lator strength may be increased by studying angles of propagation β close to the critical value β_c.[14]

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[14] The approximate expression given in Eq. (4.7a) is not valid for β close to β_c.

Correlations along a Line in the Two-Dimensional Ising Model*

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The $2n$-spin correlation function for the two-dimensional Ising model at $T=T_c$ is evaluated for the special case in which all the spins lie along a straight line, separated by many lattice constants. The resulting $2n$-spin function is simply a quotient of products of two-spin correlations. A hypothesis of reducibility of fluctuations in the critical state is introduced. This hypothesis asserts that the product of any two local fluctuating quantities in the same neighborhood of space may be effectively replaced by a finite sum of local fluctuating quantities in this neighborhood. As a result, the previously found form for the $2n$-spin function may be used to evaluate the correlation function of n energy densities when all n points lie on a line. The n-energy correlation function is simply a sum of products of two-energy correlations. The quotient form for the spin correlation plus scaling is shown to immediately imply the logarithmic specific heat.

I. INTRODUCTION

SINCE Onsager's original work,[1] many authors have discussed thermodynamic properties[2,3] and correlations[3-11] in the two-dimensional Ising model. Much of this work has concentrated on using the Onsager solution to learn about the behavior near the critical point. The concept of scaling,[12-15] for example, has arisen in part from information gained from this model. According to the scaling idea, there are two indices, described[12] as x and y, which together determine the nature of all the critical singularities.

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In the two-dimensional Ising model, x and y each have simple values: $x=15/8$, $y=1$. However, even though the Onsager solution exists as a guide, no fully satisfactory physical argument is known to be available for understanding the values of x and y. These values are only obtained by very detailed and rather untransparent calculations. One can hope, however, that such simple values of x and y can be seen as the result of some structural property of critical correlations. In this paper, I argue that the result $y=1$, which implies the logarithmic specific heat, is a natural result of a simple structure of the n-spin correlation function.

This argument is based upon an evaluation of the $2n$-spin correlation function under the conditions: (a) that the Ising model is at the critical point; (b) that all the spins lie on a single straight line; and (c) that the spins are all separated from one another by many lattice constants. Then, if the spins are ordered along the line as $\sigma_1, \sigma_{1'}, \sigma_2, \sigma_{2'}, \ldots, \sigma_n, \sigma_{n'}$, the correlation function is calculated to have the form

$$\left\langle \prod_{i=1}^{n} (\sigma_i \sigma_{i'}) \right\rangle = \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{\langle \sigma_i \sigma_{j'} \rangle}{(\langle \sigma_i \sigma_j \rangle \langle \sigma_{i'} \sigma_{j'} \rangle)^{1/2}}. \quad (1.1)$$

At first sight, it does not appear that the result (1.1) defines any critical indices. Further progress comes from the introduction of an extra idea, of the *reducibility of critical fluctuations*. Reducibility arises from the idea that there are only a limited number of independent fluctuating local variables in any phase-transition problem. Say there are s of these, $O_\nu(\mathbf{r})$ for $\nu = 1, 2, \ldots, s$. A product of two O_ν's at neighboring positions is

again a local fluctuating quantity, which then must be a linear combination of the O_ν's. In symbols, whenever $\mathbf{r}$ and $\mathbf{r'}$ are sufficiently close together,

$$O_\nu(\mathbf{r})O_{\nu'}(\mathbf{r'}) = A_{\nu\nu'}(\mathbf{r}-\mathbf{r'})$$
$$+ \sum_{\nu''} A_{\nu\nu',\nu''}(\mathbf{r}-\mathbf{r'})O_{\nu''}\left(\frac{\mathbf{r}+\mathbf{r'}}{2}\right). \quad (1.2)$$

Here the A's are simply numbers and the O_ν's are fluctuating operators.

Equation (1.2) permits us to reduce to simpler forms correlations of $2n$ spins whenever pairs of spins lie very close together. If the reducibility equation (1.2) is applied to a product of two spins, the main term in the sum involves the operator $\mathcal{E}(r)$, which is physically the energy density minus its critical value. Thus Eqs. (1.1) and (1.2) may be combined to yield expressions for energy-energy correlations. This analysis yields two notable results. First, the energy-energy correlation function is evaluated *just from (1.1) and (1.2) with the result that at the critical point*

$$\langle \mathcal{E}(1)\mathcal{E}(2)\rangle \sim 1/|\mathbf{r}_1-\mathbf{r}_2|^2. \quad (1.3)$$

Equation (1.3) implies $y = 1$ and a logarithmic singularity in the specific heat for this two-dimensional system. The other result is that whenever the points 1, 2, $\cdots$, $2n$ lie many lattice constants from one another along a single straight line, we have

$$\langle \mathcal{E}(1)\mathcal{E}(2)\cdots \mathcal{E}(2n)\rangle = \sum_{\text{perm}} \langle \mathcal{E}(1)\mathcal{E}(2)\rangle$$
$$\times \langle \mathcal{E}(3)\mathcal{E}(4)\rangle \cdots \langle \mathcal{E}(2n-1)\mathcal{E}(2n)\rangle \quad (1.4)$$

at the critical point.

Section II is devoted to the derivation of Eq. (1.1). Since this derivation is highly technical, many readers will wish to skip this part. Section III discusses reducibility [Eq. (1.2)] and its application to the two-dimensional Ising model. Section IV includes information about energy correlations and includes the deviations of the results (1.3) and (1.4).

II. EVALUATION OF MULTIPLE-SPIN CORRELATIONS

Consider $2n$ spins on the x axis at the points $1 = (0,k_1)$, $1' = (0,k_1')$, $\cdots$, $n' = (0,k_n')$ arranged so that

$$k_1 \leq k_1' \leq k_2 \leq \cdots \leq k_n'. \quad (2.1)$$

The condition $k \in \{i\}$ means that k lies between k_i and k_i', or, more precisely, that we have

$$k \in \{i\} \quad \text{when} \quad k_i < k \leq k_i'. \quad (2.2)$$

An evaluation of the $2n$-spin correlation function is most easily obtained via the Toeplitz determinant method[6] as applied by Wu.[5] For completeness, we mention a derivation of our beginning formula starting from the work of Ref. 4. The $2n$-spin case can be

handled by the method described in Sec. 3.1 of Ref. 4. In direct analogy to Eq. (I.3.13), we find that

$$\langle \prod_i \sigma_i\sigma_{i'}\rangle = \exp\tfrac{1}{4}\text{tr} \ln(1-2\eta g\eta)^2$$
$$= [\det(1-2\eta g\eta)]^{1/2}. \quad (2.3)$$

The last line follows because

$$\text{tr} \ln X = \ln \det X$$

for any X. The quantity g is a matrix in j, k space and a 2×2 "spin" space, while η is diagonal in coordinate space. In fact, we have

$$\eta(j,k) = \delta_{j,0}, \quad \text{if } k \in \{i\} \text{ for any } \{i\}$$
$$= 0, \quad \text{otherwise.} \quad (2.4)$$

The relevant part of g is then diagonal in j and takes the form

$$g(0k;0k') = \tfrac{1}{2}\delta_{kk'} + \int_{-\pi}^{\pi} \frac{dp_y}{4\pi} e^{ip_y(k-k')}[\Phi(p_y)]^{\tau_3\tau_2} \quad (2.5)$$

according to (I.3.21). Here τ_3 and τ_2 are standard Pauli-spin matrices, while $\Phi(p_y)$ is defined by Eq. (I.3.18). In this definition, Φ depends upon two parameters called A and B in the notation of I and of Refs. 2 and 3. At the critical point, $B = 1$, and then Φ takes the form

$$\Phi(p_y) = -\left(\frac{Ae^{ip_y}-1}{A-e^{ip_y}}\right)^{1/2}. \quad (2.6)$$

The remaining parameter A is a measure of the asymmetry of the lattice. When $A = 1$, the coupling in the y direction is much stronger than the coupling in the x direction; when $A = \infty$, the coupling in the x direction is much stronger. At $A = 3+2\sqrt{2}$, the couplings in both directions are equal.

It is difficult to evaluate the trace in Eq. (2.3) exactly for any finite value of the asymmetry parameter A. However, Wu has argued[5] that the A dependence of the two-spin correlation function is very simple. In the limit of large separations in the y direction, he suggests that the total A dependence of $\langle \sigma_i\sigma_{i'}\rangle$ is a multiplicative factor $[(A+1)/(A-1)]^{1/4}$.

To generalize Wu's conjecture, consider

$$\ln\langle \prod_i \sigma_i\sigma_{i'}\rangle = \tfrac{1}{4} \text{tr} \ln(1-2\eta g\eta)^2. \quad (2.7)$$

In the asymptotic limit of large separations, the main term on the right-hand side of Eq. (2.7) might be expected to come from the large $k-k'$ form of $g(0k,0k')$, which in turn comes from small p_y. However, at $p_y = 0$, Eq. (2.6) implies that $\Phi(p_y)$ is independent of A. Therefore, we might expect the main term in the asymptotic expansion of (2.7) to be independent of A.

We therefore write

$$\ln\langle\prod_i \sigma_i\sigma_{i'}\rangle = \ln\langle\prod_i \sigma_i\sigma_{i'}\rangle_0 + \text{correction term},$$

where the subscript zero indicates a correlation function at a particular value of A, picked to be $A = \infty$.

The correction to g arising from terms which do not vanish as $A \to \infty$ occurs only for $p_y \neq 0$, and hence is a relatively short-ranged function of $k-k'$. Since this correction is short-ranged, the entire correction term cannot involve correlations between different regions $\{i\}$ and $\{j\}$. Hence, the correction term is of the form of a sum over the different regions:

$$\text{correction term} = \sum_i \left[\tfrac{1}{4}\, \text{tr} \ln(1-2\eta_i g \eta_i)^2\,\big|_A \right.$$

$$\left. -\tfrac{1}{4}\, \text{tr} \ln(1-2\eta_i g \eta_i)\,\big|_{A=\infty}\right].$$

However, Wu[5] has already argued that if $k_{i'}-k_i\gg 1$, the correction term for two spins is $\tfrac{1}{4}\ln(A+1)/(A-1)$. Consequently, the correction for $2n$ spins is n times as large. As a final result, we find that

$$\langle\prod_i \sigma_i\sigma_{i'}\rangle = [(A+1)/(A-1)]^{n/4}\langle\prod_i \sigma_i\sigma_{i'}\rangle_0. \quad (2.8)$$

Here the subscript zero denotes the average at $A = \infty$. In this limit, the average can be computed, since[9] at $A = \infty$, we have

$$[1-2g(0k;0k')] = -\frac{1}{\pi}\frac{\tau_3}{k-k'+\tfrac{1}{2}\tau_3}\tau_2$$

$$= h_{kk'} \quad (2.9a)$$

and

$$\langle\prod_i \sigma_i\sigma_{i'}\rangle_0 = [\det{}_\eta(-h_{kk'})]^{1/2}, \quad (2.9b)$$

where $\det_\eta$ covers the part of k space in which η is different from zero. Wu has, in fact, evaluated a determinant precisely like this one. The result is [see Wu[5] Eqs. (4.7) and (4.8)]

$$\langle\prod_i \sigma_i\sigma_{i'}\rangle_0 = \prod_k\left(\frac{2}{\pi}\prod_{k'<k}\frac{1}{1-[2(k-k')]^{-2}}\right). \quad (2.10)$$

In Eq. (2.10) the products cover the region in which $\eta \neq 0$.

To simplify Eq. (2.10), the products are grouped according to the different regions $\{i\}$ denoted by Eq. (2.1). Then, we have

$$\langle\prod_i \sigma_i\sigma_{i'}\rangle_0 = \prod_i F_i \prod_{j<i} F_{ij}, \quad (2.11)$$

with

$$F_i = \prod_{k\in\{i\}}\frac{2}{\pi}\prod_{k'\in\{i\},k'<k}\frac{1}{1-[2(k-k')]^{-2}}, \quad (2.12)$$

$$F_{ij} = \prod_{k\in\{i\}}\prod_{k'\in\{j\}}\frac{1}{1-[2(k-k')^{-2}]}. \quad (2.13)$$

The F's can be expressed in more physical terms. By considering the case of only two spins, we see that

$$F_i = \langle\sigma_i\sigma_{i'}\rangle_0, \quad (2.14)$$

while the four-spin case gives

$$F_{ij} = \frac{\langle\sigma_i\sigma_{i'}\sigma_j\sigma_{j'}\rangle_0}{\langle\sigma_i\sigma_{i'}\rangle_0\langle\sigma_j\sigma_{j'}\rangle_0}. \quad (2.15)$$

Therefore, Eq. (2.11) reduces to

$$\langle\prod_i \sigma_i\sigma_{i'}\rangle_0 = \prod_i\langle\sigma_i\sigma_{i'}\rangle_0\prod_{j<i}\frac{\langle\sigma_i\sigma_{i'}\sigma_j\sigma_{j'}\rangle_0}{\langle\sigma_i\sigma_{i'}\rangle_0\langle\sigma_j\sigma_{j'}\rangle_0}. \quad (2.16)$$

Therefore, for example, we obtain

$$\langle\sigma_1\sigma_{1'}\sigma_2\sigma_{2'}\sigma_3\sigma_{3'}\rangle_0$$

$$= \frac{\langle\sigma_1\sigma_{1'}\sigma_2\sigma_{2'}\rangle_0\langle\sigma_1\sigma_{1'}\sigma_3\sigma_{3'}\rangle_0\langle\sigma_2\sigma_{2'}\sigma_3\sigma_{3'}\rangle_0}{\langle\sigma_1\sigma_{1'}\rangle_0\langle\sigma_2\sigma_{2'}\rangle_0\langle\sigma_3\sigma_{3'}\rangle_0}. \quad (2.17)$$

Nevertheless, the four-spin correlation function can be further simplified. In Eq. (2.17) we set $2=1'$ and $2'=3$. Then, since the case $A=\infty$ is still an Ising model, we get $\sigma_1^2=\sigma_3^2=1$ and Eq. (2.17) becomes

$$\langle\sigma_1\sigma_{3'}\rangle_0 = \frac{\langle\sigma_1\sigma_3\rangle_0\langle\sigma_1\sigma_{1'}\sigma_3\sigma_{3'}\rangle_0\langle\sigma_{1'}\sigma_{3'}\rangle_0}{\langle\sigma_1\sigma_{1'}\rangle_0\langle\sigma_{1'}\sigma_3\rangle_0\langle\sigma_3\sigma_{3'}\rangle_0}.$$

Consequently, we obtain

$$\frac{\langle\sigma_i\sigma_{i'}\sigma_j\sigma_{j'}\rangle_0}{\langle\sigma_i\sigma_{i'}\rangle_0\langle\sigma_j\sigma_{j'}\rangle_0} = \frac{\langle\sigma_i\sigma_{j'}\rangle_0\langle\sigma_j\sigma_{i'}\rangle_0}{\langle\sigma_i\sigma_j\rangle_0\langle\sigma_j\sigma_i\rangle_0}, \quad (2.18)$$

and Eq. (2.16) reads

$$\langle\prod_i \sigma_i\sigma_{i'}\rangle_0 = \prod_i\langle\sigma_i\sigma_{i'}\rangle_0\prod_{j<i}\frac{\langle\sigma_i\sigma_{j'}\rangle_0\langle\sigma_{i'}\sigma_j\rangle}{\langle\sigma_i\sigma_j\rangle_0\langle\sigma_{i'}\sigma_{j'}\rangle_0}. \quad (2.19)$$

Equation (2.19) is an exact result for correlations when $A = \infty$ and all the spins lie on the x axis. However, Eq. (2.8) enables us to relate each correlation function with finite A to a correlation with $A = \infty$ for the special case in which all the spins are separated by a distance much larger than a lattice constant. By applying Eqs. (2.19) and (2.8) we see that Eq. (2.19) remains true for correlations with finite A. After a rearrangement of terms, we derive Eq. (1.1):

$$\langle\prod_i(\sigma_i\sigma_{i'})\rangle = \prod_i\prod_j\frac{\langle\sigma_i\sigma_{j'}\rangle}{(\langle\sigma_i\sigma_j\rangle\langle\sigma_{i'}\sigma_{j'}\rangle)^{1/2}}. \quad (1.1)$$

The derivation of Eq. (1.1) requires that the spins all lie on the x axis. However, because of the rotational invariance of the critical state,[4] one can expect that

Eq. (1.1) will hold true whenever the spins are widely separated on *any* line in the lattice, even when that line is not parallel to one of the principal axes.

III. REDUCIBILITY OF FLUCTUATIONS

To use Eq. (1.1) to derive other types of critical correlations, we introduce the concepts of local critical variables and of the reducibility of fluctuations in products of these variables.

It is possible to define several thermodynamic variables which have anomalously large fluctuations near the critical point. For example, in the Ising model the magnetization M and the energy $\mathcal{3C}$ each have divergent fluctuations at the critical point. To see this divergence, we need only note that the constant-H specific heat and magnetic susceptibility, each divergent quantities, are proportional, respectively, to the mean-squared fluctuations in $\mathcal{3C}$ and M.[16]

Both of these extensive fluctuating variables can be expressed as a sum over all coordinates of local fluctuating variables. The magnetization is proportional to the sum of spins at the different sites and the energy is proportional to a sum of a local energy density. We find it convenient to use an energy density which has zero expectation value at the critical point, so we write for the case of equal couplings in the two directions

$$\mathcal{E}(j,k) = -\tfrac{1}{2}J\sigma_{j,k}(\sigma_{j+1,k}+\sigma_{j-1,k}+\sigma_{j,k+1}+\sigma_{j,k-1})$$
$$+\tfrac{1}{2}J\langle\sigma_{j,k}(\sigma_{j+1,k}+\sigma_{j-1,k}+\sigma_{j,k+1}+\sigma_{j,k-1})\rangle, \quad (3.1)$$

where angular brackets denote an expectation value at the critical point. These densities of critical fluctuating quantities σ_1 and $\mathcal{E}(1)$ are local variables in the sense that they depend only upon spins in the neighborhood of point 1. In general, we write a density of a fluctuating quantity as $O_\nu(\mathbf{r})$, with ν being an index which defines which quantity we have under consideration.

We wish to assert that there are effectively only a finite number—say, s—of independent fluctuating quantities. This assertion is an unproved hypothesis whose consequences we wish to develop. First, we must state quite precisely what we mean. Let $O(\mathbf{R})$ be a local variable at $\mathbf{R}$ which depends only upon spins in the neighborhood of R. That is,

$$O(\mathbf{R}) \text{ depends upon } \sigma_{r'} \text{ only if } |\mathbf{R}-\mathbf{r}'| \lesssim R_0 \quad (3.2)$$

for some fixed R_0. Let X be an operator which depends only upon spins far from R, i.e.,

$$X \text{ depends upon } \sigma_r \text{ only if } |\mathbf{R}-\mathbf{r}| > R_0\lambda$$
$$\text{with } \lambda \gg 1. \quad (3.3)$$

Then all averages of the form $\langle O(\mathbf{R})X\rangle$ may be eval-

uated as

$$\langle O(\mathbf{R})X\rangle = A_0\langle X\rangle + \sum_{\nu=1}^{s} A_\nu\langle O_\nu(\mathbf{R})X\rangle$$

$$+\text{terms which go to zero as } \lambda \to \infty, \quad (3.4)$$

where the A's are the same for all X's. In this sense, any $O(\mathbf{R})$ is expandable in our basic operators as

$$O(\mathbf{R}) = A_0 + \sum_{\nu=1}^{m} A_\nu O_\nu(\mathbf{R}), \quad (3.5)$$

with constant coefficients A_ν.

But now consider a product of basic variables of the form

$$O_\nu(\mathbf{R}+\tfrac{1}{2}\mathbf{r})O_{\nu'}(\mathbf{R}-\tfrac{1}{2}\mathbf{r}). \quad (3.6)$$

If $r \lesssim R_0$, this is a local variable in the sense of Eqs. (3.2)–(3.4). Hence, it must be expandable in the form (3.5). We conclude that the product variable (3.6) is reducible in the form of a sum of basic variables.

$$O_\nu(\mathbf{R}+\tfrac{1}{2}\mathbf{r})O_{\nu'}(\mathbf{R}-\tfrac{1}{2}\mathbf{r})$$
$$= A_{\nu\nu',0}(\mathbf{r}) + \sum_{\nu''} A_{\nu\nu',\nu''}(\mathbf{r})O_{\nu''}(\mathbf{R}), \quad (3.7)$$

where the A's are constants, not fluctuating variables.[17] Equation (3.7) is our basic assertion of reducibility.

In our applications, we shall always choose the $O_\nu(\mathbf{R})$ such that the average of $O_\nu(\mathbf{R})$ at the critical point will vanish. Then, we obtain

$$A_{\nu\nu',0}(\mathbf{r}) = \langle O_\nu(\mathbf{R}+\tfrac{1}{2}\mathbf{r})O_{\nu'}(\mathbf{R}-\tfrac{1}{2}\mathbf{r})\rangle_c. \quad (3.8)$$

In this paper, we apply Eq. (3.7) to the reduction of a product of two spins. We assume that the two most singular local variables that can appear in a product of two σ's are σ_r and $\mathcal{E}(r)$, so that Eq. (3.7) reads

$$\sigma_{\mathbf{R}+\mathbf{r}/2}\sigma_{\mathbf{R}-\mathbf{r}/2} = \langle\sigma_{\mathbf{R}+\mathbf{r}/2}\sigma_{\mathbf{R}-\mathbf{r}/2}\rangle_c\{1+B(\mathbf{r})\sigma_{\mathbf{R}}+C(\mathbf{r})\mathcal{E}(\mathbf{R})$$
$$+\text{less singular operators}\}. \quad (3.9)$$

A further simplification arises because of the necessity for symmetry of all critical-point correlations under the change of sign of all spins. The left-hand side of (3.9) is even under this symmetry operation; the right-hand side can only be even if $B(r)$ vanishes.

Our conclusion is that

$$\sigma_{\mathbf{R}+\mathbf{r}/2}\sigma_{\mathbf{R}-\mathbf{r}/2} = \langle\sigma_0\sigma_r\rangle[1+C(\mathbf{r})\mathcal{E}(\mathbf{R})$$
$$+\text{less singular terms}]. \quad (3.10)$$

This conclusion is not new. It has been used by several previous authors.[9,11,18,19] Furthermore, parts of the general idea of reducibility have been previously stated by Green.[20]

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IV. ENERGY CORRELATIONS

According to Eq. (1.1), at the critical point

$$F_{12} \equiv \frac{\langle \sigma_1 \sigma_{1'} \sigma_2 \sigma_{2'} \rangle}{\langle \sigma_1 \sigma_{1'} \rangle \langle \sigma_2 \sigma_{2'} \rangle} = \frac{\langle \sigma_1 \sigma_{2'} \rangle \langle \sigma_2 \sigma_{1'} \rangle}{\langle \sigma_1 \sigma_2 \rangle \langle \sigma_{1'} \sigma_{2'} \rangle}, \quad (4.1)$$

when all the points are far separated along a line. Now let the pair of points 1 and 1' as well as the pair 2 and 2' be much closer to one another than the separation between these pairs:

$$R_{12} = \left| \tfrac{1}{2} (\mathbf{r}_1 + \mathbf{r}_{1'}) - \tfrac{1}{2} (\mathbf{r}_2 + \mathbf{r}_{2'}) \right|. \quad (4.2)$$

In that case, we can reduce the product of neighboring spins according to Eq. (3.10) and find that

$$F_{12} \approx 1 + C(|\mathbf{r}_1 - \mathbf{r}_{1'}|) C(|\mathbf{r}_2 - \mathbf{r}_{2'}|) \langle \mathscr{E}(\mathbf{R}_1) \mathscr{E}(\mathbf{R}_2) \rangle, \quad (4.3)$$

with

$$\mathbf{R}_1 = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_{1'}), \quad \mathbf{R}_2 = \tfrac{1}{2}(\mathbf{r}_2 + \mathbf{r}_{2'}). \quad (4.4)$$

The right-hand side of (4.1) is expandable in a power series in $|\mathbf{r}_1 - \mathbf{r}_{1'}|/R_{12}$ and $|\mathbf{r}_2 - \mathbf{r}_{2'}|/R_{12}$. To second order in these variables, we find that

$$F_{12} \approx 1 + |\mathbf{r}_1 - \mathbf{r}_{1'}| |\mathbf{r}_2 - \mathbf{r}_{2'}| (d^2/dR_{12}{}^2)$$
$$\times \ln\langle \sigma(\mathbf{R}_1) \sigma(\mathbf{R}_2) \rangle. \quad (4.5)$$

A comparison of (4.3) with (4.5) now yields the relations

$$C(|\mathbf{r}_1 - \mathbf{r}_2|) = |\mathbf{r}_1 - \mathbf{r}_2| c, \quad (4.6)$$

$$\langle \mathscr{E}(\mathbf{R}_1) \mathscr{E}(\mathbf{R}_2) \rangle = (1/c^2)(d^2/dR_{12}{}^2) \ln\langle \sigma(\mathbf{R}_1) \sigma(\mathbf{R}_2) \rangle. \quad (4.7)$$

Since the spin-spin correlation function obeys a power law at the critical point,

$$\langle \sigma(\mathbf{R}_1) \sigma(\mathbf{R}_2) \rangle \sim 1/R_{12}{}^\eta, \quad (4.8)$$

we find that

$$\langle \mathscr{E}(\mathbf{R}_1) \mathscr{E}(\mathbf{R}_2) \rangle = (\eta/c^2) 1/R_{12}{}^2. \quad (4.9)$$

According to scaling,[12] the energy-energy correlation function $\sim R_{12}{}^{-2(d-y)}$ at the critical point. Hence $y = 1$ follows as a consequence of the structure of spin correlations on a line, Eq. (1.1), and the reduction equation (3.10).

Multiple energy correlations can be evaluated by the same technique. According to Eq. (1.1),

$$\langle \prod_i \sigma_i \sigma_{i'} \rangle / \prod_i \langle \sigma_i \sigma_{i'} \rangle = \prod_{i<j} F_{ij}, \quad (4.10)$$

where F_{ij} is of the form (4.1). As each i and i' approach one another, the left-hand side of (4.10) can be reduced with the aid of (3.10) and the right-hand side evaluated with the aid of (4.3). We find

$$\langle \prod_i [1 + C(|\mathbf{r}_i - \mathbf{r}_{i'}|) \mathscr{E}(\mathbf{R}_i)] \rangle = \prod_{i<j} [1 + C(|\mathbf{r}_i - \mathbf{r}_{i'}|)$$
$$\times C(|\mathbf{r}_j - \mathbf{r}_{j'}|) \langle \mathscr{E}(\mathbf{R}_i) \mathscr{E}(\mathbf{R}_j) \rangle]. \quad (4.11)$$

From equating the coefficients of $\prod_i C(|\mathbf{r}_i - \mathbf{r}_{i'}|)$ in Eq. (4.11) we find that

$$\left\langle \prod_{i=1}^n \mathscr{E}(\mathbf{R}_i) \right\rangle = 0, \quad \text{for } n \text{ odd}$$

$$= \sum_{\text{perm}} \langle \mathscr{E}(\mathbf{R}_1) \mathscr{E}(\mathbf{R}_2) \rangle \langle \mathscr{E}(\mathbf{R}_3) \mathscr{E}(\mathbf{R}_4) \rangle \cdots$$

$$\langle \mathscr{E}(\mathbf{R}_{n-1}) \mathscr{E}(\mathbf{R}_n) \rangle, \quad \text{for } n \text{ even}, \quad (4.12)$$

which is the result (1.4). This result holds at the critical point whenever the $\mathbf{R}_i$ lie well separated along a single line.