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Variational Principle in Spin-Wave Theory: Application to the Theory of Magnetostatic Surface Waves*

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We have formulated a variational principle suitable for discussion of long-wavelength spin waves in ferromagnets. The form we discuss takes account of the fact that in the presence of magnetic dipole interactions an integral operator with a non-Hermitian kernel is encountered. We show that our form of the variational principle fully reproduces the bulk spin-wave dispersion relation and the magnetostatic surface-mode dispersion relation for a semi-infinite geometry, with magnetization parallel to the surface. We apply the variational principle to a discussion of the effect of exchange on the Damon-Eshbach magnetostatic surface modes. We find a new surface branch that lies between the bulk manifold and the Damon-Eshbach branch. The two branches intersect at a finite wave vector k_c that depends strongly on the direction of propagation. For $k>k_c$, we find no surface-mode solutions. The properties of the new lower branch are discussed in detail.

I. INTRODUCTION

RECENTLY, a number of studies of surface spin waves and their influence on the properties of Heisenberg ferromagnets and antiferromagnets have appeared.[1–5] These investigations consider a semi-infinite array of spins and include in the theory the effect of a free surface on the spin motion. In most of the studies,[1,2,4,5] the interaction between the spins is assumed to be an exchange interaction of the Heisenberg form. Under a variety of circumstances, the presence of the free surface gives rise to surface magnons in which the spin deviation associated with the mode is localized near the surface. In these studies, the frequency of the surface mode is found to lie below the bulk spin-wave band. Since the work just cited ignores dipolar interactions between the spins, the theories are valid only for spin waves with wavelength sufficiently short that the dominant contribution to the spin-wave energy comes from the exchanged interactions. While dipolar interactions were included in Ref. 3, the spin-wave spectrum was studied only for thin films with the order of 30 atomic layers. Consequently, no useful information was obtained about the very long-wavelength surface waves, where the penetration length of the mode in the semi-infinite crystal is large compared to 30 atomic layers.

A number of years ago, Damon and Eshbach[6] studied surface spin waves in the limit of very long wavelengths, where the exchange interactions can be ignored, and the dipolar interaction predominates. These authors found that in this region, the surface wave frequency lies *above* that of the bulk modes.

The purpose of the present paper is to examine the properties of long-wavelength surface magnons in the presence of both dipolar and exchange interactions. We are motivated by the contrast between the two regimes mentioned above. Evidently, when the wavelength of the surface magnon is sufficiently short that the dominant contribution to the excitation energy comes from exchange interactions, the surface wave frequency lies below the bulk band. However, in the dipole dominated regime, the surface mode lies above. The behavior of the surface mode in the transition region, where both exchange and dipolar interactions are of comparable strength, should thus be interesting.

We proceed by writing the equations of motion for the spin density in a semi-infinite slab in the presence of dipolar interactions. We confine our attention to the long-wavelength regime by replacing dipolar lattice sums by integration. A variational principle is then formulated. Since, as we have previously pointed out,[3] one encounters a non-Hermitian dynamical matrix in the theory of spin waves in the presence of dipolar interactions; it is necessary to formulate the variational principle in a form suitable for this case. This involves the use of both the left and right eigenvectors of the appropriate dynamical matrix. After demonstrating that the variational principle reproduces the Damon-Eshbach result for the surface-magnon dispersion relation in the absence of exchange, as well as the standard dispersion relation for bulk waves, we proceed to include the effect of exchange by introducing the appropriate terms in the equations of motion and modifying the wave functions in a manner described below.

We then present detailed studies of the surface-magnon dispersion relations in the presence of exchange by utilizing the variational principle. Since one encounters a sixth-order polynomial in the theory, it is unfortunately necessary to resort to numerical methods of solution. We find two distinct surface-magnon branches. The frequency of the upper branch ap-

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proaches the Damon-Eshbach limit as the surface-magnon wavelength goes to infinity. The frequency of the second branch approaches the top of the bulk band [the frequency $\omega \to \gamma(HB)^{1/2}$] in this limit. As the wave vector increases in magnitude for a fixed direction of propagation, the two branches approach each other and intersect at some critical wave vector k_c that depends strongly on the propagation direction. We find no surface modes above the bulk band for wave vector $k > k_c$. This result suggests that the Damon-Eshbach modes, and the second branch just described, are confined to a finite region of the two-dimensional phase space appropriate to the surface-mode problem. The surface branch does not pass through the bulk band to emerge below the band to be identified with the modes discussed in Refs. 1–5 in the exchange dominated regime.

In their original work, Damon and Eshbach[6] found that the surface magnons existed only for a limited set of propagation directions. Suppose the magnetization $\mathbf{M}$ is parallel to the z axis, which lies in the surface, and β is an angle that measures the direction of propagation with respect to an axis (the x axis) perpendicular to $\mathbf{M}$, also in the surface. Damon and Eshbach[6] found no surface solutions with the direction of propagation such that $\cos\beta < (H/B)^{1/2}$. We find that as β increases, and $\cos\beta \to (H/B)^{1/2}$ from above, the critical wave vector $k_c \to 0$. Thus, as β increases toward Damon-Eshbach's critical value, the cutoff wavelength $\lambda_c = (2\pi/k_c)$ becomes longer; for β very close to $\beta_c = \cos^{-1}[(H/B)^{1/2}]$, only very long-wavelength surface modes exist. Crudely speaking, as β increases, the amount phase space in which the surface-mode solutions occur decreases continuously, to vanish when $\beta = \beta_c$.

In Sec. II, we cast the equations of motion into a form convenient for our purposes, and we formulate the variational principle for the case where only dipolar interactions are present. In Sec. III, we illustrate the use of the variational principle by recovering some standard results from it. In Sec. IV, we include exchange and apply the theory to a discussion of the effect of exchange on the surface-mode frequencies.

The form of the variational principle employed in the present work should be useful for a wide variety of problems, such as the influence of sample geometry on the spin-wave spectrum for sample shapes not easily treated in an exact analytic manner when both dipolar and exchange interactions are present.

II. EQUATIONS OF MOTION: FORMULATION OF VARIATIONAL PRINCIPLE

We consider a semi-infinite array of spins placed in an external magnetic field H and coupled together with magnetic dipole interactions. We consider a crystal in which the site symmetry is cubic. The surface will be chosen parallel to the x-z plane, with the magnetization and the external field H both parallel to the z axis. The Hamiltonian thus has the form

$$\mathcal{H} = -g\mu_B H \sum_{\mathbf{l}} S^z(\mathbf{l}) + \tfrac{1}{2}g^2\mu_B^2 \sum_{\mathbf{lm}}{}' \frac{1}{R_{lm}^5} \times \{R_{lm}^2 \mathbf{S}(\mathbf{l})\cdot\mathbf{S}(\mathbf{m}) - 3[\mathbf{R}_{lm}\cdot\mathbf{S}(\mathbf{l})][\mathbf{R}_{lm}\cdot\mathbf{S}(\mathbf{m})]\}. \quad (2.1)$$

In this expression $\mathbf{R}_{lm} = \mathbf{R}(\mathbf{l}) - \mathbf{R}(\mathbf{m})$, and the prime on the second summation indicates the term with $\mathbf{l} = \mathbf{m}$ is excluded.

We next consider the equation of motion of the operator $S^{(+)}(\mathbf{n}) = S^{(x)}(\mathbf{n}) + iS^{(y)}(\mathbf{n})$. With $\hbar = 1$, one has

$$i\dot{S}^{(+)}(\mathbf{n}) = [S^{(+)}(\mathbf{n}), \mathcal{H}].$$

The equation of motion will be linearized by replacing $S^z(\mathbf{n})$ by $+S$ everywhere and ignoring the small terms of the form $S^x(\mathbf{n})S^y(\mathbf{n})$, which are quadratic in the deviation of the spins from the z direction. If we define

$$D_{ij}(\mathbf{R}_{lm}) = (S/R_{lm}^5)[3(\hat{x}_i\cdot\mathbf{R}_{lm})(\hat{x}_j\cdot\mathbf{R}_{lm}) - R_{lm}^2\delta_{ij}] \equiv -S(\hat{x}_i\cdot\boldsymbol{\nabla}_m)(\hat{x}_j\cdot\mathbf{R}_{lm}/R_{lm}^3), \quad (2.2)$$

then one finds

$$i\dot{S}^{(+)}(\mathbf{n}) = g\mu_B[H + \sum_m{}' D_{zz}(\mathbf{R}_{nm})]S^{(+)}(\mathbf{n}) + \tfrac{1}{2}g\mu_B \sum_{\mathbf{m}}{}' D_{zz}(\mathbf{R}_{nm})S^{(+)}(\mathbf{m}) + \tfrac{1}{2}g\mu_B \sum_m{}' \times [D_{yy}(\mathbf{R}_{nm}) - D_{xx}(\mathbf{R}_{nm}) - 2iD_{xy}(\mathbf{R}_{nm})]S^{(-)}(\mathbf{m}). \quad (2.3)$$

The equation for $S^{(-)}(\mathbf{n})$ is easily derived by taking the Hermitian conjugate of Eq. (2.3).

We next treat the dipole sums in an approximation valid in the long-wavelength limit, where the spin amplitude $S^{(+)}(\mathbf{m})$ varies slowly from site to site. We do this by replacing sums over the discrete lattice by integrations. We integrate over the volume of the sample, excluding a small sphere centered about the site $\mathbf{l}$ that is omitted from the sums.

Let us consider one of the sums as an example of this procedure. Let

$$\mathcal{S}_{zz} = \tfrac{1}{2}\sum_{\mathbf{m}}{}' D_{zz}(\mathbf{R}_{lm})S^{(+)}(\mathbf{m}).$$

If one regards $S^{(+)}$ as a continuous function of position and empolys the form of $D_{zz}(\mathbf{R}_{lm})$ given in Eq. (2.2), then

$$\mathcal{S}_{zz} = -\frac{nS}{2}\hat{z}\cdot\int d^3\mathbf{r}' S^{(+)}(\mathbf{r}')\boldsymbol{\nabla}'\frac{\hat{z}\cdot(\mathbf{R}_l - \mathbf{r}')}{|\mathbf{R}_l - \mathbf{r}'|^3},$$

where n is the number of lattice sites/unit volume, and the integration is over the volume of the sample, excluding a small sphere of radius ϵ centered at $\mathbf{R}_l$. Performing an integration by parts yields

$$S_{zz}=-\frac{nS}{2}\hat{z}\cdot\int d\mathbf{A}'\,S^{(+)}(\mathbf{r}')\frac{\hat{z}\cdot(\mathbf{r}'-\mathbf{R}_l)}{|\mathbf{R}_l-\mathbf{r}'|^3}+\frac{nS}{2}\int d^3\mathbf{r}'\frac{\hat{z}\cdot(\mathbf{r}'-\mathbf{R}_l)}{|\mathbf{R}_l-\mathbf{r}'|^3}(\hat{z}\cdot\nabla')S^{(+)}(\mathbf{r}').$$

The area integration is over the surface of the sample, as well as the surface of the small sphere surrounding the point $\mathbf{R}_l$. Since the area element $d\mathbf{A}'$ for the surface of the film is normal to $\hat{z}$, no contribution from the integration over the sample surface is obtained for this particular sum. From the small sphere, one receives the contribution $-(\frac{1}{3}4\pi)S^{(+)}(\mathbf{R}_l)$ to the surface integral. Thus, one has

$$S_{zz}=-\frac{2\pi}{3}nSS^{(+)}(\mathbf{R}_l)-\frac{nS}{\hat{z}}\int d^3\mathbf{r}'(\hat{z}\cdot\nabla')\frac{1}{|\mathbf{R}_l-\mathbf{r}'|}\frac{\partial}{\partial z'}S^{(+)}(\mathbf{r}').$$

The sums

$$\sum_m{}' D_{xy}(\mathbf{R}_{lm})S^{(-)}(\mathbf{m}),\quad \sum_m{}' D_{xx}(\mathbf{R}_{lm})S^{(-)}(\mathbf{m})$$

may be evaluated in a similar fashion. One finds no contribution from the integral over the sample surface. To evaluate $\sum_m{}'D_{yy}(\mathbf{R}_{lm})S^{(-)}(\mathbf{m})$, one notes that

$$\sum_{\mathbf{m}} D_{yy}(\mathbf{R}_{lm})S^{(-)}(\mathbf{m})=-\sum_{\mathbf{m}}[D_{xx}(\mathbf{R}_{lm})+D_{zz}(\mathbf{R}_{lm})]S^{(-)}(\mathbf{m}).$$

Combining these results, we find the equation of motion assumes the form

$$i\dot{S}^{(+)}(\mathbf{r})=g\mu_B(H+2\pi M_s)S^{(+)}(\mathbf{r})-2\pi g\mu_B M_s S^{(-)}(\mathbf{r})-\tfrac{1}{2}g\mu_B M_s\int d^3\mathbf{r}'d_{zz}(\mathbf{r}-\mathbf{r}')S^{(+)}(\mathbf{r}')+g\mu_B M_s\int d^3\mathbf{r}'\times[d_{xx}(\mathbf{r}-\mathbf{r}')+\tfrac{1}{2}d_{zz}(\mathbf{r}-\mathbf{r}')-id_{yx}(\mathbf{r}-\mathbf{r}')]S^{(-)}(\mathbf{r}'),\quad(2.4)$$

where

$$d_{ij}(\mathbf{r}-\mathbf{r}')=[(\hat{x}_i\cdot\nabla')(|\mathbf{r}-\mathbf{r}'|^{-1})](\hat{x}_j\cdot\nabla'),$$

and the saturation magnetization $M_s=ng\mu_BS$.

We desire to employ Eq. (2.4) to study spin waves in a medium with the semi-infinite geometry described earlier in this section. For this configuration, the presence of the surface does not destroy translational invariance with respect to the two directions ($\hat{x}$ and $\hat{z}$) parallel to the surface. The solutions thus will have the Bloch form

$$S^{(\pm)}(\mathbf{r})=e^{i\mathbf{k}_{\parallel}\cdot\mathbf{r}_{\parallel}}\mathcal{S}^{(\pm)}(y),\quad(2.5)$$

where $\mathbf{k}_{\parallel}=k_x\hat{x}+k_z\hat{z}$ is a two-dimensional wave vector parallel to the surface, and $\mathbf{r}_{\parallel}=x\hat{x}+z\hat{z}$.

Upon substituting the form in Eq. (2.5) into (2.4) and the corresponding equation of motion for $S^{(-)}$, one may reduce the integral equations with integrations over the single coordinate y perpendicular to the surface In carrying out this procedure, one encounters integrals of the form

$$\mathcal{G}_\alpha=\int d^2\mathbf{r}_{\parallel}'\,e^{i\mathbf{k}_{\parallel}\cdot(\mathbf{r}_{\parallel}'-\mathbf{r}_{\parallel})}\frac{\partial}{\partial r_\alpha'}\frac{1}{|\mathbf{r}'-\mathbf{r}|},\quad(2.6)$$

where $\mathbf{r}=\mathbf{r}_{\parallel}+y\hat{y}$. We consider the particular case with the subscript $\alpha=x$ as an example.

One may evaluate $\mathcal{G}_x$ by first employing the Fourier representation

$$\frac{1}{|\mathbf{r}'-\mathbf{r}|}=\frac{4\pi}{(2\pi)^3}\int d^3\mathbf{q}\frac{\exp[-i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})]}{q^2}.$$

Inserting this form into Eq. (2.6) allows the $r_{\parallel}'$ integration to be performed at once, leaving

$$\mathcal{G}_x=-2ik_x\int_{-\infty}^{+\infty}\frac{dq_y}{q_y^2+k_{\parallel}^2}e^{-iq_y(y'-y)},=-2\pi i(k_x/k_{\parallel})e^{-k_{\parallel}|y'-y|}.$$

Similarly, one easily shows that

$$\mathcal{G}_z=-2\pi i(k_z/k_{\parallel})e^{-k_{\parallel}|y'-y|}.$$

To evaluate $\mathcal{G}_y$, a slightly different procedure will be empolyed. One has

$$\mathcal{G}_y=-(y-y')\int d^2r_{\parallel}'\frac{e^{i\mathbf{k}_{\parallel}\cdot(\mathbf{r}_{\parallel}'-\mathbf{r}_{\parallel})}}{|\mathbf{r}'-\mathbf{r}|^3},=-(y-y')\int_0^\infty\int_0^{2\pi}\frac{d\rho\,\rho d\theta}{[\rho^2+(y-y')^2]^{3/2}}e^{ik_{\parallel}\rho\cos\theta},=-2\pi(y-y')\int_0^\infty\frac{d\rho\,\rho J_0(k_{\parallel}\rho)}{[\rho^2+(y-y')^2]^{3/2}}=2\pi\,\mathrm{sgn}(y-y')\,e^{-k_{\parallel}|y-y'|}.$$

We define

$$\gamma_s(y-y')=e^{-k_{\parallel}|y-y'|}\quad(2.7a)$$

and

$$\gamma_a(y-y')=\mathrm{sgn}(y-y')e^{-k_{\parallel}|y-y'|}.\quad(2.7b)$$

Then, upon inserting the forms in Eqs. (2.5) and (2.7) into the equation of motion for $S^{(+)}$, we obtain

$$i\dot{\mathcal{S}}^{(+)}(y)=g\mu_B(H+2\pi M_s)\mathcal{S}^{(+)}(y)-2\pi g\mu_B M_s\mathcal{S}^{(-)}(y)-\pi g\mu_B M_s\frac{k_z^2}{k_{\parallel}}\int_0^\infty dy'\gamma_s(y-y')\mathcal{S}^{(+)}(y')+2\pi g\mu_B M_s k_x\int_0^\infty\gamma_a(y-y')\mathcal{S}^{(-)}(y')dy'+\pi g\mu_B M_s(2k_x^2+k_z^2)k_{\parallel}^{-1}\int_0^\infty dy'\gamma_s(y-y')\mathcal{S}^{(-)}(y').$$

It will be convenient for us to construct the equations of motion for the x and y components of spin deviation $\mathcal{S}^{(x)}(y)$ and $\mathcal{S}^{(y)}(y)$. We first assume solutions that vary with time like $e^{-i\Omega t}$, then we form an equation for $\mathcal{S}^{x}(y)$ and $\mathcal{S}^{y}(y)$ by combining the equations of motion of $\mathcal{S}^{(+)}$ and $\mathcal{S}^{(-)}$.

Let

$$s^x(y)\equiv \mathcal{S}^{(y)}(y)\,,$$
$$s^y(y)\equiv i\mathcal{S}^{(y)}(y)\,.$$

Also let $\cos\beta=(k_x/k_{\parallel})$, $\omega_H=g\mu_B H$, $\omega_M=4\pi g\mu_B M_s$, and $\omega_B=\omega_H+\omega_M$. Then we find that $s^x(y)$ and $s^y(y)$ satisfy equations of the form

$$\Omega s^x(y)=-\omega_B s^y(y)+\tfrac{1}{2}\omega_M k_{\parallel}\left[\int_0^\infty dy'\gamma_s(yy')s^y(y') -\cos\beta\int_0^\infty dy'\gamma_a(yy')s^x(y')\right], \quad (2.8a)$$

$$\Omega s^y(y)=-\omega_H s^x(y)-\tfrac{1}{2}\omega_M k_{\parallel}\cos\beta\left[\cos\beta\int_0^\infty dy' \times\gamma_s(yy')s^x(y')-\int_0^\infty dy'\gamma_a(yy')s^y(y')\right]. \quad (2.8b)$$

Equations (2.8) provide a convenient form for the equations of motion, since the coefficient of each term is a real number. We may write Eqs. (2.8) in the form

$$\Omega s^i(y)=\int_0^\infty dy'\sum_j \Gamma_{ij}(y,y')s^j(y')\,, \quad (2.9)$$

where i and j refer to the Cartesian subscripts x and y, and

$$\Gamma_{xx}(yy')=-\tfrac{1}{2}\omega_M k_{\parallel}\cos\beta\;\gamma_a(y,y')=-\Gamma_{yy}(yy')\,, \quad (2.10a)$$

$$\Gamma_{xy}(yy')=-\omega_B\delta(y-y')+\tfrac{1}{2}\omega_M k_{\parallel}\gamma_s(y,y')\,, \quad (2.10b)$$

$$\Gamma_{yx}(yy')=-\omega_H\delta(y-y')-\tfrac{1}{2}\omega_M k_{\parallel}\cos^2\beta\;\gamma_s(y,y')\,. \quad (2.10c)$$

Fuchs and Kliewer[7] have derived a set of equations similar to Eqs. (2.8) in their study of surface optical phonons in ionic crystals. The structure of Eqs. (2.8) differs markedly from the equations encountered by Fuchs and Kliewer. The principle difference is that in our problem, the kernal $\Gamma_{ij}(y,y')$ is non-Hermitian in the presence of dipolar interactions (i.e., $\omega_M\neq 0$). One notes from Eqs. (2.10) that $\Gamma_{ij}(y,y')\neq\Gamma_{ji}^*(y'y)$ unless $\omega_M=0$. In the theory of lattice dynamics, one encounters a Hermitian kernel in the integral equations. We have previously pointed out[3] that one encounters non-Hermitian operators to diagonalize in spin-wave theory in the presence of dipolar coupling between the spins. This is a consequence of the fact that if the dipolar interactions between the spins are sufficiently strong, the ferromagnetic ground state may be unstable[8]; the instability manifests itself by the appearance of complex spin-wave frequencies.

One may obtain the Damon-Eshbach surface spin-wave dispersion relation by explicit examination of the solutions of Eq. (2.9). One assumes solutions with s_x, s_y proportional to e^{-qy} and eliminates q by requiring this functional form satisfy Eq. (2.9) identically. However, we shall find it more convenient, when exchange is introduced, to work with the variational principle constructed from Eq. (2.9). Thus, we proceed to formulate the variational principle, and then demonstrate that the results of Damon and Eshbach follow, when exchange interactions are neglected.

First, consider the case of a set of N coupled linear integral equations with a Hermitian kernel

$$\lambda f^i(y)=\sum_{j=1}^N\int dy' K_{ij}(yy')f_j(y')\,, \quad (2.11)$$

where λ is the eigenvalue, and $K_{ij}(yy')=K_{ji}^*(y'y)$.

One constructs a functional $\tilde{\lambda}$ of the form

$$\tilde{\lambda}=\int dy'dy\sum_{i,j=1}^N g_i^*(y)K_{ij}(y,y')g_j(y')\Big/\sum_{i=1}^N\int dy\, g_i^*(y)g_i(y)\,. \quad (2.12)$$

If the vector $\mathbf{g}(y)$ is chosen equal to the eigenvector $\mathbf{f}(y)$, then $\tilde{\lambda}$ is the eigenvalue λ of the integral equation. If, however, the vector $\mathbf{g}(y)$ differs from the true eigenvector $\mathbf{f}(y)$ by an amount $\delta\mathbf{g}(y)$, one finds from standard discussions[9] that one obtains an estimate of the eigenvalue λ, with an error of order $(\delta g)^2$. If one seeks the lowest eigenvalue, then insertion of a guess for the eigenvector $\mathbf{f}(y)$ into Eq. (2.12) produces an upper bound to the true eigenvalue.

Now consider generalization of the variational principle to the case where the kernel is non-Hermitian, as in Eq. (2.9). Let us introduce the left eigenvector $s_L^i(y)$ associated with the eigenvalue Ω that satisfies

$$s_L^i(y)\Omega=\int_0^\infty dy'\sum_j s_L^j(y')\Gamma_{ji}(y',y)\,. \quad (2.13)$$

The left eigenvector in Eq. (2.13) is distinct from the right eigenvector that appears in Eq. (2.9), if the kernel Γ is not Hermitian. To avoid confusion, we denote the right eigenvector that appears in Eq. (2.9) by $s_R^i(y)$ in the subsequent discussion. We shall see in a moment that a simple relationship between $s_L^i(y)$ and $s_R^i(y)$ exists for the present problem.

[7] R. Fuchs and K. Kliewer, Phys. Rev. **140**, 2076 (1965).

[8] M. H. Cohen and F. Keffer, Phys. Rev. **99**, 1135 (1955).

[9] See, for example, J. Mathews and R. Walker, in *Mathematical Methods of Physics* (W. A. Benjamin, Inc., New York, 1965), p. 317.

We then form the quantity

$$\tilde{\Omega}=\int_0^\infty dy\,dy'\sum_{ij} s_L{}^i(y)^*\Gamma_{ij}(yy')s_R{}^j(y')\Big/\int_0^\infty dy\sum_i s_L{}^i(y)^* s_R{}^i(y)\,. \quad (2.14)$$

If one inserts the proper left and right eigenvectors into Eq. (2.14), then it is easily seen that $\tilde{\Omega}$ equals the eigenvalue Ω. Furthermore, if one inserts approximate forms for $\mathbf{s}_L$ and $\mathbf{s}_R$ in Eq. (2.14), then one obtains an estimate for the eigenvalue that contains an error proportional to $(\delta\mathbf{s}_{L,R})^2$. The functional exhibited in Eq. (2.14) thus provides a generalization of the variational principle suitable for the present problem.

Let us next consider the form of the equation satisfied by the left eigenvector. Explicitly, upon rearranging Eq. (2.13) slightly, one has

$$\Omega s_L{}^x(y)=\int_0^\infty dy'\Gamma_{xx}(y'y)s_L{}^x(y')+\int_0^\infty dy'\Gamma_{yx}(y'y)s_L{}^y(y')$$

and

$$\Omega s_L{}^y(y)=\int_0^\infty dy'\Gamma_{xy}(y'y)s_L{}^x(y')+\int_0^\infty dy'\Gamma_{yy}(y'y)s_L{}^y(y')\,.$$

Upon employing the properties of $\Gamma_{ij}(yy')$ under interchange of (iy), (jy'), one finds

$$\Omega s_L{}^y(y)=\int_0^\infty dy'\Gamma_{xy}(yy')s_L{}^x(y')+\int_0^\infty dy'\Gamma_{xx}(yy')s_L{}^y(y')\,,$$

$$\Omega s_L{}^x(y)=\int_0^\infty dy'\Gamma_{yy}(yy')s_L{}^x(y')+\int_0^\infty dy'\Gamma_{yx}(yy')s_L{}^y(y')\,.$$

After comparing this result with the equation satisfied by the right eigenvector, the following simple relationship is obtained

$$s_L{}^x(y)\equiv s_R{}^y(y)\,, \quad (2.15a)$$

$$s_L{}^y(y)\equiv s_R{}^x(y)\,. \quad (2.15b)$$

If one is given a right eigenvector, a left eigenvector is obtained by merely interchanging s^x and s^y in the column vector.

At this point, we should mention that Sparks[10] has also employed a variational principle to study spin waves in materials of finite spatial extent. However, Sparks has not taken account of the non-Hermitian character of Γ_{ij} in formulating the variational principle. The form given in Eq. (2.12) was utilized in this work. In the general case, one must employ the proper form of the variational principle to obtain meaningful results from the theory. However, we note that Sparks has applied his form of the theory only to a discussion of the limit where the magnetic field is large and the effect of the dipolar interactions small, i.e., $\omega_H\gg\omega_M$. In this limit, if one examines the form of the eigenfunctions, one sees that to a good approximation $s^x\approx s^y$. In this case, the distinction between left and right eigenvectors is unimportant, and the form given in Eq. (2.12) provides a suitable basis for the discussion.

More precisely, we shall see in Sec. III that for the case of bulk excitations, when $\omega_H\gg\omega_M$, Sparks's form of the variational principle gives the spin-wave frequency correctly only to first order in ω_M. For the surface waves, which in the Damon-Eshbach regime owe their existence in a fundamental way to the presence of dipolar interactions between the spins, one must employ the full form of the variational principle to obtain meaningful results.

III. SOME APPLICATIONS OF VARIATIONAL PRINCIPLE

In Sec. II, we have seen that one may compute the spin-wave excitation spectrum of the dipolar spin array by considering extremal values of a functional $\tilde{\Omega}$ that one may write in the form

$$\tilde{\Omega}=\int_0^\infty dy\,dy'\Gamma_{xx}(y,y')\times[s_R{}^{y*}(y)s_R{}^x(y')-s_R{}^x(y)^*s_R{}^y(y')]+s_R{}^x(y)^*\Gamma_{yx}(y,y')s_R{}^x(y')+s_R{}^y(y)^*\Gamma_{xy}(yy')s_R{}^y(y')\times\left\{2\mathrm{Re}\left[\int_0^\infty dy\,s_R{}^{y*}(y)s_R{}^x(y)\right]\right\}^{-1}. \quad (3.1)$$

In this section, we shall apply the variational principle to two standard situations in order to illustrate its use and to demonstrate the importance of employing the complete form of the variational principle.

A. Bulk Spin-Wave Excitations

Let us consider a macroscopic disk of ferromagnetic material that is infinite in two directions (the x and z directions) and which has thickness L in the y direction, where L is some macroscopic length. As in Sec. II, the magnetization is taken parallel to the z axis. One may study spin waves in this structure by using the variational principle in the form given by Eq. (3.1), except the upper limits of the y integration are taken to be L instead of infinity.

[10] M. Sparks, B. Tittmann, J. Mee, and C. Newkirk, Fourteenth Annual Conference on Magnetism and Magnetic Materials, New York, 1968 (unpublished), paper HA-1.

Let us consider the following form for the trial function

$$s_R{}^x(y)=e^{ik_yy}, \quad s_R{}^y(y)=\eta e^{ik_yy},$$

where η is a complex number. If periodic boundary conditions are applied in the y direction, then

$$e^{ik_yL}=1.$$

One finds that

$$\int_0^L dy\,dy' e^{-ik_y(y-y')}\gamma_s(yy')=(2k_{11}/k^2)L$$

and

$$\int_0^L dy\,dy' e^{-ik_y(y-y')}\gamma_a(yy')=-2i(k_y/k^2)L.$$

If we recall the definition of $\Gamma_{ij}(y-y')$ and note that

$$k^2=k_x{}^2+k_y{}^2+k_z{}^2, \quad k_x=k_{11}\cos\beta, \quad k_z=k_{11}\sin\beta,$$

and write

$$\eta=\epsilon e^{i\theta},$$

then we obtain

$$\tilde{\Omega}=\frac{1}{2\epsilon\cos\theta}\left[\omega_H+\epsilon^2\omega_B+\omega_M\left(\frac{k_x{}^2}{k^2}-\frac{k_{11}{}^2}{k^2}\epsilon^2\right)-2\epsilon\omega_M\frac{k_xk_y}{k^2}\sin\theta\right].$$

One seeks the values of the variational parameters ϵ and θ which make $\tilde{\Omega}$ an extremum.

Consider first the dependence on θ. If we define

$$A=(2\epsilon)^{-1}[\omega_H+\epsilon^2\omega_B+\omega_M(k_x{}^2/k^2-k_{11}{}^2\epsilon^2/k^2)],$$
$$B=+\omega_M(k_xk_y/k^2),$$

then

$$\tilde{\Omega}=\cos^{-1}\theta(A-B\sin\theta).$$

We set

$$\partial\tilde{\Omega}/\partial\theta=0=-B+(\sin\theta/\cos^2\theta)(A-B\sin\theta).$$

Thus,

$$\sin\theta=B/A, \quad \text{and} \quad \tilde{\Omega}=\sqrt{(A^2-B^2)}.$$

We next set $\partial\tilde{\Omega}/\partial\epsilon=0$ to determine ϵ. Note that only A depends on ϵ. Thus, $\tilde{\Omega}$ is stationary when A is stationary. Hence, ϵ is found from

$$0=\frac{\partial A}{\partial\epsilon}=\frac{1}{2}\left(\omega_B-\omega_M\frac{k_{11}{}^2}{k^2}\right)-\frac{1}{2\epsilon^2}\left(\omega_H+\omega_M\frac{k_x{}^2}{k^2}\right).$$

Thus,

$$\epsilon^2=\frac{\omega_H+\omega_M(k_x{}^2/k^2)}{\omega_B-\omega_M(k_{11}{}^2/k^2)}\equiv\frac{\omega_H+\omega_M(k_x{}^2/k^2)}{\omega_H+\omega_M(k_y{}^2/k^2)}.$$

This gives

$$A=[\omega_H+\omega_M(k_x{}^2/k^2)][\omega_H+\omega_M(k_y{}^2/k^2)]^{1/2},$$

and finally we obtain the well-known dispersion relation

$$\tilde{\Omega}=\{\omega_H[\omega_H+\omega_M(1-k_z{}^2/k^2)]\}^{-1/2}. \qquad (3.2)$$

The present form of the variational principle, thus, properly reproduces the bulk spin-wave dispersion relation. It is interesting to compare the result of Eq. (3.2) with the prediction of the variational principle in the form employed by Sparks,[10] which ignores the non-Hermitian character of the kernel Γ. Sparks's form is obtained by replacing the left eigenvector in Eq. (2.14) by the right eigenvector. One finds an expression for $\tilde{\Omega}$ which agrees with the exact result of Eq. (3.2) only to first order in ω_M, when $\omega_H\gg\omega_M$. As we mentioned above, this is the only limit considered by Sparks, who examined the effect of the finite size of a ferromagnetic disc on the nature of the spin-wave spectrum. We can easily see why Sparks's form produces the correct result in the limit $\omega_H\gg\omega_M$. In this limit, one sees $\epsilon^2\cong1$, so the left and right eigenvectors are equal.

B. Damon-Eshbach Surface Modes

We next return to the semi-infinite geometry and consider solutions of the form

$$s_R{}^x(y)=e^{-qy}, \quad s_R{}^y(y)=\eta e^{-qy},$$

with q real and positive. We consider both η and q to be variational parameters. One easily sees that

$$\int_0^\infty dy\,dy' e^{-q(y+y')}\gamma_a(y,y')\equiv0.$$

Then again with $\eta=\epsilon e^{i\theta}$, we obtain

$$\tilde{\Omega}=(-1/2\epsilon\cos\theta)[\omega_H+\epsilon^2\omega_B+\alpha\omega_M(\cos^2\beta-\epsilon^2)],$$

with

$$\alpha=k_{11}/(k_{11}+q).$$

Since $q\geq0$, one has $0\leq\alpha\leq1$. The first condition $d\tilde{\Omega}/d\theta=0$ allows us to choose $\theta=\pi$ or $\cos\theta=-1$. Then, $\tilde{\Omega}=(1/2\epsilon)[\omega_H+\epsilon^2\omega_B+\alpha(\cos^2\beta-\epsilon)]$. Then, taking $d\tilde{\Omega}/d\epsilon=0$, one has

$$\epsilon^2=(\omega_H+\alpha\omega_M\cos^2\beta)/(\omega_B-\alpha\omega_M). \qquad (3.3)$$

The condition

$$\partial\tilde{\Omega}/\partial\alpha=0$$

gives

$$\alpha=\tfrac{1}{2}(\omega_B\cos^2\beta-\omega_H)/\omega_M\cos^2\beta \qquad (3.4)$$

and

$$\tilde{\Omega}=\tfrac{1}{2}[(\omega_H/\cos\beta)+\omega_B\cos\beta] \qquad (3.5)$$

Equation (3.5) is the dispersion relation obtained by Damon and Eshbach for magnetostatic surface modes in the semi-infinite geometry. The parameter α must be >0, since q is real and positive. Thus, one obtains solutions only for directions of propagation such that

$$\cos\beta>\cos\beta_c=(\omega_H/\omega_B)^{1/2}.$$

Note, that, except for the special case $\beta=0$ (propagation perpendicular to the magnetization), one must employ the proper form of the variational principle which takes full account of the non-Hermitian character of the kernel Γ. This point is made clear by inserting the form of α into the expression for ϵ^2. One finds

$$\epsilon=|s_R{}^y/s_R{}^x|=\cos\beta\,.$$

Thus, for $\beta\neq 0$, ϵ differs significantly from unity even for the case $\omega_H\gg\omega_M$. The existence of the magnetostatic surface modes is intimately tied up with the presence of dipolar coupling between the spins, even in the high-field limit. We now proceed to examine the effect of exchange on the surface-mode frequency.

IV. ESTIMATE OF EFFECT OF EXCHANGE ON MAGNETOSTATIC SURFACE MODES

In this section, we apply the form of the variational principle derived in Sec. III to a study of the effect of exchange on the Damon-Eshbach surface modes.

First, consider the effect of including exchange on the equations of motion. It is well known that in the long-wavelength limit, one may include the effect of exchange by replacing the magnetic field ω_H by the quantity $\omega_H-D\nabla^2$, where

$$\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2\,.$$

One may, thus, include the effect of exchange in the variational principle by replacing ω_H everywhere by $\omega_H+D[k_{\parallel}{}^2-(\partial^2/\partial y^2)]$, since we always consider solutions to the equations of motion which have the Bloch form in the x and z directions.

In addition to modifying the equations of motion for $y\neq 0$, the fact that spins in the surface region are coupled to fewer neighboring spins than those in the bulk impose conditions on the form of the solution. If one begins with spins on a discrete lattice with short-range exchange interactions between the spins, and if the long-wavelength form of the equations of motion are examined, then in the absence of surface pinning fields one deduces that the boundary condition

$$\left.\frac{\partial s^x}{\partial y}\right|_{y=0}=\left.\frac{\partial s^y}{\partial y}\right|_{y=0}=0 \tag{4.1}$$

must be imposed on the solution. This form of the boundary condition has been derived by Kittel.[11] We note that in recent years, it has been demonstrated that the surface pinning fields in carefully prepared films are very small.[12] Thus, we ignore the effect of surface pinning fields in the present discussion.

Formally, the boundary condition in Eq. (4.1) may be derived by including exchange interactions in the discussion of the equations of motion for the discrete semi-infinite lattice of spins considered in Sec. II. Upon taking the long-wavelength limit, one encounters terms proportional to $\delta(y)a(\partial s_x/\partial y)$ and $\delta(y)a(\partial s_y/\partial y)$ in the equations of motion, where a is the lattice constant. The presence of these terms impose the boundary condition of Eq. (4.1) on the solution.

Actually, the boundary conditions in Eq. (4.1) are approximations to the exact boundary conditions. If, for example, one retains the next higher terms in the derivatives of s_x and s_y with respect to the spatial coordinates, then for a simple cubic lattice of spins with nearest-neighbor exchange interaction J, next-nearest-neighbor interaction J_2, and a free (100) surface, one finds that the boundary condition becomes

$$\frac{\partial s_{x,y}}{\partial y}-\frac{a}{(J_1+4J_2)}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)s_{x,y}=0\,.$$

When this boundary condition is combined with the equation of motion for $y\neq 0$, then in the absence of dipolar coupling between the spins, one may derive the long-wavelength form of the surface-magnon dispersion relation given in Ref. 1. We will only be concerned with a study of very long-wavelength surface magnons in this work. Thus, we will approximate the exact boundary condition by the long-wavelength form given in Eq. (4.1).

The boundary condition can be incorporated into the theory by choosing the appropriate variational function. For $y\neq 0$, the equation of motion has the form given in Eq. (2.9), with ω_H replaced by

$$\omega_H+D(k_{\parallel}{}^2-\partial^2/\partial y^2)\,.$$

It is easy to see that the equations still admit solutions of the exponential form e^{-qy}. In an exact treatment, the boundary condition insists that near the surface the solution be modified so that s_x and s_y have vanishing slope at $y=0$.

We incorporate this condition into the theory by choosing a variation function of the form

$$s_R{}^x(y)=f(y)\,,\qquad s_R{}^y(y)=\eta f(y)\,, \tag{4.2}$$

with

$$f(y)=1-\beta y^2\,,\quad 0<y<t$$
$$=\gamma e^{-qy}\,,\quad t<y<\infty\,.$$

We choose γ and β so that $f(y)$ and $\partial f/\partial y$ are continuous at $y=t$. One finds explicitly that

$$\gamma=e^{qt}/(1+\tfrac{1}{2}qt)\,,$$
$$\beta=\tfrac{1}{2}(q/t)e^{qt}/(1+\tfrac{1}{2}qt)\,.$$

We expect the macroscopic equations of motion including the phenomenological exchange term $-D\nabla^2$ should provide a valid representation of the exact equations of motion everywhere, except within some distance from the surface the order of a few lattice constants.

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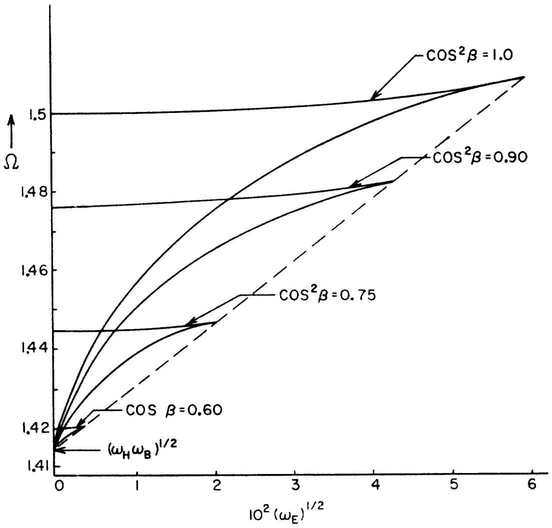

FIG. 1. Frequencies of the two surface-magnon branches, plotted as a function of $\sqrt{\omega_E}$ for various propagation directions β. We have chosen units with $\omega_H=1$ for this plot and have also chosen $\omega_H=\omega_M$.

Thus, in the long-wavelength limit, where the wavelength $\lambda_{\parallel}=(2\pi/k_{\parallel})$ is a macroscopic length, we feel that the wave function should be well represented by the simple exponential everywhere, except for a region near the surface whose spatial extent is small compared to $\lambda_{\parallel}$. Thus, we consider only the limit

$$qt\ll 1\,.$$

In this limit, one verifies by explicit computation that the integrals over $\gamma_a(y,y')$ and $\gamma_s(y,y')$ are unchanged by this small change in the wave function to lowest order in qt.

Consider next the operator $\partial^2/\partial y^2$. We need the expectation value

$$\left\langle\frac{\partial^2}{\partial y^2}\right\rangle=\int_0^\infty dy\, f(y)\frac{\partial^2}{\partial y^2}f(y)\Big/\int_0^\infty dy\, f^2(y)\,.$$

Straightforward integrations show that

$$\int_0^\infty dy\, f^2(y)=\frac{1}{2q}(1+\text{order of } qt)$$

and

$$\int_0^\infty dy\, f(y)\frac{\partial^2}{\partial y^2}f(y)=\tfrac{1}{2}q\gamma\, e^{-2qt}-2\beta t(1-\tfrac{2}{3}\beta t^2)$$

$$\approx -\tfrac{1}{2}q(1+\text{order of } qt)\,.$$

Thus,

$$\langle(\partial^2/\partial y^2)\rangle=-q^2[1+\text{order of } qt]\cong -q^2\,.$$

This shows that for a variational function of the form given in Eq. (4.2), one replaces ω_H in the discussions of Sec. III by $\omega_H+D(k_{\parallel}^2+q^2)$. When $qt\ll 1$, the form of $\tilde{\Omega}$ is independent of the precise value of the parameter t. Thus, we repeat the theory of the magnetostatic surface modes of Sec. III, treating the attenuation constant q and the parameter η as variational parameters.

Again, we let

$$\eta=\epsilon\, e^{i\theta}\,,$$

as in Sec. III. As before, the phase angle θ appears only in the denominator of the expression for $\tilde{\Omega}$ in the form of $\cos\theta$. Choosing θ so that $\tilde{\Omega}$ is an extremum leads to the choice $\theta=\pi$, as in Sec. III.[13] With this choice of θ, one then has

$$\tilde{\Omega}=(2\epsilon)^{-1}[\omega_H+\epsilon^2\omega_B+(1+\epsilon^2)\omega_E(1+\varphi^2) +\omega_M(\cos^2\beta-\epsilon^2)/(1+\varphi)]\,. \quad (4.3a)$$

We have introduced the parameters

$$\varphi=q/k_{\parallel}\,,$$

$$\omega_E=Dk_{\parallel}^2\,.$$

One determines ϵ and φ from the conditions

$$d\tilde{\Omega}/\partial\epsilon=0\,, \quad (4.3b)$$

$$\partial\tilde{\Omega}/\partial\varphi=0\,. \quad (4.3c)$$

Condition (4.3c) gives an expression for ϵ^2 in terms of φ,

$$\epsilon^2=[\omega_M\cos^2\beta-2\varphi(1+\varphi^2)\omega_E]/ \times[\omega_M+2\varphi(1+\varphi)^2\omega_E]\,. \quad (4.4)$$

Condition (4.3a) gives a second expression for ϵ^2,

$$\epsilon^2=\frac{\omega_H+\omega_M\cos^2\beta(1+\varphi)^{-1}+\omega_E(1+\varphi^2)}{\omega_B-\omega_M(1+\varphi)^{-1}+\omega_E(1+\varphi^2)}\,. \quad (4.5)$$

Upon equating Eqs. (4.4) and (4.5), we obtain an equation that determines the parameter φ. Unfortunately, we are led to a polynomial of sixth order in φ. We can write the resulting polynomial in the form

$$\sum_{n=0}^{6} a_n\varphi^n=0\,. \quad (4.6)$$

Explicitly we find

$$a_6=4\omega_E^2\,,$$

$$a_5=12\omega_E^2\,,$$

$$a_4=2\omega_E(\omega_H+\omega_B+8\omega_E)\,,$$

$$a_3=\omega_E[\omega_H(1+\cos^2\beta)+4(\omega_B+2\omega_H)+16\omega_E]\,,$$

$$a_2=\omega_E[\omega_M(1+3\cos^2\beta)+2(\omega_B+5\omega_H)+12\omega_E]\,,$$

$$a_1=\omega_M[\omega_H-\omega_B\cos^2\beta+\omega_E(1+\cos^2\beta)]+4\omega_E(\omega_E+\omega_H)\,,$$

$$a_0=\omega_M(\omega_H+\omega_E)(1-\cos^2\beta)+\omega_M^2\cos^2\beta\,.$$

[13] We could also choose $\theta=0$. This would lead to a negative value of $\tilde{\Omega}$. We have pointed out in Ref. 3 that for a given value of k_x and k_z for each positive frequency solution of the exact equations of motion, one can also derive a negative frequency solution by the appropriate symmetry operation for a slab geometry.

For given values of the parameters ω_E, ω_H, ω_M, and β, one can solve for the real positive roots of Eq. (4.6). The frequencies of the surface magnons may then be computed by employing Eqs. (4.4) and (4.3a). Unfortunately, we have not been able to find analytic expressions for the roots, except in the limit as $\omega_E \to 0$. Thus, except for this region, we have found it necessary to resort to numerical solutions of Eq. (4.6).

Some typical results of this study are illustrated in Fig. 1. For this figure, we have chosen units in which $\omega_H = 1$. We have taken $\omega_M = 1$, and the surface-magnon dispersion relations are plotted for various values of the parameter $\cos^2\beta$. We plot the frequencies as a function of $\sqrt{\omega_E}$. Since $\omega_E = Dk_{\parallel}^2$, this is equivalent to a plot of Ω versus $k_{\parallel}$, except for the scale.

One notes that for a given value of β such that $\cos\beta > (\omega_H/\omega_B)^{1/2}$ ($=1/\sqrt{2}$ for the parameters employed in Fig. 1), one has two surface spin-wave frequencies for each value of ω_E. The polynomial in Eq. (3.6) has two real roots for each ω_E. As $\omega_E \to 0$, the frequency of the upper branch approaches the Damon-Eshbach frequency

$$\Omega_{DE} = \tfrac{1}{2}(\omega_H/\cos\beta) + \tfrac{1}{2}\omega_B \cos\beta .$$

In addition to the Damon-Eshbach branch, we find a second branch with frequency below $\Omega_{\rm DE}$, but above the $\mathbf{k}=0$ upper limit $(\omega_H\omega_B)^{1/2}$ of the bulk manifold. As $\omega_E \to 0$, we find the frequency of this second branch approaches $(\omega_H\omega_B)^{1/2}$ from above, for all values of $\cos\beta$. This result will be derived analytically from the above equations in the subsequent discussion. As ω_E increases, these two surface branches intersect at some real wave vector $k_c(\beta)$, which depends strongly on the direction of propagation β. In particular, as β approaches the critical value $\beta_c = \cos^{-1}[(\omega_H/\omega_B)^{1/2}]$ beyond which Damon and Eshbach find no magnetostatic surface wave, we find that $k_c \to 0$. Thus, crudely speaking, the amount of phase space available to the magnetostatic modes decreases continuously to zero as the critical angle is approached.

Let us consider next the limit as $\omega_E \to 0$. If one examines the coefficients a_n in Eq. (4.6), one finds that all coefficients a_n with $n \geq 2$ approach zero. In the limit as $\omega_E \to 0$, one obtains the Damon-Eshbach solution from the terms that remain. Suppose that we have a very small value of ω_E. The numerical study shows that the polynomial in Eq. (4.6) has a root with φ near the Damon-Eshbach value. However, there is a second root in which $\varphi \gg 1$, when $\omega_E \ll \omega_H$. We can extract an analytic expression for this second root, when ω_E is small.

Equation (4.6) may be written in the form suitable for examination of the roots with $\varphi \gg 1$;

$$4\omega_E^2\varphi^6[1+\text{order of } (1/\varphi)] + 2\omega_E(\omega_H+\omega_B)\varphi^4[1+\text{order of } (1/\varphi)] + \varphi\omega_M(\omega_H - \omega_B\cos^2\beta)[1+\text{order of } (1/\varphi)] = 0.$$

Thus, when $\varphi \gg 1$, we may replace the full expression by the approximate form

$$\omega_E^2\varphi^5 + \tfrac{1}{2}\omega_E(\omega_H+\omega_B)\varphi^3 = \tfrac{1}{4}\omega_M(\omega_B\cos^2\beta - \omega_H) .$$

Let $x = \omega_E\varphi^3$. Then rearrangement of this equation gives

$$x = \omega_M(\omega_B\cos^2\beta - \omega_H)/2(\omega_B+\omega_H) - 2x^2/\varphi(\omega_H+\omega_B) .$$

When $\varphi \gg 1$, we can drop the second term on the right-hand side of this last form. This yields an expression for the second root of the polynomial valid when $\varphi \gg 1$

$$\varphi \cong [\omega_M(\omega_B\cos^2\beta - \omega_H)/2\omega_E(\omega_H+\omega_B)]^{1/3}, \quad \omega_E \ll 1 . \tag{4.7a}$$

Upon noting that $\omega_E = Dk_{\parallel}^2$ and $\varphi = (q/k_{\parallel})$ for the attenuation constant q for the lower branch, one finds

$$q \cong k_{\parallel}^{1/3}[\omega_M(\omega_B\cos^2\beta - \omega_H)/2D(\omega_H+\omega_B)]^{1/3}. \tag{4.7b}$$

Thus, for a given value of D, the penetration length of the lower branch into the crystal $\to$ infinity as $k_{\parallel}^{-1/3} = (\lambda_{\parallel}/2\pi)^{1/3}$ as the wavelength $\lambda_{\parallel}$ of the surface branch in the surface becomes infinite.

When $\varphi \gg 1$, the parameter ϵ^2 becomes

$$\epsilon^2 \cong (\omega_M\cos^2\beta - 2\omega_E\varphi^3)/(\omega_M + 2\omega_E\varphi^3) .$$

Upon inserting the result of Eq. (4.7a) into this expression, one finds

$$\epsilon^2 = (\omega_H/\omega_B)$$

and

$$\lim_{k_{\parallel}\to 0} \tilde{\Omega} = (\omega_H\omega_B)^{1/2} + \text{terms of order } (k_{\parallel}^{2/3}) .$$

The fact that the frequency of the mode approaches $(\omega_H\omega_B)^{1/2}$ as $k_{\parallel} \to 0$ is a consequence of Eq. (4.7b) which states that $q \alpha k_{\parallel}^{1/3}$ in this limit, we present a heuristic argument that supports this point. Consider the frequency of a bulk spin wave

$$\Omega = (\omega_H\{\omega_H + \omega_M[1 - k_z^2/(k_x^2+k_y^2+k_z^2)]\})^{1/2}.$$

The change in energy associated with a disturbance that decays like e^{-qy} is found from this result by replacing k_y by $iq = iCk_{\parallel}^{1/3}$, where C is a constant and $k_{\parallel} = \sqrt{(k_x^2+k_z^2)}$. Then, the excitation energy is

$$\Omega = (\omega_H\{\omega_H + \omega_M[1 - k_z^2/(k_{\parallel}^2 - C^2k_{\parallel}^{2/3})]\})^{1/2}.$$

As $k_{\parallel} \to 0$, this becomes

$$\Omega = \{\omega_H[\omega_B + (\omega_M/C^2)k_{\parallel}^{4/3}\sin^2\beta]\}^{1/2} \Rightarrow (\omega_H\omega_B)^{1/2} \quad \text{as } k_{\parallel} \to 0 .$$

There are a few more comments about the behavior of the lower branch that are relevant in the regime of small ω_E. If one examines the formula for $\tilde{\Omega}$ as $\omega_E \to 0$, one notes that the contribution to the excitation energy that comes *directly* from the exchange interactions (i.e., from the operator $-D\nabla^2$ averaged over the variational function) vanishes as $k_{\shortparallel} \to 0$. One notes that as $\varphi = q/k_{\shortparallel} \to \infty$ these terms contribute to $\tilde{\Omega}$ an amount proportional to $\omega_E \varphi^2 =$ order of $(k_{\shortparallel}^2 \times k_{\shortparallel}^{-4/3}) =$ order of $k_{\shortparallel}^{2/3}$. The new branch has its origin in the boundary condition that forces the eigenfunction to come in to $y=0$ with vanishing slope. It is curious that in the long-wavelength limit, the statement of the boundary condition is independent of the strength of the exchange interactions, in the absence of surface pinning fields.

It is easily seen that our theory of the lower branch is meaningful only if exchange interactions of sufficient strength are present in the system—i.e., one cannot mathematically take the limit $D \to 0$ in the present formalism. This is because we have assumed that the equations of motion derived in Sec. II are valid everywhere, except for a region near the surface whose spatial extent is small compared to $1/q$. Let us thickness of the region, where the macroscopic equations break down, be denoted by l. Then one clearly requires that $ql \ll 1$ for our theory to be valid. But if, for fixed $k_{\shortparallel}$, one formally takes the limit $D \to 0$ in Eq. (4.7b), one sees that $q \to \infty$. When the length q^{-1} becomes microscopic, it is clear that the present treatment breaks down. From the numerical estimates presented below, we will see that q^{-1} is a macroscopic length if parameters characteristic of typical magnetic materials are employed.

Now let us examine some of the numerical values of the parameters that enter the preceding discussion. First, consider the critical wave vector k_c, where the two branches intersect. From Fig. 1, we have for $\beta=0$ (propagation perpendicular to the magnetization),

$$(\omega_E/\omega_H)_c^{1/2} \cong 6\times 10^{-2}.$$

Recall that we have employed units in which $\omega_H=1$, in labeling the axis of the figure. If we write $\omega_E = Dk_{\shortparallel}^{(c)2}$, where $k_{\shortparallel}^{(c)} = 2\pi/\lambda^{(c)}$, then the "cutoff" wavelength $\lambda^{(c)}$ is given by

$$\lambda^{(c)} \cong 10^2 (D/\omega_H)^{1/2}.$$

The frequency ω_H is the Larmor frequency of a spin in an external magnetic field H. Explicitly, $\omega_H = g(e/2mc)H$. If we take $g=2$ and $H=100$ Oe, then $\omega_H = 1.6\times 10^{10}$ rad/sec. For any given microscopic model, one can express D in terms of the exchange interactions between the spins and the lattice constant of the crystal. For a simple cubic lattice of spins with nearest-neighbor exchange coupling between spins, one may express D in terms of the molecular-field Curie temperature T_c by the relation

$$D = [k_B T_c / 2\hbar(S+1)] a^2.$$

If we assume $T_c \approx 300$°K, $S=\frac{3}{2}$, and $a \approx 4$ Å, then

$$D \cong 1.6\times 10^{-2} \text{ cm}^2 \text{ rad/sec}.$$

These numbers give the estimate

$$\lambda^{(c)} \cong 10^{-4} \text{ cm}.$$

Thus, the cutoff wavelength $\lambda^{(c)}$ is quite short, when compared to the wavelengths encountered in microwave experiments. Hence, it would be quite difficult to excite magnetostatic modes with wavelength near the critical value $\lambda^{(c)}$ by direct methods, unless one works near the critical angle β_c where k_c is small.

Let us next consider the value of the attenuation length $1/q$ associated with the lower branch, for wavelengths of interest in microwave studies. We have seen that the cutoff wavelength $\lambda^{(c)}$ is small compared to, say, 1 cm. Thus, we can estimate q^{-1} from Eq. (4.7a), which is valid for wavelengths $\lambda \gg \lambda^{(c)}$. Upon rearranging Eq. (4.7a), we have

$$q^{-1} = \lambda^{1/3} [D(\omega_H + \omega_B)/\pi\omega_M(\omega_B \cos^2\beta - \omega_H)]^{1/3}.$$

If we use the parameters employed in Fig. 1, where $\omega_M = \omega_H$, $\omega_B = 2\omega_H$, and, furthermore, if we take $\cos^2\beta = 1$, then

$$q^{-1} \approx \lambda^{1/3} (D/\omega_H)^{1/3}.$$

For the value of D and ω_H employed above, one has

$$q^{-1} \approx 10^{-4} \lambda^{1/3} \text{ (cm)}.$$

If $\lambda = 1$ cm, then

$$(q^{-1})_{\text{lower branch}} \cong 10^{-4} \text{ cm}.$$

We can compare this value of the penetration depth with that associated with the Damon-Eshbach branch for the same wavelength and parameters. From Sec. III, one finds for the upper (Damon-Eshbach) branch

$$(q^{-1})_{\text{DE}} = (\lambda/2\pi)(\omega_B \cos^2\beta - \omega_H)/(\omega_M \cos^2\beta + \omega_H \sin^2\beta).$$

For the parameters employed above, with $\beta=0$, and $\lambda=1$ cm, one has

$$(q^{-1})_{\text{DE}} \cong 10^{-1} \text{ cm}.$$

Thus, for wavelengths of the order of 1 cm, the surface mode associated with the new lower branch is far more tightly bound to the surface than the Damon-Eshbach mode. This means that the coupling of the mode associated with the lower branch to an external microwave field is small compared to the oscillator strength of the Damon-Eshbach mode in this wavelength regime. It is easily seen that the ratio of oscillator strengths is simply proportional to the ratio $(q^{-1})_{\text{lower branch}}/(q^{-1})_{\text{DE}}$. We have just estimated this ratio to be $\approx 10^{-3}$, for $\lambda \cong 1$ cm. This estimate indicates that it will be quite difficult to observe the mode associated with the lower branch with conventional microwave techniques, since the oscillator strength is small. Note that the pentration length $(q^{-1})_{\text{lower branch}}$ increases as the critical angle $\beta_c = \cos^{-1}[(\omega_B/\omega_H)^{1/2}]$ is approached. Thus, the oscillator strength may be increased by studying angles of propagation β close to the critical value β_c.[14]

[14] The approximate expression given in Eq. (4.7a) is not valid for β close to β_c.

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Correlations along a Line in the Two-Dimensional Ising Model*

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The $2n$-spin correlation function for the two-dimensional Ising model at $T=T_c$ is evaluated for the special case in which all the spins lie along a straight line, separated by many lattice constants. The resulting $2n$-spin function is simply a quotient of products of two-spin correlations. A hypothesis of reducibility of fluctuations in the critical state is introduced. This hypothesis asserts that the product of any two local fluctuating quantities in the same neighborhood of space may be effectively replaced by a finite sum of local fluctuating quantities in this neighborhood. As a result, the previously found form for the $2n$-spin function may be used to evaluate the correlation function of n energy densities when all n points lie on a line. The n-energy correlation function is simply a sum of products of two-energy correlations. The quotient form for the spin correlation plus scaling is shown to immediately imply the logarithmic specific heat.

I. INTRODUCTION

SINCE Onsager's original work,[1] many authors have discussed thermodynamic properties[2,3] and correlations[3–11] in the two-dimensional Ising model. Much of this work has concentrated on using the Onsager solution to learn about the behavior near the critical point. The concept of scaling,[12–15] for example, has arisen in part from information gained from this model. According to the scaling idea, there are two indices, described[12] as x and y, which together determine the nature of all the critical singularities.

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In the two-dimensional Ising model, x and y each have simple values: $x=15/8$, $y=1$. However, even though the Onsager solution exists as a guide, no fully satisfactory physical argument is known to be available for understanding the values of x and y. These values are only obtained by very detailed and rather untransparent calculations. One can hope, however, that such simple values of x and y can be seen as the result of some structural property of critical correlations. In this paper, I argue that the result $y=1$, which implies the logarithmic specific heat, is a natural result of a simple structure of the n-spin correlation function.

This argument is based upon an evaluation of the $2n$-spin correlation function under the conditions: (a) that the Ising model is at the critical point; (b) that all the spins lie on a single straight line; and (c) that the spins are all separated from one another by many lattice constants. Then, if the spins are ordered along the line as $\sigma_1, \sigma_{1'}, \sigma_2, \sigma_{2'}, \ldots, \sigma_n, \sigma_{n'}$, the correlation function is calculated to have the form

$$\langle \prod_{i=1}^{n} (\sigma_i\sigma_{i'})\rangle = \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{\langle\sigma_i\sigma_{j'}\rangle}{(\langle\sigma_i\sigma_j\rangle\langle\sigma_{i'}\sigma_{j'}\rangle)^{1/2}}. \quad (1.1)$$

At first sight, it does not appear that the result (1.1) defines any critical indices. Further progress comes from the introduction of an extra idea, of the *reducibility of critical fluctuations*. Reducibility arises from the idea that there are only a limited number of independent fluctuating local variables in any phase-transition problem. Say there are s of these, $O_\nu(\mathbf{r})$ for $\nu=1, 2, \ldots, s$. A product of two O_ν's at neighboring positions is