}{2}). \end{aligned} \quad (4.3)$$

The spin operators defined by (4.3) obey the usual angular momentum commutation relations within the manifold of states for which the Hamiltonian $\mathcal{H}$ is finite, namely the manifold of states with $a_i^\dagger a_i\leq 1$. Note that the z component of the spin density is analogous to the Bose particle density.

The Hamiltonian (4.1) does not conserve momentum because of the presence of umklapp scattering terms in the Fourier transform of the interaction potential. In this respect the lattice gas is more closely analogous to the physical system of helium in fine pores, where the motion of the normal fluid is damped out by the viscosity. Below the λ point, helium in fine pores has one propagating mode at long wavelengths which is known as fourth sound[32]: a mode associated with fluctuations in the particle density and the superfluid velocity, but with no entropy transport. Above the λ point, there is no long-wavelength propagating mode.

For pure bulk helium, there is an additional conserved quantity, the momentum density. Below the λ point there are consequently two propagating modes, first and second sound, which contribute to the order-parameter correlation function.[10,11]

5. PLANAR ANTIFERROMAGNET

The Hamiltonian of the planar antiferromagnet[13] is identical in form to that of the planar ferromagnet, Eq. (2.2). The coupling constants $J_{ij}^\perp$ must have different values, however, so that the ground state of the antiferromagnetic system will have a magnetization in the x-y plane which changes sign from one unit cell to the next. Let us divide the lattice into two sublattices A and B, and let us define new variables

$$\mathbf{Q}_i=\eta_i\mathbf{S}_i, \quad (5.1)$$

where

$$\begin{aligned} \eta_i&=+1 \text{ on sublattice } A,\\ \eta_i&=-1 \text{ on sublattice } B. \end{aligned} \quad (5.2)$$

By definition, the ground state of the planar antiferromagnet has

$$\langle Q_i^x\rangle+i\langle Q_i^y\rangle\equiv N_\perp e^{i\varphi}/\mathfrak{N}\neq 0. \quad (5.3)$$

In finite field H_z, we may also have

$$\langle S_i^z\rangle=M_z\neq 0. \quad (5.4)$$

The three variables Q_i^x, Q_i^y, and S_i^z obey exactly the same commutation relations as the three variables S_i^x, S_i^y, and S_i^z:

$$[Q_i^x,Q_i^y]=(i/\hbar)S_i^z, \quad \text{etc.} \quad (5.5)$$

Furthermore, if the Hamiltonian (2.1) is rewritten in terms of the variables, Q_i^x, Q_i^y, and S_i^z, it retains the same form as (2.2)

$$H=-\sum_{ij}\{J_{ij}^z S_i^z S_j^z+\eta_i\eta_j J_{ij}^\perp[Q_i^x Q_j^x+Q_i^y Q_j^y]\}-\sum_i S_i^z H_z. \quad (5.6)$$

The coupling constants $\eta_i\eta_j J_{ij}^\perp$ will be ferromagnetic, if the original coupling constants $J_{ij}^\perp$ were antiferromagnetic. It is clear, therefore, that the planar antiferromagnet is isomorphic to the ferromagnet, and the static and dynamic properties of the two systems must be exactly the same after the appropriate change of variables. The component of the total magnetization M_z is a constant of the motion for the planar antiferromagnet. The equilibrium states of the system, below the Néel temperature, are determined by the values of M_z, φ, and the internal energy E. The value of $N_\perp$ is determined by E and M_z

$$N_\perp=N_0(E,M_z). \quad (5.7)$$

Let us define the staggered magnetization density

$$\mathbf{n}(\mathbf{r})\equiv\langle\sum_i \mathbf{Q}_i\delta(\mathbf{r}-\mathbf{r}_i)\rangle, \quad (5.8)$$

while the total magnetization density $\mathbf{m}(r)$ is defined by (2.10a) as before. We shall assume the δ functions in (5.8) and (2.10a) to be spread out in space over a unit cell (including one site from each sublattice), so that $\mathbf{m}(r)$ and $\mathbf{n}(r)$ can be slowly varying in space, with

$$m_x(\mathbf{r})\approx m_y(\mathbf{r})\approx 0. \quad (5.9)$$

[32] See Ref. 4, Chap. 14.

Let us define

$$n_x(\mathbf{r})+in_y(\mathbf{r})\equiv n_\perp(\mathbf{r})e^{i\varphi(\mathbf{r})}. \tag{5.10}$$

The hydrodynamic states of the planar antiferromagnet are determined by the functions $\epsilon(\mathbf{r})$, $m_z(\mathbf{r})$, and $\varphi(\mathbf{r})$. The equations of motion for these functions are exactly the same as for the case of the planar ferromagnet. Similarly, the form of the entropy (2.26)–(2.28) is the same as before. Equations (2.47) apply unchanged to the planar antiferromagnet. The hydrodynamics therefore predicts spin waves with a velocity given by (2.41) and damping given by (2.50). Of course, the spin waves are now associated with oscillations in the direction of the staggered magnetization $\mathbf{n}(\mathbf{r})$, rather than of the total magnetization $\mathbf{m}(\mathbf{r})$, in the x-y plane.

The linear response functions and correlation functions for the planar antiferromagnet are identical to those for the planar ferromagnet given in Sec. 3 and the Appendix, except for the replacement of m_x by n_x, m_y by n_y, and M_0 by N_0.

6. ISOTROPIC ANTIFERROMAGNET

The isotropic or Heisenberg antiferromagnet possesses a Hamiltonian similar to (2.2), but with $J_{ij}{}^\perp=J_{ij}{}^z$:

$$\mathcal{H}_0=-\sum_{ij}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j. \tag{6.1}$$

We assume that there are two sublattices A and B, with opposite average magnetizations in the ground state. We define the total magnetization density $\mathbf{m}(\mathbf{r})$ and the staggered magnetization density $\mathbf{n}(\mathbf{r})$ by (2.10a) and (5.8), as before. In the absence of a magnetic field, the Hamiltonian (2.2) is invariant under an arbitrary uniform rotation of all the spins. Thus the staggered magnetization can point in any direction in space. A true thermal equilibrium state, in the absence of an applied magnetic field, will have

$$\mathbf{m}(\mathbf{r})=0, \tag{6.2}$$

$$\mathbf{n}(\mathbf{r})=N_0(E)\hat{n}, \tag{6.3}$$

and the state is determined by the energy E and the arbitrary direction $\hat{n}$ of the staggered magnetization. As a consequence of the rotational invariance of (6.1), the total magnetization of the system is a constant of the motion. It is thus possible to have an infinitely long-lived (equilibrium) state in which $\mathbf{m}(\mathbf{r})$ is nonzero (and constant in space). In such a case, the orientation of the staggered magnetization is no longer arbitrary; the staggered magnetization will always tend to line up perpendicular to the total magnetization[33]

$$\mathbf{m}\cdot\mathbf{n}=0. \tag{6.4}$$

We shall restrict ourselves to small fluctuations about true thermal equilibrium in zero magnetic field, and hence, $\mathbf{m}\cdot\mathbf{n}$ will always be small. We shall assume the average $\mathbf{n}$ to lie in the x direction, in order to conform to the notation of the planar case. Let us define the variables

$$\varphi_z(\mathbf{r})=n_y(\mathbf{r})/N_0, \tag{6.5}$$
$$\varphi_y(\mathbf{r})=-n_z(\mathbf{r})/N_0.$$

We shall describe long-wavelength fluctuations in the system by specifying the functions $\varphi_y(\mathbf{r})$ and $\varphi_z(\mathbf{r})$, as well as the conserved variables $\epsilon(\mathbf{r})$ and $\mathbf{m}(\mathbf{r})$. If the system has evolved in the absence of external fields for a time long compared to τ, then we assume that past history is forgotten and that the state of the system is uniquely determined by these functions.

Let us define the vector fields

$$\mathbf{v}^\alpha=\nabla\varphi_\alpha,\quad \alpha=y,z. \tag{6.6}$$

The equations of motion for ϵ, $\mathbf{m}$, and $\mathbf{v}^\alpha$ have the form

$$\frac{\partial\mathbf{v}^\alpha}{\partial t}=-\nabla\psi_\alpha,\qquad \alpha=y,z \tag{6.7a}$$

$$\frac{\partial m_\alpha}{\partial t}=-\nabla\cdot\mathbf{j}^{m_\alpha},\quad \alpha=x,y,z \tag{6.7b}$$

$$\frac{\partial\epsilon}{\partial t}=-\nabla\cdot\mathbf{j}^\epsilon, \tag{6.7c}$$

where ψ^α, j^{m_α}, and j^ϵ are to be written as gradient expansions of $\mathbf{v}^\alpha$, $\mathbf{m}$, and $(\epsilon-E)$. We assume that deviations from equilibrium are sufficiently small so that we need only keep terms linear in $\mathbf{v}^\alpha$, $\mathbf{m}$, and $(\epsilon-E)$. The entropy S may be expressed in the form (2.26) with

$$s=S_0(\epsilon)-\tfrac{1}{2}T^{-1}[(m_y{}^2+m_z{}^2)/\chi+m_x{}^2/\chi_{\parallel}+\rho_s(|\mathbf{v}^y|^2+|\mathbf{v}^z|^2)]. \tag{6.8}$$

Here, χ is the susceptibility for a uniform applied magnetic field perpendicular to the axis of sublattice magnetization, $\chi_\parallel$ is the susceptibility for a field parallel to this axis, and ρ_s is the stiffness constant for changes in direction of sublattice magnetization.[34]

We may now proceed exactly as in the previous sections, taking advantage of the symmetries of the system. [The useful symmetries include, in addition to the invariance under simultaneous rotations of all the spins, an invariance with respect to the interchange

[33] L. Néel, Ann. Phys. (Paris) 5, 232 (1936).

[34] The quantity χ, sometimes denoted by $\chi_\perp$, is the thermodynamic susceptibility. The susceptibility $\chi_\parallel$ is not directly defined thermodynamically, since the isotropic antiferromagnet will not be stable with its axis of sublattice magnetization lined up parallel to a uniform applied field. The situation can be stabilized, however, by applying a very weak staggered field along the x axis, and it seems probable that $\chi_\parallel$ can be correctly defined by taking the appropriate limit as the uniform and staggered fields to go to zero. In any case, we show below that the variable m_x is completely decoupled from the other variables at long wavelengths, so that the value of $\chi_\parallel$ does not affect the spin waves.

of the magnetizations of sublattice A and sublattice B, i.e., invariance under a change of sign of $\mathbf{n}(\mathbf{r})$, with $\mathbf{m}(\mathbf{r})$ held constant.] We find

$$\frac{\partial \epsilon}{\partial t}=\kappa\nabla^2 T=\kappa C^{-1}\nabla^2\epsilon, \tag{6.9a}$$

$$\frac{\partial m_x}{\partial t}=\frac{K_{\parallel}}{\chi_{\parallel}}\nabla^2 m_x, \tag{6.9b}$$

$$\frac{\partial m_\alpha}{\partial t}=\rho_s\nabla\cdot\mathbf{v}^\alpha+K_\perp\chi^{-1}\nabla^2 m_\alpha, \qquad \alpha=y,\ z \tag{6.9c}$$

$$\frac{\partial \mathbf{v}^\alpha}{\partial t}=\chi^{-1}\nabla m_\alpha+\zeta\rho_s\nabla(\nabla\cdot\mathbf{v}^\alpha), \qquad \alpha=y,\ z \tag{6.9d}$$

where κ, $K_{\parallel}$, $K_\perp$, and ζ are transport coefficients whose values are not given by the theory. The coefficient κ is the thermal conductivity, whereas $K_{\parallel}$ and $K_\perp$ may be interpreted as transport coefficients for the components of spin parallel and perpendicular to the axis of sublattice magnitization. The coefficient ζ, like K_{33} in Sec. 2, is somewhat analogous to a second viscosity coefficient in liquid helium. The normal modes of the antiferromagnet consist of a diffusive mode for the energy, a diffusive mode for the x component of the total magnetization, and two spin-wave modes. The two polarizations of the spin waves couple oscillations in $\hat{n}_y$ and $\hat{n}_z$ with oscillations in m_z and m_y, respectively. The spin waves have a real frequency ck and a damping $\frac{1}{2}Dk^2$, where c is equal to $(\rho_s/\chi)^{1/2}$, as before [Eq. (2.41)], and

$$D=K_\perp\chi^{-1}+\rho_s\zeta. \tag{6.10a}$$

The thermal diffusion constant D_T is given by Eq. (2.49), while the diffusion constant for fluctuations in m_x may be written

$$D_{\parallel}=K_{\parallel}/\chi_{\parallel}. \tag{6.10b}$$

The existence of a linear spin-wave spectrum in the antiferromagnet is of course well known, and the low-temperature spin-wave velocity has been given by many authors.[17,19–21,35,36] In the present theory, the relation between c and the stiffness constant ρ_s, Eq. (2.41), is maintained at arbitrary temperatures. Since χ is believed to remain finite at the Néel point, Eq. (2.41) implies that c vanishes as $\rho_s^{1/2}$ as $T\to T_N$.[37] Equation (2.41) for the spin-wave velocity was essentially found at arbitrary temperatures by Tani,[19] who neglected the damping and assumed that sum rules analogous to Eqs. (3.9) and (3.11), are exhausted by simple poles at $\omega=\pm\omega_k$. A similar derivation by Bar'yakhtar and Popov,[38] also led to a linear dispersion relation, but with an incorrect velocity. These last authors have used the sum rule

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega\,\mathrm{Im}\chi_{m_z m_z}(\mathbf{k},\omega)=\sum_{ij}J_{ij}\langle S_i^x S_j^x+S_i^y S_j^y\rangle\left|1-e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}\right|^2,$$

which is *not* exhausted by the spin-wave poles at finite temperatures. In fact, the spin-wave velocity found in this way remains finite at T_N.

The hydrodynamic prediction of a damping of spin waves proportional to k^2 is a new result, which disagrees with a number of earlier microscopic calculations,[19–21] but agrees with more recent work by Harris, Kumar and the authors.[22]

Proceeding again as in Sec. 3, we may find the long-wavelength forms of the various susceptibilities and correlation functions. Among the correlation functions of interest, we have the following results for $k\to 0$:

$$C_{n_y n_y}(\mathbf{k},\omega)=C_{n_z n_z}(\mathbf{k},\omega)=\frac{2N_0^2 k_B T[c^2Dk^4+\rho_s\zeta k^2(\omega^2-c^2k^2)]}{\rho_s k^2[(\omega-ck)^2+(\frac{1}{2}Dk^2)^2][(\omega+ck)^2+(\frac{1}{2}Dk^2)^2]}, \tag{6.11a}$$

$$C_{m_y m_y}(\mathbf{k},\omega)=C_{m_z m_z}(\mathbf{k},\omega)=\frac{2\chi k_B T[c^2Dk^4+\chi^{-1}K_\perp k^2(\omega^2-c^2k^2)]}{[(\omega-ck)^2+(\frac{1}{2}Dk^2)^2][(\omega+ck)^2+(\frac{1}{2}Dk^2)^2]}, \tag{6.11b}$$

$$C_{\epsilon\epsilon}(\mathbf{k},\omega)=\frac{2Ck_B T^2 D_T k^2}{\omega^2+(D_T k^2)^2}, \tag{6.11c}$$

$$C_{m_x m_x}(\mathbf{k},\omega)=\frac{2\chi_{\parallel}k_B T D_{\parallel}k^2}{\omega^2+(D_{\parallel}k^2)^2}. \tag{6.11d}$$

[35] L. Hulthén, Proc. Roy. Acad. Sci. Amsterdam **39**, 190 (1963); P. W. Anderson, Phys. Rev. **86**, 694 (1952); R. Kubo, *ibid.* **87**, 568 (1952); J. Ziman, Proc. Phys. Soc. (London) **65**, 540 (1952); **65**, 548 (1952); T. Nakamura, Progr. Theoret. Phys. (Kyoto) **7**, 539 (1952).

[36] A. B. Harris, Phys. Rev. Letters **21**, 602 (1968).

[37] As remarked in Ref. 6 [Eqs. (2.4), (5.3), (5.4)], ρ_s^{-1} is proportional to a correlation length for order-parameter fluctuations. (Note that in Ref. 6, the direction of alignment of the order parameter was chosen as the z direction.) Thus the relation between the spin-wave velocity and a correlation length is a purely hydrodynamic result. If, in addition, we employ *scaling* hypotheses, we can obtain relations between c and other thermodynamic quantities near T_N.

[38] V. G. Bar'yakhtar and V. A. Popov, Fiz. Tverd. Tela **10**, 773 (1968) [English trans.: Soviet Phys.—Solid State **10**, 605 (1968)].

Corrections to the above formulas should be smaller by order k^2, as measured by the areas under the frequency spectra.

The autocorrelation functions for the energy and for the total magnetization parallel to the axis of antiferromagnetic alignment are characterized by diffusive modes, while the autocorrelation functions for the total magnetization and staggered magnetization perpendicular to the antiferromagnetic alignment have a pair of Lorentzian-like peaks at the spin-wave frequencies $\omega=\pm ck$. The dynamic cross correlation functions of m_z with n_y, and of m_y with n_z are also nonzero, and are dominated by the spin-wave peaks. All the remaining cross correlation functions among the six variables (ϵ, m_x, m_y, m_z, n_y, and n_z) are zero by symmetry. For fluctuations in the staggered magnetization parallel to the axis of sublattice magnetization, we may write

$$C_{n_x n_x}(\mathbf{k},\omega)=\left(\frac{\partial N}{\partial \epsilon}\right)^2 C_{\epsilon\epsilon}(\mathbf{k},\omega)+\tilde{R}(\mathbf{k},\omega), \quad (6.12)$$

where $\tilde{R}(\mathbf{k},\omega)$ presumably has a broad frequency spectrum, while the first term on the right-hand side has a narrow peak of width $D_T k^2$. The second term is not negligible compared to the first, however, and will be much larger than the first if $C_{n_x n_x}(\mathbf{k})$ diverges as k^{-1}, as is predicted by the spin-wave approximation.[31]

In the paramagnetic state, the average value of $\mathbf{n}(\mathbf{r})$ is zero. The variables $\mathbf{v}_y$ and $\mathbf{v}_z$ are therefore meaningless. The correct hydrodynamic equations for m_x, m_y, m_z, and ϵ are obtained from Eqs. (6.8) by setting $\rho_s=0$. The x, y, and z directions are now physically equivalent, and there is a single spin-diffusion constant, D. The correlation functions for the total magnetization are

$$C_{m_x m_x}(\mathbf{k},\omega)=C_{m_y m_y}(\mathbf{k},\omega)=C_{m_z m_z}(\mathbf{k},\omega)$$
$$=2\chi k_B T D k^2/[\omega^2+(Dk^2)^2]. \quad (6.13)$$

Equation (6.13) was given previously by Kadanoff and Martin.[24] The energy autocorrelation function is again given by (6.11c).

The staggered magnetization $\mathbf{N}$ is not conserved by the Hamiltonian, and we expect it to relax to the equilibrium value $\mathbf{N}=0$, in a finite time τ. The corelation functions $C_{n_x n_x}(\mathbf{k},\omega)$, $C_{n_y n_y}(\mathbf{k},\omega)$, and $C_{n_z n_z}(\mathbf{k},\omega)$, which are now equal to each other, will therefore have a broad spectrum of finite width $1/\tau$, in the limit $\mathbf{k}\to 0$. The exact shape of the frequency distribution cannot be predicted by the hydrodynamic theory.

7. ISOTROPIC FERROMAGNET

The isotropic ferromagnet differs from the other cases that we have considered in that the order parameter, for the broken symmetry, is itself a constant of the motion. The Hamiltonian for the ferromagnet has the same form as that of the isotropic antiferromagnet Eq. (6.1), with the difference that the coupling constants favor ferromagnetic alignment in the ground state. The steady states of the system both above and below the Curie point are characterized by the energy E and the three components of the total spin $\mathbf{S}$. For energies E below a critical energy E_c, there exist macroscopically homogeneous equilibrium states only for values of the spin equal to or greater than the equilibrium magnetization in zero field, $M_0(E)\neq 0$. This inequality imposes a constraint on the possible values of $\mathbf{m}$ and ϵ. We assume, nonetheless, that for permissible values of the variables, the entropy density in the ferromagnetic state can be expanded in the simple form

$$s=S_0(\epsilon)+f\{m_x\}$$
$$-\tfrac{1}{2}(\rho_s/TM_0^2)[|\nabla m_y|^2+|\nabla m_z|^2]+\cdots, \quad (7.1)$$

where f is a functional of m_x-M_0. We are considering small deviations from true thermal equilibrium in zero field, with the magnetization aligned, once again, in the x direction. We assume also that we may write for the time dependence of m_y and m_z

$$\frac{\partial \mathbf{m}}{\partial t}=-\nabla\cdot\mathbf{j}^{\mathrm{m}}, \quad (7.2)$$

$$\mathbf{j}^{\mathrm{m}y}=A\nabla m_y+B\nabla m_z+O(\nabla^3), \quad (7.3a)$$

$$\mathbf{j}^{\mathrm{m}z}=C\nabla m_z+D\nabla m_y+O(\nabla^3). \quad (7.3b)$$

It is a consequence of symmetry that j^{m_y} and j^{m_z} have no terms linear in ϵ or (m_x-M_0). Furthermore, symmetry requires that in Eq. (7.2), $A=C$ and $B=-D$. In order to determine the quantities A and B, suppose that we add to the Hamiltonian a magnetic field $\mathbf{H}(\mathbf{r})$. This must change (7.2) to read

$$\frac{\partial \mathbf{m}}{\partial t}=-\nabla\cdot\mathbf{j}^{\mathrm{m}}+\mathbf{m}\times\mathbf{H}. \quad (7.4)$$

Furthermore, if the magnetic field is sufficiently weak, the coefficients in (7.3) will not be affected by the presence of the field. Let us suppose that $\mathbf{H}$ has the form

$$\mathbf{H}(\mathbf{r})=a\hat{y}\cos\mathbf{k}\cdot\mathbf{r}, \quad (7.5)$$

where k and ak^{-2} are considered small. We assume that for such a weak field, the modification of Eq. (7.1) is also negligible to lowest order. (Cf. the Appendix.) By maximizing the entropy for a given value of the total energy,

$$E_{\mathrm{total}}=\int[\epsilon(\mathbf{r})-\mathbf{H}(\mathbf{r})\cdot\mathbf{m}(\mathbf{r})]d\mathbf{r}, \quad (7.6)$$

we find that the ferromagnet will come to equilibrium with magnetization

$$m_y(\mathbf{r})=M_0^2\rho_s^{-1}k^{-2}\cdot a\cos\mathbf{k}\cdot\mathbf{r},$$
$$m_z(\mathbf{r})=0, \quad (7.7)$$
$$m_x(\mathbf{r})=M_0+O(a^2).$$

The requirement that $\partial\mathbf{m}/\partial t$ equals zero at equilibrium gives

$$A=C=0, \qquad (7.8)$$
$$B=-D=\rho_s/M_0.$$

Thus, in the absence of the applied field $\mathbf{H}$, equations (7.2) and (7.8) lead to the existence of spin waves with frequency

$$\omega(\mathbf{k})=(\rho_s/M_0)k^2 \qquad (7.9)$$

and no damping to order k^2. The terms of order (∇^3) in Eq. (7.3) should lead to frequency corrections and to damping of order k^4.[39]

Equation (7.9) for the spin-wave frequency is just the Landau-Lifshitz formula,[14] which may be obtained by simple sum-rule arguments, neglecting damping terms. Similar results have been obtained by many authors.[17] The present derivation justifies the neglect of damping at small k, and in addition predicts a damping proportional to k^4. The microscopic calculations of Ref. 23 find a $k^4 \ln k$ behavior, which, if correct, would be an indication that the gradient expansion already breaks down at this order. The hydrodynamic prediction of the damping may be less reliable in the ferromagnet than in the antiferromagnet, since it is necessary that the expansion of $\partial\mathbf{m}/\partial t$ be carried to a higher order in k than is necessary in the antiferromagnet.

In view of the probable singular dependence of the entropy on the magnitude of the order parameter (m_x-M_0),[31] it seems difficult to predict the time dependence of fluctuations in the x component of $\mathbf{m}$. If one assumes that the dependence of s and $\partial m_x/dt$ on (m_x-M_0) is nonsingular, however, one is led to a pair of coupled diffusion modes for heat and the x component of the magnetization.

In the paramagnetic state, of course, the hydrodynamics is exactly the same as in Sec. 6. There is an independent diffusive mode for ϵ and for each of the three components of $\mathbf{m}$. The only qualitative difference between the ferromagnetic and antiferromagnet systems above T_c is in the temperature dependence of the magnetic susceptibility and transport coefficients, as T_c is approached.[6]

8. DOMAIN OF VALIDITY OF HYDRODYNAMICS: LONG-WAVELENGTH LIMIT

The hydrodynamic theory is believed to be exact for the ideal models considered, in the limit of long-wavelengths, at any fixed temperature in the range $0<T<T_c$. This means that there are correction terms to the hydrodynamic expressions of relative order $[k/k_c(T)]^2$, which become important when k reaches the finite temperature-dependent value $k_c(T)$. Unfortunately, the present theory does not tell us the value of k_c.

One upper bound for k_c is the wave vector at which the damping rate becomes equal to the real part of the spin-wave frequency. For a linear dispersion relation this yields

$$k_c \leq D/c. \qquad (8.1)$$

Another upper bound for k_c is the point at which finite wave-vector corrections must be taken into account in the static susceptibilities, such as $\chi_{m_z}(\mathbf{k})$ and $\chi_{n_z}(\mathbf{k})$, which enter the real part of the frequency via χ and ρ_s. The hydrodynamic theory, however, does not give us the values of the parameters which enter into Eq. (8.1) or $\chi_{m_z}(\mathbf{k})$. For these one must turn to some other theory. We shall briefly review the available information for the isotropic antiferromagnet.

For a classical antiferromagnet at zero temperature we have the exact relation

$$\omega(\mathbf{k})=N_0[\chi_{m_z}(\mathbf{k})\chi_{n_z}(\mathbf{k})]^{-1/2}, \qquad (8.2)$$

with zero damping for the spin waves. (Quantum mechanically, there are corrections to this formula for short wavelengths, but these corrections are of the order of the zero-piont energy which is believed to be small in all cases.[17]) Since the susceptibilities in Eq. (8.2) are functions of $k/k_{\max}$, where $k_{\max}$ is the distance to the Brillouin zone edge, we have for the real part of the spin-wave frequency, at low temperatures

$$k_c \sim k_{\max}=\pi/a, \qquad (8.3)$$

where a is the lattice spacing. The range of validity of hydrodynamics for the damping of spin waves, however, is much smaller than (8.3) at low temperatures. The microscopic calculation of Ref. 22 predicts a cutoff

$$k_c \sim k_{\max}(ST/T_c)^3, \quad \text{for} \quad T \ll T_c/S, \qquad (8.4)$$

in a quantum-mechanical antiferromagnet of spin quantum number S.[40] As the spin S becomes large, i.e., as the system approaches a classical antiferromagnet, one finds, again at low temperatures,

$$k_c \sim k_{\max}, \quad \text{for} \quad T_c/S \ll T \ll T_c. \qquad (8.5)$$

At finite temperatures, say, $T \approx \frac{3}{4}T_c$, one expects that k_c should be of order $k_{\max}$ (e.g., $k_c \gtrsim \frac{1}{10}k_{\max}$).

Near the critical temperature T_N, the static correlation functions have corrections of order $(k\xi)^2$, where the correlation length ξ is believed to diverge roughly as $(T_N-T)^{-2/3}$. Thus the value of k_c obtained from the real part of the spin-wave frequency [Eq. (8.3)] is of order ξ^{-1} near T_N. According to the dynamic scaling hypothesis,[6] the same k_c should be obtained from the damping via Eq. (8.1), that is, $k_c \sim (D/c) \propto \xi^{-1}$.

[39] Consistent with a nonvanishing damping of order k^4 is the assumption that the field (7.5) introduces corrections in Eq. (7.1) of order H_y^2.

[40] For wave vectors larger than (8.4) but smaller than $k_{\max}\cdot(ST/T_c)$, the damping is proportional to $k^2 \ln k$, indicating a breakdown of hydrodynamics.

9. APPLICATIONS TO REAL SYSTEMS

Unlike superfluid helium for which the Bose liquid is an exceedingly good model, real magnetic materials differ in several respects from the simple systems considered thus far. Nevertheless, we believe that the hydrodynamic theory does apply to real materials, and a number of its assertions may be tested by experiment. By far the most promising systems are the antiferromagnets, such as $RbMnF_3$,[41,42] whose Hamiltonians are almost isotropic with respect to the direction of sublattice magnetization. We shall therefore confine our discussion primarily to these systems.

The spin-correlation function for a crystal can be measured in principle by experiments on the inelastic scattering of neutrons.[43,44] The autocorrelation function for the total magnetization at long wavelengths is proportional to the scattering cross section for small momentum transfers or for momentum transfers close to a reciprocal lattice vector. The staggered magnetization can be determined from the scattering cross section close to a magnetic lattice vector:

$$\frac{d^2\sigma}{d\Omega d\omega}(\mathbf{q}=\mathbf{k}+\mathbf{G}+\tfrac{1}{2}\mathbf{G}_0) \propto \sum_\alpha{}' \left[C_{n_\alpha n_\alpha}(\mathbf{k},\omega)+(2\pi)^{-4}\langle N_\alpha\rangle^2\delta(\mathbf{k})\delta(\omega)\right], \quad (9.1)$$

where $\hbar\mathbf{q}$ is the momentum transfer, $\hbar\omega$ is the energy transfer, $\mathbf{G}$ is a reciprocal lattice vector, and $\frac{1}{2}\mathbf{G}_0$ is a wave vector characterizing the antiferromagnetism. In an experiment using unpolarized neutrons, the sum on the right-hand side of (8.1) is taken over the two coordinate directions perpendicular to the momentum transfer $\mathbf{q}$. In a cubic crystal with many magnetic domains, the x direction, (i.e., the direction of sublattice magnetization,) will lie in random directions relative to $\mathbf{q}$, so that the scattering cross section measures an average of the autocorrelation function for n_x, n_y and n_z. For small k, the correlation functions for n_y, and n_z diverge as k^{+2}, whereas the correlation function for n_x is much smaller, probably diverging as k^{-1}.[31] Therefore, for small k we may neglect the scattering due to fluctuations in the magnitude of $\mathbf{n}$, and write

$$\frac{d^2\sigma}{d\Omega d\omega} \propto (2\pi)^4 N_0^2\delta(\mathbf{k})\delta(\omega)+2C_{n_y n_y}(\mathbf{k},\omega). \quad (9.2)$$

Further information is obtainable, in principle, if one can align the magnetic domains, via strain or magnetic fields, and if one uses polarized neutrons.

Several of the hydrodynamic predictions about the correlation function $C_{n_y n_y}$ can be tested by the inelastic scattering technique, provided measurements can be made at sufficiently long wavelengths. First, we may test that the energy dependence of the correlation function can be characterized by spin-wave peaks centered at an energy ck, linear in k. Second, we may check that the width of the peak is proportional to k^2. Finally, we may verify that the spin-wave velocity is given by Eq. (2.41) in terms of ρ_s and χ. The magnetic susceptibility χ may be measured by the usual methods, provided the applied field is sufficiently strong for all domains to line up with staggered magnetization perpendicular to the applied field. In principle, the stiffness constant ρ_s may itself be measured by another neutron scattering experiment, a quasielastic scattering experiment[44] in which one integrates over the energy transfer ω. From (9.2) and (6.11a), we find

$$\frac{\partial\sigma}{\partial\Omega}(\mathbf{q}=\mathbf{k}+\mathbf{G}+\tfrac{1}{2}\mathbf{G}_0) \propto N_0^2\left[(2\pi)^3\delta(\mathbf{k})+\frac{2k_BT}{\rho_s k^2}+\cdots\right]. \quad (9.3)$$

Thus ρ_s is obtained by comparing the weight under the magnetic Bragg peak with the quasielastic cross section in the vicinity of the peak.

The Hamiltonian (6.1) describes a model in which the spins are localized on their lattice sites. The only properties of the Hamiltonian we have used in the hydrodynamic theory, however, are its symmetries and the antiferromagnetic nature of its ground state. The hydrodynamic theory would be unchanged if we use an itinerant model for the electrons. It is similarly unchanged if we consider a metal with free electrons and localized spins present as long as the Hamiltonian is invariant under uniform rotation of all the spin degrees of freedom. (It may also be necessary to require that the electron mean free path be sufficiently short compared to the wavelengths under consideration so that we do not have to introduce a new degree of freedom corresponding to an almost conserved crystal momentum. In practice, the effective interaction between spins in a metal tends to be of rather long range,[45,46] and the limiting wave vector k_c for the validity of hydrodynamics will be correspondingly small.)

The models we have considered differ from real crystals in several other respects. First of all, real crystals always have anisotropic terms in the energy which are not invariant under simultaneous rotation of all the spins. Second, the Hamiltonian of a real system must include the effects of phonons and of impurities.

[41] D. T. Teany, M. J. Freiser, and R. W. H. Stevenson, Phys. Rev. Letters **9**, 212 (1962).
[42] R. Nathans, F. Menzinger, and S. J. Pickart, J. Appl. Phys. **39**, 1237 (1968).
[43] L. Van Hove, Phys. Rev. **93**, 1374 (1954).
[44] See, for example, B. Jacrot and T. Riste, in *Thermal Neutron Scattering*, edited by P. A. Egelstaff (Academic Press Inc., New York, 1965), Chap. 6.
[45] See, for example, T. Kasyua, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1966), Vol. IIB, p. 215.
[46] M. F. Collins, V. J. Minkiewicz, R. Nathans, L. Passell, and G. Shirane, Phys. Rev. **179**, 417 (1969).

If the spin-phonon interaction and the perturbations due to impurities are invariant under uniform rotation of the spins, then these terms will have no effect on the results of the previous sections. As long as the crystal is homogeneous on a macroscopic scale (i.e., on the scale of the wavelength of the spin wave we are considering), the existence of short-range disorder due to impurities will not affect the reasoning we have used to derive the hydrodynamic theory. (Indeed, we have already disregarded the short-range disorder of the spins themselves.) Of course, the presence of impurities can affect the values of the thermodynamic constants and transport coefficients entering the hydrodynamic equations. If there are macroscopic inhomogeneities in the density of impurities, then the coefficients in the hydrodynamic equations will be spatially varying, and a spin wave propagating through the crystal will be diffracted by the inhomogeneities. As a test of the self-consistency of the hydrodynamic point of view, we may calculate, using the Born approximation, the scattering of a hydrodynamic spin wave by the long-wavelength spatial variations of ρ_s arising from a random distribution of impurities. Let us assume

$$\rho_s(\mathbf{r})=\rho_s^0+\gamma c(\mathbf{r}), \tag{9.4}$$

where γ is a constant, and $c(\mathbf{r})$ is the local concentration of impurities. We shall assume χ to be independent of the impurity concentration. The decay rate Γ for a spin wave, due to the scattering, must be proportional to the square of the magnitude of the variations in ρ_s, and from dimensional considerations we may write

$$\Gamma_k/\omega_k \propto \langle \delta\bar{\rho}_s^2\rangle/\rho_s^2, \tag{9.5}$$

where ω_k is the frequency of the spin wave, and $\delta\bar{\rho}_s$ is the difference between the value of ρ_s averaged over a particular region of volume k^{-3} and the value of ρ_s averaged over the crystal as a whole. The angular brackets denote the expectation value over the distribution of impurities. If the value of ρ_s is given by (9.4), then $\langle \delta\bar{\rho}_s^2\rangle$ will be proportional to k^3. The scattering of spin waves by inhomogeneities in ρ_s will therefore be negligible in the long-wavelength limit, compared to the damping $\frac{1}{2}Dk^2$ arising from microscopic processes.

The effects of phonons are similar to the effects of impurities, if the electron-phonon interaction is invariant under uniform rotation of all the spins. We may disregard the effects of short-wavelength phonons, because the phonons do not change the symmetry of the Hamiltonian. To take into account long-wavelength phonons we must introduce new macroscopic variables into the problem, which we may choose to be the displacement field for the nuclei and the momentum density of the crystal. We have in addition a new set of symmetries for the Hamiltonian: invariance with respect to displacements of the crystal as a whole, as well as the Galilean transformation properties for motion of the crystal with a finite velocity. Corresponding to the two new vector fields, we have three new oscillating normal modes: phonons of three polarizations. Because the Hamiltonian is invariant under uniform rotation of all the spins, it is clear that there can be no linear coupling between the spin-wave modes and the phonon displacements. The linearized equations of motion for the spins will, therefore, have the same form as before, and all the results of the previous sections will carry through unchanged.

Every real material has in its Hamiltonian relativistic terms which are not invariant under uniform rotation of the spins. In an itinerant-electron antiferromagnet, such terms will arise from spin-orbit splitting of the band structure at various parts of the Brillouin zone. In a localized spin model, anisotropy results from the effects of the crystal field on the state of the magnetic ion. (This effect also depends indirectly on the spin-orbit interaction, which is responsible for the coupling of the electronic spin to the orbital wave function on the ion.) In addition, for itinerant or localized models alike, there will be a magnetic dipole-dipole interaction and there may be a nuclear hyperfine interaction,[47] neither of which is invariant under uniform rotation of the electronic spins.

In some systems, the spin-nonconserving terms are as large as the spin-conserving terms, and conclusions derived from considerations of an isotropic model are not likely to be applicable. On the other hand, there are many systems in which the spin-nonconserving terms are much smaller than the spin-conserving terms. In these systems, approximation by means of an isotropic model should be relevant. Unfortunately, we know that it is precisely in the limit of long wavelengths and low frequencies that small spin-nonconserving terms make themselves felt. We therefore expect that in a real system with a small anisotropy, the hydrodynamics should be applicable only for a window of wave vectors

$$k_0 \ll k \ll k_c. \tag{9.6}$$

The lower limit k_0 is determined by the anisotropy of the system and presumably goes to zero with the anisotropy. The upper cutoff k_c is independent of the anisotropy for small anisotropy, and is just the cutoff considered earlier.

In the presence of anisotropy, we expect a dispersion relation of the form[17]

$$\omega_k \propto (\rho_s k^2+H_A N_0)^{1/2}\chi^{-1/2} \tag{9.7}$$

[47] The interaction between the electronic and nuclear spin actually conserves the total spin angular momentum of the system. Thus, in the true limit of long wavelengths and low frequencies, the hyperfine interaction would not destroy the validity of hydrodynamic equations for an antiferromagnet with no other anisotropic forces. (It would lead to considerable renormalization of the spin-wave velocity at very low temperatures, however.) At the frequencies appropriate to neutron diffraction, however, the nuclei would not be expected to follow the motion of the electronic spins, and deviations from hydrodynamic behavior may result from the hyperfine field at low temperatures.

at low temperatures and long wavelengths, where H_A is the anisotropy field for the crystal. Thus we estimate $k_0 \approx N_0^{1/2} H_A^{1/2} \rho_s^{-1/2}$ as the lower bound for the validity of the hydrodynamic formula for the real part of ω_k. The microscopic calculations of Ref. 22 suggest that this value of k_0 applies to the spin-wave damping as well.

An antiferromagnet which is known to have a very small anisotropy is $RbMnF_3$.[41] The anisotropy field for this material has been found to be 5 Oe at low temperatures, and the average hyperfine field is $(9/T)$ Oe, where T is the absolute temperature in Kelvins. These figures should be compared with the exchange field of 9×10^5 Oe. This gives a value of $k_0 < 10^{-2}\, k_{\max}$, where $k_{\max}$ is the value of k at the edge of the Brillouin zone. In practice, experiments using neutron diffraction will easily satisfy $k \gg k_0$. Thus $RbMnF_3$ seems to be a good candidate to verify the hydrodynamic predictions, if the instrumental resolution of the neutron measurements can be made high enough to observe a spin-wave damping in the region $k \ll k_c \approx \frac{1}{2} k_{\max}$ for $(T/T_N) \sim 0.4$–0.8, for instance.

The possibility of experimentally checking the hydrodynamic damping of spin waves in a ferromagnet seems much less promising than for the antiferromagnet. In the first place, since the damping goes as k^4, the energy resolution would have to be very much better for the ferromagnet at a given value of the exchange constant. Second, the anisotropic dipolar forces are more important in the ferromagnet at long wavelengths. These difficulties might be surmountable in transition metal ferromagnets which have high transition temperatures, but here another difficulty arises. The k^4 terms in the real part of the spin-wave frequency tend to be large[45,46] in metallic ferromagnets, due to the relatively long-range nature of the effective spin-spin interaction, so it may not be possible to get into the hydrodynamic region with neutrons, even if the dipolar forces can be neglected.

10. ASSUMPTION OF MICROSCOPIC RELAXATION TIME FOR MAGNITUDE OF ORDER PARAMETER REEXAMINED

In the body of this paper we have assumed that relaxation to a state of local equilibrium occurs in a microscopic (wave-vector-independent) time τ. We shall now reexamine this assumption, considering explictly the case of the planar ferromagnet.

The characteristic relaxation time for a fluctuation in the magnitude of the order parameter $m_\perp(\mathbf{r})$ when m_z and ϵ are constant may be written in the form[43]

$$\tau_{\mathbf{k}}^{-1} = \Lambda(\mathbf{k}) / \chi_{m_x m_x}(\mathbf{k}), \tag{10.1}$$

where $\Lambda(\mathbf{k})$ is analogous to a transport coefficient. In the molecular field approximation, $\chi_{m_x m_x}(\mathbf{k})$ is finite as $k \to 0$. If we assume that $\Lambda(\mathbf{k})$ is also finite and nonzero, then we find that $\tau_{\mathbf{k}}$ is finite as $k \to 0$. On the other hand, on the basis of spin-wave calculations, there is strong reason for believing that $\chi_{m_x m_x}(\mathbf{k})$ is not finite at $k=0$, but rather[31]

$$\chi_{m_x m_x}(\mathbf{k}) \propto k^{-1}. \tag{10.2}$$

If we assume that $\Lambda(\mathbf{k})$ is nonetheless finite, then we predict

$$\tau_{\mathbf{k}} \propto k^{-1}. \tag{10.3}$$

The situation here is very similar to the behavior of spin fluctuations above the transition temperature as $T \to T_c$. Van Hove[43] argued that the transport coefficients should generally remain finite at the critical point, whereas susceptibilities diverge, and he therefore predicted critical slowing down of spin fluctuations with $\tau \propto \chi$. It is now believed that Van Hove's arguments are not quantitatively correct; transport coefficients are also usually singular at the critical point, and the divergence of τ is usually slower than the divergence of χ.[6] If the analogous situation holds for $\tau_{\mathbf{k}}$ as a function of k at a fixed temperature below T_c, then we would have

$$\tau_{\mathbf{k}} \propto k^{-x}, \tag{10.4}$$

with $0 < x < 1$.[48]

If $\tau_{\mathbf{k}}$ has such a nonanalytic behavior at $k=0$, it becomes difficult to say with any certainty whether or not the conclusions of hydrodynamics will be affected. It is not clear how the equations of motion will be modified by the deviations of $m_\perp$ from its local equilibrium value, since these equations may also have a singular dependence on $m_\perp$ in the limit $k \to 0$. Nonetheless, we may illustrate the types of behavior to be expected by studying a model with the simplest possible assumptions for this dependence. For small enough deviations from equilibrium, at a fixed value of k, it is sufficient to consider the linearized equations of motion. Let us assume that these can be written as

$$\frac{\partial m_\perp(\mathbf{k})}{\partial t} = -\frac{1}{\tau_{\mathbf{k}}}[m_\perp(\mathbf{k}) - m_0(\mathbf{k})], \tag{10.5a}$$

$$\frac{\partial m_z(\mathbf{k})}{\partial t} = -ik\rho_s v(\mathbf{k}) + O(k^2), \tag{10.5b}$$

$$\frac{\partial v(\mathbf{k})}{\partial t} = -ik\{\chi^{-1} m_z(\mathbf{k}) - b[m_\perp(\mathbf{k}) - m_0(\mathbf{k})]\} + O(k^2), \tag{10.5c}$$

where b is a constant and

$$m_0(\mathbf{k}) = m_z(\mathbf{k}) \partial M_0(\epsilon, m_z)/\partial m_z. \tag{10.6}$$

(We neglect here the dependence of M_0 and of k on the energy density ϵ.) If $\tau_{\mathbf{k}}$ has the form 10.4 with $0 < x < 1$, then we find that the velocity of the spin wave is

[48] The static response function for m_x is infinite at $k=0$ only for an infinitesimal field H_x. For a finite field, the magnetization is predicted to follow the relation $M_x - M_0(E, M_z) \propto H_x^{1/2}$. Therefore it seems almost certain that the relaxation time for a finite uniform fluctuation in $M_\perp$ is finite, but it is possible that the relaxation rate approaches zero as $M_\perp$ approaches M_0.

unchanged from the hydrodynamic result Eq. (1.2), but the damping rate of the spin-wave amplitude is given by

$$\begin{aligned}\tfrac{1}{2}\Gamma_{\mathbf{k}} &= \tfrac{1}{2}k^2\tau_{\mathbf{k}}\rho_s b(\partial M_0/\partial m_z)\\ &\propto k^{2-x}.\end{aligned} \tag{10.7}$$

If the exponent in (10.4) is greater than unity, then we find a renormalized spin-wave frequency

$$\omega(\mathbf{k}) = \rho_s^{1/2}(\chi^{-1} + b\ \partial M_0/\partial m_z)^{1/2}k\,, \tag{10.8}$$

and a nonhydrodynamic damping rate

$$\begin{aligned}\tfrac{1}{2}\Gamma_{\mathbf{k}} &= \tfrac{1}{2}\tau_{\mathbf{k}}{}^{-1}\chi b\ \partial M_0/\partial m_z\\ &\propto k^x.\end{aligned} \tag{10.9}$$

In either of the above cases, the damping is of higher order in k than the real part of the frequency for small enough k. If $x=1$, however, the frequency is renormalized and the damping is the *same* order in k as the frequency.

For the case of the planar ferromagnet with small deviations from true thermal equilibrium in zero magnetic field H_z, symmetry considerations require that both the coefficient b and the coefficient $\partial M_0/\partial m_z$ be zero. Similar considerations require the vanishing of the corresponding coefficients for an isotropic antiferromagnet (or ferromagnet) in zero external field. Thus for these cases there should not be any variation in the magnitude of the order parameter linear in the spin-wave amplitudes. Hence, even if the relaxation time $\tau_{\mathbf{k}}$ does diverge at $\mathbf{k}=0$, it seems very unlikely that this will affect the validity of the hydrodynamic equations for m_y, or the resulting form for the correlation function $C_{m_y m_y}$. On the other hand, superfluid helium is analogous to the planar ferromagnet in finite magnetic field H_z, where the coefficients b and $\partial M_0/\partial m_z$ do not vanish by symmetry.

The equations of two-fluid dydrodynamics have been verified in many ways in helium.[49] In particular, the damping of second sound has been found to vary as k^2, in a number of different experiments over a large range of temperatures.[50] In terms of the above discussion, there are three possible reasons why two-fluid hydrodynamics is valid for second sound: (i) The relaxation time $\tau_{\mathbf{k}}$ for fluctuations in the magnitude of the order parameter may simply not diverge at $k=0$. (ii) The relaxation time may diverge, but the coefficients analogous to b or $\partial M_0/\partial m_z$ in Eqs. (10.5) may not in fact be independent of k, but may go to zero as $k \to 0$ sufficiently fast so that the damping of second sound still goes as k^2. [In support of this possibility, we may note that Eq. (10.2) implies that the term in the entropy quadratic in $m_\perp - M_0(\epsilon, m_z)$ should have a coefficient proportional to k in the limit $k \to 0$.] This possibility seems to the authors to be the most likely one. (iii) A third possibility is that two-fluid hydrodynamics is *not* valid for infinitesimal second-sound amplitudes at finite k, but only becomes valid in the limit where k goes to zero first, and then the amplitude of the second-sound wave becomes small. (Cf. footnote 48.) For example, one might hypothesize that two-fluid hydrodynamics only gives the correct second-sound damping when the amplitude of the superfluid velocity $\mathbf{v}_s$ is large compared to the rms value of $\mathbf{v}_s$ due to thermal fluctuations in the second-sound amplitude at all wave vectors smaller than k. [Roughly this requires $v_s > (k_B T k^3/\rho_s)^{1/2}$.] This inequality is always satisfied in the conventional attenuation experiment, where one measures the decay of macroscopic second-sound waves injected into the sample. It would be of great interest to perform an experiment which would directly test two-fluid hydrodynamics in the other limit, namely, for an infinitesimal perturbation at finite wave vector k. An example is the measurement of the width of the Brillouin peak of light scattered by second sound in a sample of helium at thermal equilibrium. This experiment would be exceedingly difficult in pure He^4 because the coupling of light to the second-sound mode is very small, but it should be possible in mixtures of He^4 and He^3.[51]

Any anomalous damping of second sound due to the divergence of $\tau_{\mathbf{k}}$ should also be reflected in a damping of first sound which does not go as k^2. However, the coefficient of the anomalous term in the damping of first sound would probably be much smaller than in the case of second sound. This is because the compressibility of liquid helium is relatively insensitive to the exact conditions of the compression (e.g., the adiabatic compressibility is almost the same as the isothermal), and thus the effect on the pressure of a change in the magnitude of the order parameter is probably small.

The hydrodynamics of an ordinary crystal is in many ways similar in principle to the hydrodynamics of a superfluid or a magnetic system. In a crystal, the nonzero expectation values of the Fourier components $\rho_{\mathbf{G}}$ of the density at wave vectors $\mathbf{G}$ on the reciprocal lattice break the translational symmetry of the Hamiltonian. Corresponding to this broken symmetry, there are two propagating long-wavelength normal modes of the system which do not exist above the melting temperature, namely, the transverse acoustic modes. In equilibrium, the magnitude of the order parameter $\langle\rho_{\mathbf{G}}\rangle$ is given in the harmonic approximation by $|\langle\rho_{\mathbf{G}}\rangle| = e^{-W_{\mathbf{G}}}$, where $e^{-2W_{\mathbf{G}}}$ is the Debye-Waller factor.[52] This expectation value $|\langle\rho_{\mathbf{G}}\rangle|$ depends, of course, on the density and temperature of the crystal.

[49] See J. Wilks, *The Properties of Liquid and Solid Helium* (Clarendon Press, Oxford, England, 1967).

[50] W. B. Hanson and J. R. Pellam, Phys. Rev. **95**, 321 (1954); J. A. Tyson, Phys. Rev. Letters **21**, 1235 (1968), and (private communication).

[51] L. P. Gor'kov and L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. **33**, 634 (1957) [English transl.: Soviet Phys.—JETP **6** 486 (1958)]; B. N. Ganguly and A. Griffin, Can. J. Phys. **46**, 1895 (1968).

[52] C. Kittel, *Quantum Theory of Solids* (John Wiley & Sons, Inc., New York, 1963), Chap. 19.

If a long-wavelength longitudinal sound wave propagates through the crystal, the local temperature and density will vary periodically. One may first ask whether the local magnitude of the order parameter follows these changes completely, or whether readjustment takes place at a rate $\tau_{\mathbf{k}}^{-1}$ which approaches zero in the limit where the wave vector $\mathbf{k}$ and the amplitude of the sound wave approach zero. One may then ask whether a divergence in the equilibration time $\tau_{\mathbf{k}}$ would produce a nonhydrodynamic behavior of the damping of longitudinal or transverse sound waves, analogous to an anomalous damping of first or second sound in helium. Such a possibility could be detected, in principle, if one could compare the intrinsic damping of sound waves of sufficiently long wavelength as measured by Brillouin scattering techniques and as measured or extrapolated from conventional ultrasonic attenuation measurements.

The difficulty with the Brillouin measurement is that the amplitude of thermal vibration in a crystal is always rather small compared to the interatomic spacing, even near the melting point, so that intrinsic phonon lifetimes are very long. Furthermore, the upper limit k_c for the hydrodynamic damping of sound may be smaller than the minimum wave vector practically measurable with the Brillouin technique.[53]

Another experiment of interest would be to measure directly via inelastic scattering of neutrons the relaxation time $\tau_{\mathbf{k}}$ for fluctuations in the magnitude of the staggered magnetization in a nearly isotropic antiferromagnet such as $RbMnF_3$. Unfortunately, although it is possible in principle to separate longitudinal from transverse fluctuations by a properly designed experiment, this is probably exceedingly difficult in practice because the transverse fluctuations are much larger than the magnitude fluctuations as $k \to 0$.

11. NONLINEAR INSTABILITIES

Our principal reason for studying the hydrodynamics of spin systems has been to predict the form of the frequency-dependent *linear* response functions as measured, for instance, by neutron diffraction. It is also interesting, however, to inquire how the equations of linear hydrodynamics would break down for fluctuations of *finite* amplitude. In particular, we wish to examine the possibility of nonlinear instabilities which can cause a qualitative departure from the predictions of linear hydrodynamics.

In the usual discussions of two-fluid hydrodynamics for liquid helium, the equations are assumed to break down when the difference between the superfluid and normal velocities exceeds a critical velocity v_s.[49] Above this velocity, an originally uniform superflow will decay rapidly through the formation of vortex rings or other turbulence. More recent theories[54,55] suggest that, in principle, there should be a decay of uniform superflow for any nonzero relative velocity. For small velocities, the decay mechanism requires the thermally activated creation of a very large vortex ring, and the decay rate will be proportional to $\exp(-A(T)/|v_n-v_s|)$ i.e., it is zero to all orders in $|v_n-v_s|$. Thus the equations of linear hydrodynamics are quantitatively and qualitatively correct in the limit $k \to 0$ and $v_s \to 0$ for any finite value of the ratio v_s/k. (See Sec. 10 above, however, where we discuss the possibility of the breakdown of two-fluid hydrodynamics for $v_s/k \to 0$.) The condition that the total phase oscillations be small compared to π (i.e., $v_s/k \ll 1$) is certainly *not* necessary for the validity of two-fluid hydrodynamics.

We believe that in the *planar* ferromagnet or antiferromagnet the conditions for nonlinear instabilities leading to decay of a uniform gradient in the phase of the order parameter are similar to those in the superfluid. Thus a hypothetical state in which there are a small number of phase loops around a large torus of magnetic material could persist for astronomically long times. The set of equilibrium values of the order parameter is a multiply connected region for these systems (i.e., a circle in the complex plane), and the persistent current state could only decay through the appearance of a discontinuity in the phase, such as a vortex core. In the isotropic ferromagnet and antiferromagnet, on the other hand, the equilibrium range of the order parameter is simply connected (i.e., the surface of a sphere). In that case, decay of a finite persistent current can occur without the necessity for a large deviation from local equilibrium, and without the necessity of passing over a finite free energy barrier. For example, in an antiferromagnet an order parameter initially in a state of uniform phase gradient with $\mathbf{N}$ lined up in the x-y plane can decay continuously to an equilibrium state of uniform sublattice magnetization in the z direction, via

$$n_x(\mathbf{r},t)+in_y(\mathbf{r},t)=s(t)N_0e^{i\mathbf{q}\cdot\mathbf{r}}, \tag{11.1a}$$

$$n_z(\mathbf{r},t)=\pm(1-|s|^2)^{1/2}N_0, \tag{11.1b}$$

with $|s(t)|=1$ initially, and $s(t) \to 0$ for large t. According to Sec. 6 above, the decay rate of state (11.1) is proportional to q^2 when $|s|$ is small, and it may be a reasonable conjecture that it is also proportional to $q^2(=v^2/s^2)$ at the initial time, when $s \approx 1$. The decay process of (11.1) must be regarded as an instability, since there is no *a priori* preference for the + sign or the − sign in (11.1b).

A necessary condition for the qualitative validity of hydrodynamics in the isotropic antiferromagnet is

[53] A. S. Pine in *Proceedings of the International Conference on Light Scattering in Solids, New York University, 1968*, edited by G. B. Wright (Springer-Verlag, New York, 1969), and (private communication).

[54] S. V. Iordanskii, Zh. Eksperim. i Teor. Fiz. **48**, 708 (1965) [English transl.: Soviet Phys.—JETP **21**, 467 (1965)].

[55] J. S. Langer and M. E. Fisher, Phys. Rev. Letters **19**, 560 (1967).

probably the condition that v/k be small, i.e., that the angle of deviation of $\mathbf{n}(\mathbf{r})$ from its average value should everywhere be small compared to π.

In the ferromagnet, a decay of spin current analogous to Eq. (11.1) would be inconsistent with the conservation of the z component of spin, and the decay must occur nonuniformly around the torus. In fact, a state initially in the ferromagnetic equivalent of (11.1) (with $|s|\neq 0$ and q very small) can never decay into a state of uniform magnetization, but rather will eventually decay to a state of form (11.1) with $|s|\neq 0$ and q the minimum possible nonzero value, i.e., q equal to 2π divided by the circumference of the torus. Reasoning along these lines suggests that the linear hydrodynamic description of the decay of a spin wave may only make sense if s is sufficiently close to zero so that the difference between $(1-|s|^2)^{1/2}$ and unity is negligible compared to q^2; i.e., in the limit $s/q\rightarrow 0$.

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APPENDIX: RELATION BETWEEN ρ_s AND STATIC SUSCEPTIBILITY

Let us consider the thermal equilibrium state of the planar ferromagnet when we apply a weak spatially varying magnetic field in the y direction:

$$H_y(\mathbf{r})=h_\perp \cos\mathbf{k}\cdot\mathbf{r}. \tag{A1}$$

To find this state we must maximize the entropy while holding fixed the total energy

$$E_{\text{total}}=\int[\epsilon(\mathbf{r})-H_z m_z(\mathbf{r}) -H_y(\mathbf{r})m_\perp(\mathbf{r})\sin\varphi(\mathbf{r})]d\mathbf{r}. \tag{A2}$$

Let us assume that the entropy density is given by (2.28) and that $m_\perp(r)$ is given by (2.12); then we can maximize the entropy by the methods of variational calculus. We find a solution

$$\epsilon(\mathbf{r})=\text{const}+O(h_\perp^2), \tag{A3a}$$

$$m_z(\mathbf{r})=\text{const}+O(h_\perp^2), \tag{A3b}$$

$$m_x(\mathbf{r})=\pm M_0(\epsilon,m_z)+O(h_\perp^2), \tag{A3c}$$

$$m_y(\mathbf{r})=(M_0^2/\rho_s k^2)H_y(\mathbf{r})+O(h_\perp^3), \tag{A3d}$$

from which Eq. (3.9b) follows. The constants in (A3a) and (A3b) are simply the values of ϵ and m_z for a state in true thermal equilibrium with the given total energy E_{total} in the field H_z, but with $H_y=0$. Equations (A3) will be applicable provided $|m_y(\mathbf{r})|\ll M_0$.

Expression (2.28), which gives the entropy as a functional of ϵ, m_z, and φ, is only strictly correct in the absence of an external field, such as H_y.

In fact, in order for the system to be uniquely defined by the functions ϵ, φ, and m_z, we required in Sec. 2 that the system have already evolved in the absence of fields for a time long compared to τ. In the presence of applied fields, there will generally be corrections to (2.28) and (2.12) proportional to the square of the applied field. In the limit of long wavelengths, however, the field $h_\perp$ necessary to produce a given magnetization $m_y(\mathbf{r})$ is very small,

$$h_\perp\propto k^2 m_y, \tag{A4}$$

and hence the corrections due to the applied field are believed to be negligible in this limit.

The above arguments justifying the use of (2.28) and (2.12) in the presence of the field (A1) can be restated in various forms. We have asserted that the thermodynamic equilibrium state in the presence of the weak applied field $h_\perp$ (state 1), and the *local* equilibrium state having the same value of $m_y(\mathbf{r})$ but obtained after a period of free evolution (state 2), are essentially the same physical states in the limit $k\rightarrow 0$. In particular, we assert that both these states may be represented by a state (3) in which a position-dependent spin rotation has been applied to the uniform field-free equilibrium state, so that the direction of spin quantization oscillates in the x-y plane about the x axis. The identity of states 1 and 3 may itself be verified in the molecular-field approximation and in various other simple approximations.