Modulated Exponential Decay in a Two-Level Quantum System*

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We carry out a Weisskopf-Wigner-type calculation for a system of two quantum levels which are simultaneously coupled by a quantized radiation interaction H and a classical external perturbation V. The results for the level populations properly reduce to the quantum oscillation solution when $H\to 0$, and to the exponential-decay solution when $V\to 0$. When neither H nor V is zero, the upper level undergoes a modulated exponential decay. We derive an expression for the perturbed decay rate and discover a new type of modulation factor. The result suggests possibly important modifications to the standard, phenomenological solution of the three-level $H+V$ problem, which is of practical interest.

I. INTRODUCTION

Experiments[1-3] on the system of states $2s$, $2p$, and $1s$ in atomic hydrogen reveal many interesting features of, and provide a number of important tests for, the time-dependent perturbation theory of atomic transitions. As indicated in Fig. 1, the transitions occur under the *combined* effects of an external perturbation V and the radiation interaction H. The metastable $2s$ state can be coupled to the $2p$ state via a Stark matrix element $|V|$, while $2p$ is coupled to $1s$ via the radiation coupling $|H|$, which gives rise to the $2p$ spontaneous decay rate $\gamma\propto|H|^2$. If the $2p$ state did not decay (i.e., $\gamma=0$), then the $2s$-$2p$ coupling by $|V|$ would result in the well-known "quantum oscillation" between these states, at (angular) frequency δ, and with transition amplitude $\propto|V|^2$.[4] As it is, with $\gamma\neq 0$, one expects in general some sort of quantum oscillation superimposed on an exponential decay.

A phenomenological theory describing transitions within this system due to the coupling $(H+V)$ has been given by Lamb,[5] as an extension of earlier work by Bethe.[6] Lamb used the theory, under the condition $|V|\ll(\frac{1}{2}\hbar\gamma)$, to derive line-shapes for the $2s\to 2p$ quenching resonances[7] observed to high accuracy during measurements on hydrogen fine structure.[1] The same theory has been used by Wangsness,[8] under the condition $|V|\gg(\frac{1}{2}\hbar\gamma)$, to account for the temporal intensity variations in various spectral lines of hydrogen observed during beam-foil experiments.[3,9] Wangsness concludes that for sufficiently large $|V|$, the $2p\to 1s$ exponential decay will be modulated by the $2s$-$2p$ quantum oscillation. Series has also used the theory to propose a novel method for measuring the Lamb shift.[10]

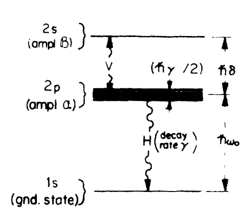

FIG. 1. Three-level system in hydrogen. The $2s$ and $2p$ levels (energy separation $\hbar\delta$) can be coupled by an external (Stark) perturbation V. The $2s$ level is metastable, but the $2p$ level decays spontaneously to the $1s$ ground state, emitting a photon of energy $\hbar\omega_0$. The decay is due to the $2p$-$1s$ coupling by the radiation interaction H, which results in the $2p$ spontaneous decay rate $\gamma\propto H^2$.

The theory of these transition processes is phenomenological in that the $2p$-$1s$ coupling by H is accounted for by simply adding a "damping term," proportional to γ, to the ordinary equations of time-dependent perturbation theory which describe the coupling of the (stationary) states by V. Thus, for the problem in Fig. 1, the equations for the $2p$ and $2s$ state amplitudes $\mathcal{A}$ and $\mathcal{B}$ are taken to be

$$i\hbar\dot{\mathcal{A}}=V_{ps}\mathcal{B}\exp(-i\delta t)-\tfrac{1}{2}i\hbar\gamma\mathcal{A}\,,\qquad i\hbar\mathcal{B}=V_{ps}^{*}\mathcal{A}\exp(+i\delta t)\,. \tag{1}$$

Here, $V_{ps}=\langle 2p|V|2s\rangle$, and the relatively small $2s$-state decay rate is ignored. In neglecting to include the $1s$ state in this description, it is implicitly assumed that the ground state is sufficiently far away ($\omega_0\gg\delta$) so as to be only weakly coupled by V. The ground state thus serves only as a sort of reservoir into which the $2p$ decays are dumped. However, it is generally true that the *same* V which couples $2p$ to $2s$ will also couple $2p$ to $1s$. The effect of this additional coupling can become important for $|V|\sim(\frac{1}{2}\hbar\gamma)$, or larger, as in the beam-foil experiments.[3]

The phenomenological Eqs. (1) can presumably be justified by writing equations of the Weisskopf-Wigner (WW) type.[11] However, if we wish to include the ground state, so as to estimate the effects of the additional $2p$-$1s$ coupling by V, as well as to insure that probability is conserved for times such that $\gamma t \gg 1$, it is necessary to carry out a full WW calculation, including a quantized radiation interaction, sums over photons, etc.[12] We expect to find that for sufficiently large $|V|$, the $2p \to 1s$ exponential decay will be modulated by both the $2p$-$2s$ and $2p$-$1s$ quantum oscillations. As a first step in understanding this additional modulation, we are led to consider the two-level $(H+V)$ problem indicated in Fig. 2. The p and s states here resemble the $2p$ and $1s$ states of the three-level $(H+V)$ problem in Fig. 1; note, however, that now these states are *simultaneously* coupled by H and V, a feature which is missing in the problem of Fig. 1.

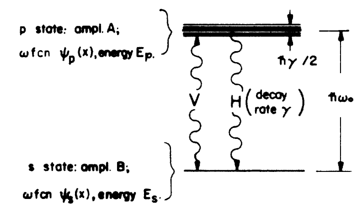

FIG. 2. Hypothetical two-level system. The p and s levels here (energy separation $\hbar\omega_0$) are analogous to the $2p$ and $1s$ levels in Fig. 1. However, now they are *simultaneously* coupled by the external perturbation V and radiation interaction H. Separately, V gives a quantum oscillation between p and s, while H gives an exponential decay of the upper state, at decay rate γ.

In this paper, we shall solve the two-level problem of Fig. 2, when neither H nor V is zero. The solution of this problem must reduce to the quantum oscillation (QO) solution[4] when $H \to 0$, and to the WW solution[11] when $V \to 0$; these separate solutions are well-known exercises in the elementary quantum mechanics of a two-level system. We shall solve the two-level $(H+V)$ problem by carrying out a WW-type calculation in which the external perturbation V is treated classically, and the radiation interaction H is quantized (as in the ordinary WW theory). We do this not only as a pedagogical extension of the elementary quantum-mechanical solutions, but also as a first step in understanding the modulated exponential decay of the three-level problem, which is of practical interest. The calculation is straightforward, but rather involved. The results show a new type of modulation factor, which suggests that important modifications to the standard, phenomenological solutions of the three-level problem may result when it is treated by a complete WW calculation.

II. ELEMENTARY SOLUTIONS

In this section, we review the standard solutions to the elementary two-level problems, namely, (A) $H=0$ but $V \neq 0$, which gives the (QO) solution, and (B) $H \neq 0$, but $V=0$, which - by means of the WW calculation – gives the WW solution, i.e., an exponential decay of the upper state. In both cases, we assume that an interaction U, either V or H, effects a superposition of states:

$$\Psi(x,t)=A(t)\,\psi_p(x)\exp(-iE_p t/\hbar)+B(t)\psi_s(x)\exp(-iE_s t/\hbar)\,, \tag{2}$$

where the $\psi_n(x)$ are orthonormal eigenfunctions of the Hamiltonian $\mathcal{H}$ which binds the two-level system, $\mathcal{H}\psi_n = E_n\psi_n$, and the time-dependent amplitudes A and B are to be found from the Schrödinger equation, $i\hbar(\partial\Psi/\partial t)=(\mathcal{H}+U)\Psi$. In both cases, we shall choose the initial conditions: $A=1$, $B=0$, at $t=0$ (i.e., the system is initially in the upper state). The notation defined here will be used later in solving the combined $(H+V)$ problem.

A. QO Solution

For the case where the radiation interaction $H=0$, but the unquantized, external perturbation $V \neq 0$, the amplitude equations are

$$i\hbar\dot{A}=V_{ps}B\exp(+i\omega_0 t)\,,\qquad i\hbar\dot{B}=V_{ps}^{*}A\exp(-i\omega_0 t)\,, \tag{3}$$

where $\omega_0 \equiv (E_p - E_s)/\hbar$ and $V_{pp}=V_{ss}=0$. We write $V_{ps}=\langle\psi_p|V|\psi_s\rangle = V\exp(-i\varphi)$,

where V is real and time-independent (by assumption), and φ is an arbitrary phase. It is convenient to define the dimensionless quantities

$$Q=[1+(2V/\hbar\omega_0)^2]^{1/2}, \qquad N=[2Q/(Q+1)]^{1/2}=(1+\mu^2)^{1/2}, \tag{4}$$

where $\mu=[(Q-1)/(Q+1)]^{1/2}\simeq V/\hbar\omega_0$, as $V\to 0$.

The parameter μ measures the strength of V relative to the binding interaction.

1. *Exact Solution in Terms of the Eigenfunctions of* $\mathcal{H}$

The solutions to Eq. (3), subject to the initial conditions $A=1$, $B=0$, at $t=0$ are *exactly*

$$A(t)\exp(-iE_p t/\hbar)=(1/N^2)[e^{-iPt}+\mu^2 e^{-iSt}], \qquad B(t)\exp(-iE_s t/\hbar)=(\mu e^{i\varphi}/N^2)[e^{-iPt}-e^{-iSt}], \tag{5}$$

where P and S are the perturbed eigenfrequencies,

$$P=(E_p+\Delta)/\hbar, \quad S=(E_s-\Delta)/\hbar, \quad \text{and} \quad \Delta=\tfrac{1}{2}(Q-1)\hbar\omega_0\simeq V^2/(E_p-E_s), \quad \text{as } V\to 0. \tag{6}$$

The state populations, as functions of time, are therefore *exactly*

$$\begin{aligned}\mathcal{P}(t)&=|A(t)|^2=1-(4\mu^2/N^2)\sin^2(\tfrac{1}{2}\omega' t),\\ \mathcal{S}(t)&=|B(t)|^2=(4\mu^2/N^2)\sin^2(\tfrac{1}{2}\omega' t),\end{aligned} \tag{7}$$

where $\omega'=P-S=Q\omega_0$ is the perturbed frequency difference. Probability is conserved, since $\mathcal{P}+\mathcal{S}=1$, for all time.

2. *Exact Solution in Terms of the Eigenfunctions of* $(\mathcal{H}+V)$

In addition to the above solution, it will be useful later to have the solution expressed in terms of the eigenfunctions of $(\mathcal{H}+V)$. We diagonalize the problem $(\mathcal{H}+V)\phi_n=E_n'\phi_n$, where $E_n'\equiv E_n\pm\Delta$ are the perturbed energies. The complete eigenfunctions are

$$\Phi_p(x,t)=\phi_p(x)e^{-iPt}, \qquad \Phi_s(x,t)=\phi_s(x)e^{-iSt},$$

where
$$\phi_p(x)=\frac{1}{N}[\psi_p(x)+\mu e^{i\varphi}\psi_s(x)], \qquad \phi_s(x)=\frac{1}{N}[e^{i\varphi}\psi_s(x)-\mu\psi_p(x)]. \tag{8}$$

It should be noted that in this diagonalization process initial conditions have not been specified, as they previously were. However, linear combinations of Φ_p and Φ_s can be formed which do correspond to the system being entirely in the p state or the s state at $t=0$:

$$\begin{aligned}\Phi_1&=\frac{1}{N}(\Phi_p-\mu\Phi_s), &&\text{for pure } \psi_p \text{ at } t=0,\\ \Phi_2&=\frac{1}{N}(\mu\Phi_p+\Phi_s), &&\text{for pure } \psi_s \text{ at } t=0.\end{aligned} \tag{9}$$

The Φ_n, and the ϕ_n, are orthonormal if the ψ_n are normalized. Later, in solving the $(H+V)$ problem, we shall use

$$\Phi(x,t)=a(t)\Phi_1(x,t)+b(t)\Phi_2(x,t), \tag{10}$$

as the general superposition of eigenstates of the V problem, which is then perturbed by $H\neq 0$.

3. *Comparison of the Two Solutions*

By substituting Eqs. (9) and (8) into Eq. (10), we can show that the superposition of Eq. (10) is identical to that of Eq. (2) if the ψ_n and Φ_n amplitudes are related by

$$A(t)\exp(-iE_p t/\hbar)=(1/N^2)[(a+\mu b)e^{-iPt}+\mu(\mu a-b)e^{-iSt}], \tag{11}$$

$$B(t)\exp(-iE_S t/\hbar)=(e^{i\varphi}/N^2)[\mu(a+\mu b)e^{-iPt}-(\mu a-b)e^{-iSt}] .$$

Comparing this with Eq. (5) shows that the QO solution, with the system initially in the upper state, corresponds to the choice $a=1$, $b=0$ for all time.

B. WW Method

For the case where the unquantized, external perturbation $V=0$, but the quantized radiation interaction $H\neq 0$, the amplitude equations are[13]

$$i\hbar\dot{A}=\sum_{\vec{k},\lambda} H^*_{sp}(\vec{k},\lambda)Be^{+i(k_0-k)ct} , \qquad i\hbar\dot{B}=H_{sp}(\vec{k},\lambda)Ae^{-i(k_0-k)ct} , \tag{12}$$

where $k_0=\omega_0/c$, $k=\omega/c=$ photon wave number, and

$$\sum_{\vec{k},\lambda}\simeq\sum_\lambda\int_{4\pi}d\Omega_{\vec{k}}\int_0^\infty dk\, w(k) .$$

The sum over photons (wave vector $\vec{k}$ and polarization λ) involves an integration over solid angles as well as integration over the photon density of states, $w(k)$, which is $vk^2/(2\pi)^3$ for photons quantized in a box of volume v. The sum occurs in the first amplitude equation, but not in the second, as the absorption $s\to p$ can result from a variety of photons in the radiation field, while the emission $p\to s$ gives rise to only a single photon.

The WW calculation proceeds by assuming that the upper state decays exponentially, so that the amplitude A may be given by $A(t)=\exp(-\frac{1}{2}\gamma_0 t)$, where γ_0 is a time-independent decay constant to be found. We write $\langle\psi_S|H|\psi_P\rangle\equiv H_{Sp}\equiv H\exp(+i\theta)$, where H is real and time-independent, and θ is an arbitrary phase. Putting this Ansatz into Eqs. (12) gives

$$B(t)=(He^{i\theta}/\hbar)z_0^{-1}[1-e^{+iz_0t}], \qquad \gamma_0=(2i/\hbar^2)\sum_{\vec{k},\lambda}H^2 z_0^{-1}[e^{-iz_0t}-1] , \tag{13}$$

where $\quad z_0\equiv(k-k_0)c+\frac{1}{2}i\gamma_0$.

The integral equation for γ_0 can be solved approximately by assuming: (i) γ_0 is small with respect to k_0c, i.e., the radiation interaction is weak compared to the binding energy; then γ_0 may be neglected in the integrand. (ii) H and $w(k)$ vary slowly with k compared to the sharp resonance at k_0 exhibited by the exponential terms. Then H and $w(k)$ may be evaluated at $k=k_0$ and taken outside the k integral. The result for the decay rate, for time such that $k_0ct\gg 1$, is

$$\gamma_0\simeq(2\pi/\hbar^2c)\sum_\lambda\int_{4\pi}d\Omega_{\vec{k}}H^2(\vec{k},\lambda)w(k)\Big|_{k=k_0} . \tag{14}$$

The population of the lower state can be found by summing over all photons which can cause this state to be populated. Using the approximation (ii), but *not* neglecting γ_0 in the integrand, gives

$$\mathcal{S}(t)=\sum_{\vec{k},\lambda}|B(t)|^2\simeq\sum_\lambda\int_{4\pi}d\Omega_{\vec{k}}[H(\vec{k},\lambda)/\hbar]^2w(k)\Big|_{k=k_0}\times\int_0^\infty dk|z_0^{-1}(1-e^{iz_0t})|^2 . \tag{15}$$

Carrying out the integration, again for $k_0ct\gg 1$, gives the state populations

$$\mathcal{S}(t)=1-\exp(-\gamma_0 t) , \qquad \mathcal{P}(t)=|A(t)|^2=\exp(-\gamma_0 t) . \tag{16}$$

As in the QO case, probability is conserved, since $\mathcal{P}+\mathcal{S}=1$, for all time.

For future reference, we note that the quantization of the radiation field is accounted for, without using direct product states, by associating[13]

$$\sum_{\vec{k},\lambda}e^{-ikct}\text{ with the absorption }(s\to p)\text{ matrix element, }H^*_{sp}=\langle\psi_p|H|\psi_s\rangle ,$$

and e^{+ikct} with the emission $(p\to s)$ matrix element, $H_{sp}=\langle\psi_s|H|\psi_p\rangle$. (17)

These rules, the exponential Ansatz, the approximations (i) and (ii) given above, and the imposition of a time scale by $k_0ct\gg1$ comprise the WW method which we apply to the combined $(H+V)$ problem in Sec. III.

III. TWO-LEVEL $(H+V)$ PROBLEM

In this section, we solve the two-level $(H+V)$ problem shown in Fig. 2, using the methods and notation outlined in Sec. II. We carry out a WW-type calculation, using the state superposition of Eq. (10). This superposition and the WW ansatz $a(t)=e^{-\Lambda t/2}$ stipulate that the system is in an eigenstate of $(\mathcal{H}+V)$ when H is turned on. As in the previous section, $\mathcal{H}$ is the binding interaction, V is the external perturbation, and H is the radiation interaction. We later remove this arbitrary time-ordering by averaging over an assumed random phase between H and V. As it must, the solution for $H=0$, and to the WW solution for $V=0$. When neither H nor V vanishes, the solution shows a new type of modulation factor, in addition to the expected QO modulation of the exponential decay.[3,8]

We make several approximations during the calculation. First, we consider single-photon processes only. Next, assuming both V and H are weak compared to $\mathcal{H}$, we carry terms of order V^2H, and no higher, in the amplitude equations. We also assume that the perturbed decay rate Γ is not much different than γ_0, except for having a slight time-dependence. Then, just as in the H problem, the upper level decays approximately exponentially. We solve the problem for a time scale $\Gamma t\gg1$ and "near resonance" for the photons involved in the $p\longleftrightarrow s$ transitions. Finally, we pass to the "isolated atom," by letting the photon quantization volume $v\to\infty$. All of these approximations are reasonable in typical physical situations.

A. Differential Equations

Using the state superposition of Eq. (10) in the Schrödinger equation

$$i\hbar\frac{\partial\Phi}{\partial t}=[(\mathcal{H}+V)+\hat{H}]\Phi \tag{18}$$

gives the amplitude equations to be solved for the time-dependent coefficients a and b,

$$\begin{aligned} i\hbar\dot{a}&=\langle\Phi_2|\hat{H}|\Phi_1\rangle^* b+\langle\Phi_1|\hat{H}|\Phi_1\rangle a\ ,\\ i\hbar\dot{b}&=\langle\Phi_2|\hat{H}|\Phi_1\rangle a+\langle\Phi_2|\hat{H}|\Phi_2\rangle b\ .\end{aligned} \tag{19}$$

Here, the caret over H indicates that the photon sums are to be included in accordance with Eq. (17). The matrix elements can be calculated in a straightforward manner, using Eqs. (8) and (9), the notation of Sec. II, and the rules of Eq. (17). If we define $\delta\equiv\theta-\varphi$ as the phase difference between H and V, and keep terms of order V^2H and no higher, we obtain, for example,

$$N^4\langle\Phi_2|\hat{H}|\Phi_1\rangle\simeq e^{i\delta}H(e^{-i\omega' t}+2\mu^2)e^{+ikct}+2\mu^2e^{-i\delta}\sum_{\vec{k},\lambda}H(1-\cos\omega' t)e^{-ikct},$$

$$\text{where}\quad \mu\simeq V/\hbar\omega_0,\quad N^4\simeq1+2\mu^2, \tag{20}$$

$$\text{and}\quad \omega'\equiv Q\omega_0\simeq(1+2\mu^2)k_0c\ .$$

We make the WW Ansatz by assuming that the amplitude of the "upper state" Φ_1, can be represented by $a(t)=\exp(-\frac{1}{2}\Lambda t)$. The real part of Λ will be the perturbed decay rate and will approach γ_0 as $V\to0$. Putting this Ansatz and the expressions for the matrix elements into the second of Eqs. (19) gives the equation to be solved for $b(t)$. As a convenience in solving for $b(t)$, we write

$$b(t)\equiv(1/N^4)b_0(t)f(t)\ ,$$

$$\text{where}\quad b_0(t)\equiv(He^{i\delta}/\hbar)z^{-1}(1-e^{izt})\ , \tag{21}$$

$$\text{and}\quad z\equiv(k-Qk_0)c+\tfrac{1}{2}i\Lambda\ .$$

The function $b_0(t)$ is analogous to the unperturbed WW amplitude $B(t)$ of Eq. (13); it is obtained by letting $\mu\to0$ in the equation for $b(t)$ and solving it with the assumption that z is independent of time. With this assumption, the equation for $b(t)$ can be converted into an equation for $f(t)$. After rewriting one term, we find

$$\dot{f}-[i\nu e^{i\delta}(1-e^{-i\omega' t})e^{ikct}]f\simeq[\exp(-\tfrac{1}{2}\Lambda t)/b_0(t)]\{\quad\}, \tag{22}$$

$$\text{where}\quad \nu\equiv\mu H/N^4\hbar\ .$$

The braces on the right-hand side contain complicated terms of order V, H, and V^2H. Fortunately, we can ignore the entire right-hand side for times such that $\gamma_0t\gg1$; thus we can solve Eq. (22) "exactly." Using $f(t=0)=1$, evaluating the resulting expression at resonance, $k=Qk_0$, and ignoring a term of order $\nu/\omega'\ll1$, we get

$$f(t)\simeq\exp(-i\nu te^{i\delta})\ . \tag{23}$$

This, with Eq. (21), is the solution for $b(t)$.

The next step is to substitute $b(t)$ and the WW Ansatz into the first of Eqs. (19) to get an equation for Λ. Using the convenient notation

$$\tau \equiv \omega' t, \quad G \equiv \Lambda\tau/2\omega',$$

$$r \equiv \epsilon\, e^{i\delta}, \quad \text{where} \quad \epsilon \equiv \nu/\omega' = \mu H/N^4\hbar\omega',$$

$$\gamma \equiv (Q^3/N^8)\gamma_0 \simeq (1+2\mu^2)\gamma_0, \tag{24}$$

allowing Λ to be time-dependent, and evaluating the first two terms at resonance ($k = Qk_0$), gives

$$\frac{dG}{d\tau} \simeq \frac{\gamma}{2\omega'}(1+2\mu^2 e^{-i\tau})e^{-ir\tau} + 4\epsilon^2 \sin^2\left(\frac{\tau}{2}\right) \times \left[\frac{\tau}{G}(e^G - 1)\right] e^{i(1-r)\tau} + ir(1-e^{-i\omega' t})e^{ikct} + i\sum_{\vec{k},\lambda} r^*(1-e^{i\tau})e^{-ikct}. \tag{25}$$

The solution for G will give the desired Λ, which is analogous to the WW decay rate in the unperturbed ($\mu = 0$) case. The approximations made in obtaining Eq. (25) are mainly that $\gamma_0 t \gg 1$, and that the various terms are evaluated near resonance, $k = Qk_0$.

B. Perturbed Decay Rate

The first two terms on the right-hand side of Eq. (25) are the dominant ones, as both exhibit factors which grow exponentially with time. We shall ignore the last two terms on the right-hand side which are small by comparison.[14] Of the remaining two terms, the second is small by comparison with the first in the limit of an isolated atom, that is, when we let the photon quantization volume v become infinite. This can be seen with the help of Eq. (14), by which we estimate the value of the radiation interaction matrix element H to be $H^2 \sim \hbar^2 c\gamma_0/vk_0^2$. Then the parameter ϵ defined in Eq. (24) vanishes when $v \to \infty$, as

$$\epsilon = |r| = \mu H/N^4\hbar\omega' \sim (\mu/\omega_0^2)(c^3\gamma_0/v)^{1/2} \to 0. \tag{26}$$

Therefore, we can consider the term in ϵ^2 in Eq. (25) to be a perturbation on the solution for G. We set $G = F + g$, where F and g satisfy

$$\frac{dF}{d\tau} = \frac{\gamma}{2\omega'}(1+2\mu^2 e^{-i\tau})e^{-ir\tau},$$

$$\frac{dg}{d\tau} \simeq 4\epsilon^2 \sin^2\left(\frac{\tau}{2}\right)\left(\frac{\tau}{G}(e^G - 1)\right) e^{i(1-r)\tau}. \tag{27}$$

We can solve for F exactly. Imposing the initial condition, $F = 0$ at $\tau = 0$, gives

$$F(\tau) = \frac{\gamma}{2\omega'}\left[\frac{1-e^{-ir\tau}}{ir} + 2\mu^2\left(\frac{1-e^{-i(1+r)\tau}}{i(1+r)}\right)\right]$$

$$\underset{r\to 0}{\simeq} \frac{\gamma\tau}{2\omega'}\left(1+\frac{2\mu^2}{\tau}[\sin\tau - i(1-\cos\tau)]\right)$$

$$\equiv F_a(\tau) \underset{\mu\to 0}{\simeq} \frac{\gamma\tau}{2\omega'}, \tag{28}$$

where the useful approximate forms are for $r \to 0$ and $\mu \to 0$. To solve for the relatively small term g, we approximate $G \simeq F$ on the right-hand side of the g equation; in fact, we use the final approximate form of F. Imposing $g = 0$ at $\tau = 0$ gives

$$g(\tau) = -\epsilon^2\alpha^2 \int_0^{\gamma\tau/2\omega'} (e^x - 1)e^{-\beta x}(1-e^{i\alpha x})^2 dx, \tag{29}$$

where $\alpha \equiv 2\omega'/\gamma \gg 1$, $\quad \beta \equiv 2ie^{-i\delta}(\nu/\gamma)$.

After performing the indicated integrations, we use the facts that $\beta \to 0$ as $v \to \infty$ (isolated atom limit), and $\alpha \gg 1$, to obtain the approximate result

$$g(\tau) \simeq \epsilon^2\alpha^2(\beta^{-1} - e^{\tau/\alpha}). \tag{30}$$

Combining the solutions for F and g give the approximate solution to Eq. (25)

$$G(\tau) \simeq F_a(\tau) - \tfrac{1}{2}(\epsilon\gamma/2\omega')\tau^2 i e^{i\delta} - \epsilon^2\alpha^2 e^{\tau/\alpha}. \tag{31}$$

The second term on the right-hand side comes from expanding the exact solution for F; it vanishes in the limit $\epsilon \to 0$. The last term is the dominant exponential term in Eq. (30) for g, which also vanishes as $\epsilon \to 0$.

We have now obtained approximate solutions of the amplitude Eqs. (19). The solution for $b(t)$ is, from Eqs. (21) and (23),

$$b(t) = (He^{i\delta}/N^4\hbar)z^{-1}(1-e^{izt})\exp(-i\nu t e^{i\delta}). \tag{32}$$

The solution for $a(t)$ is, from Eqs. (24) and (31),

$$a(t) = \exp[-G(t)] = e^{(\Lambda_a/2)t}\exp\left(\tfrac{1}{4}\epsilon\gamma\omega' t^2 i e^{i\delta} + \epsilon^2\alpha^2 e^{(\gamma/2)t}\right),$$

where
$$\Lambda_a \equiv \gamma\left\{1 + 2\mu^2\left[\left(\frac{\sin\omega' t}{\omega' t}\right) - i\left(\frac{1-\cos\omega' t}{\omega' t}\right)\right]\right\}, \tag{33}$$

with $\mu \simeq V/\hbar\omega_0$, $\omega' \simeq (1+2\mu^2)\omega_0$,

and $\gamma \simeq (1+2\mu^2)\gamma_0$.

Λ_a is the desired expression for the (complex) perturbed WW decay rate of the "upper state" Φ_1, in the superposition of Eq. (10). We can see this by noting that when $\epsilon \to 0$ (isolated atom limit), the Φ_1 amplitude becomes $a(t) \simeq \exp(-\frac{1}{2}\Lambda_a t)$. The approximations involved in these solutions are: (i) matrix elements are evaluated to order V^2H; (ii) the time scale is fixed by $\gamma_0 t \gg 1$; (iii) only resonance photons occur, i.e., $\omega \simeq \omega' = Qk_0c$; and (iv) the parameter $\epsilon \ll 1$. We have yet to pass to the limit $\epsilon \propto (\sqrt{v}\,)^{-1} \to 0$ and to average over the relative phase δ .

C. Average State Populations

We now return to Eqs. (11) which give the (ψ_p, ψ_s) state amplitudes A and B in terms of the (Φ_1, Φ_2) amplitudes a and b. By substituting the solutions for a and b into Eqs. (11), we shall derive expressions for the p and s state populations $\mathcal{P} = |A|^2$ and $\mathcal{S} = |B|^2$. We shall average these expressions over the phase δ, pass to the isolated atom limit $\epsilon \to 0$, and sum over the photons which can cause the lower state to be populated. In this way, we shall derive expressions for the time dependence of the transitions $p \to s$ when neither the radiation interaction H nor external perturbation V is zero.

Taking the absolute value of both sides in Eqs. (11) and summing over photons, we find (to order μ^2)

$$N^4|A|^2 \simeq (1+2\mu^2\cos\omega' t)|a|^2 + 2\mu^2(1-\cos\omega' t)|\tilde{b}|^2 + 2\mu\,\mathrm{Re}\chi ,$$

$$N^4|B|^2 \simeq (1+2\mu^2\cos\omega' t)|\tilde{b}|^2 + 2\mu^2(1-\cos\omega' t)|a|^2 - 2\mu\mathrm{Re}\chi ,$$

$$\text{where} \quad x(t) = (1-e^{i\omega' t})\,a^*(t)\tilde{b}(t) . \tag{34}$$

The tilde over b indicates that the photon sum is to be done. The various parameters (i.e., μ, N, ω') are to be approximated to order V^2, as in Eq. (20). We note that if probability is conserved in one representation, then it is conserved in the other, as, to order μ^2,

$$|A|^2 + |B|^2 = |a|^2 + |\tilde{b}|^2 . \tag{35}$$

The probabilities are, from Eqs. (32) and (33),

$$|a|^2 = e^{-\Gamma t}\exp(2\epsilon^2\alpha^2 e^{(\gamma/2)t} - \tfrac{1}{2}\epsilon\gamma\omega' t^2 \sin\delta) ,$$

$$\text{where} \quad \Gamma \equiv \mathrm{Re}\Lambda_a = \gamma\left(1+2\mu^2 \frac{\sin\omega' t}{\omega' t}\right) ; \tag{36}$$

$$|\tilde{b}|^2 = \sum_{\vec{k},\lambda} (H/N^4\hbar)^2 \left|\frac{1-e^{izt}}{z}\right|^2 \exp(2\nu t \sin\delta),$$

$$\text{where} \quad z \simeq (k - Qk_0)c + \tfrac{1}{2}i\Gamma . \tag{37}$$

We approximate $\Lambda \simeq \Gamma$ in Eq. (21) for z; this is correct as $\epsilon \to 0$. The phase-dependent factors in the cross term in Eqs. (34) are

$$\chi \propto e^{i\delta}\exp\{-\epsilon\omega' t[(\tfrac{1}{4}\gamma t - 1)\sin\delta + i(\tfrac{1}{4}\gamma t + 1)\cos\delta]\} . \tag{38}$$

Now we average each of the expressions in Eqs. (36)–(38) over the relative phase δ. We do this by operating with $(1/2\pi)\int d\delta\times$, where the integration extends from 0 to 2π. The phase averages in both Eqs. (36) and (37) involve integrals of the form

$$\frac{1}{2\pi}\int_0^{2\pi} \exp(C\sin\delta)\,d\delta = I_0(C), \tag{39}$$

where C is a constant $\propto \epsilon$, and I_0 is the modified Bessel function of order zero.[15] Since $I_0(0) = 1$, we readily obtain, in the limit $\epsilon \to 0$,

$$|a|^2_{\mathrm{av}} \equiv \lim_{\epsilon\to 0} \frac{1}{2\pi}\int_0^{2\pi} |a|^2 d\delta \simeq \exp(-\Gamma t) . \tag{40}$$

Carrying out the phase average, and a photon sum in the same fashion as in Eqs. (15) and (16), we also get

$$|\tilde{b}|_{\mathrm{av}}^2 \simeq 1 - \exp(-\Gamma t) \simeq 1 - |a|_{\mathrm{av}}^2 . \tag{41}$$

At this point, we have shown that probability is conserved in the context of this calculation. Finally, by integrating the right-hand side of Eq. (38), we find

$$\chi_{av} \propto -\tfrac{1}{2} i (\gamma t)^{1/2} \lim_{\epsilon \to 0} J_1[\epsilon \omega' t (\gamma t)^{1/2}] = 0 \ , \tag{42}$$

where J_1 is the ordinary Bessel function of order one.

By putting the results given by Eqs. (40)–(42) into Eq. (34), using the definition (36) of Γ, and retaining terms of order μ^2, we obtain the populations of the p and s states

$$\mathcal{P}(t) \equiv |A(t)|^2_{av} \ , \quad \mathcal{S}(t) \equiv |B(t)|^2_{av} = 1 - \mathcal{P}(t) \ ,$$

where $\mathcal{P}(t) \simeq [1 - \mathcal{Q}(t)] e^{-\gamma t} e^{-\rho\mu^2 \sin\omega' t} + \mathcal{Q}(t)[1 - e^{-\gamma t} e^{-\rho\mu^2 \sin\omega' t}]$,

and
$$\mathcal{Q}(t) \equiv 4\mu^2 \sin^2 \frac{\omega' t}{2} \ , \quad \rho \equiv 4 \frac{\hbar\gamma_0/2}{\hbar\omega_0} \ , \tag{43}$$

with $\mu \simeq V/\hbar\omega_0$, $\omega' \simeq (1 + 2\mu^2)\omega_0$, and $\gamma \simeq (1 + 2\mu^2)\gamma_0$, as before. We call $\mathcal{Q}(t)$ a "QO function", as it appears in the QO solution of Eq. (7). The parameters μ and ρ measure, respectively, the strength of the external perturbation V and radiation interaction H with respect to the unperturbed binding energy $\hbar\omega_0$. Finally, ω' and γ are the perturbed counterparts of the p-s state separation ω_0 and the p-state decay rate γ_0. We can easily see that when $H \to 0$, we get the QO solution of Eq. (7), namely, $\mathcal{P} \simeq 1 - \mathcal{Q}$, while, when $V \to 0$, we get the WW solution of Eq. (16), namely, $\mathcal{P} \simeq \exp(-\gamma_0 t)$.

IV. CONCLUSIONS

The expressions we have derived for the state populations in the two-level $(H + V)$ problem show the combined effects of a quantum oscillation superimposed on an exponential decay. The over-all dependence on time is indicated in Fig. 3. These solutions are valid so long as the effects of the coupling due to both the external perturbation V and the radiation interaction H are small compared to the level separation $\hbar\omega_0$, i.e., the parameters μ and ρ in Eq. (43) are small. Thus, our solution is *not* valid near a level crossing point. On the other hand, the solutions are valid for V either large or small compared to $\frac{1}{2}(\hbar\gamma_0)$.

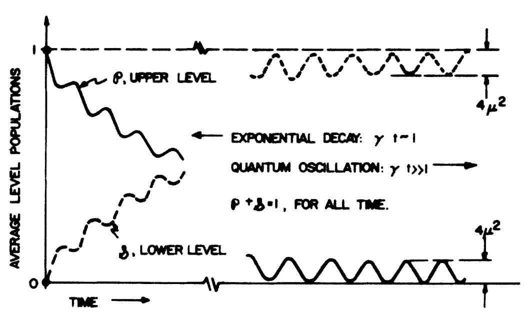

FIG. 3. Average (fractional) level populations for the problem shown in Fig. 2. The functions $\mathcal{P}$ and $\mathcal{S}$ are derived in the text. At early times, the exponential decay of the upper level is modulated by two quantum oscillation factors. Later, the levels exhibit a characteristic quantum oscillation of amplitude $4\mu^2$, at (angular) frequency $\omega' \simeq (1 + 2\mu^2)\omega_0$.

Our results for the two-level p-s system (Fig. 2) cannot be compared directly with previous results[8,10] for the three-level $2s$-$2p$-$1s$ system (Fig. 1). There are two reasons for this. First, the three-level calculations are done in the context of the phenomenological theory discussed in Sec. I, rather than the full WW formalism we have used. Second, in the three-level calculations, no allowance is made for any *pair* of levels to be simultaneously coupled by H and V. We can speculate that the second difference here might lead to different factors of order V^2 appearing in the results. For example, if the three-level problem were done by a WW-type calculation, including the $2p$-$1s$ quantum oscillation (at frequency ω_0) as well as the $2s$-$2p$ quantum oscillation (at frequency δ), one might expect to see QO functions $\mathcal{Q}(t, \omega_0)$ as well as $\mathcal{Q}(t, \delta)$ occurring as factors. On the other hand, the first difference here – a difference in method of calculation – might well lead to the appearance of entirely different terms of order $V^2\gamma$. This is indicated by the observation that in a phenomenological description such as Eqs. (1), an implicit choice has been made with regard to the relative phase of V and H (the latter being represented by its associated decay rate γ_0 only); consequently, the cross terms are likely to be different.

Series[10] has solved Eqs. (1) for the three-level problem in Fig. 1. If we let the unperturbed $2p$-$2s$ separation be ω_0 rather than $-\delta$, so as to draw an analogy with the p-s problem in Fig. 2, Series's result for the population of the upper level may be quoted as[16]

$$|a|^2 \simeq (1-2\mu^2)\exp(-\gamma_0 t) - \mathcal{Q}(t)\exp(-\tfrac{1}{2}\gamma_0 t) + 2\mu^2(1-2\rho\,\sin\omega' t)\exp(-\tfrac{1}{2}\gamma_0 t), \qquad (44)$$

in the same notation, and to the same degree of approximation, as in Eq. (43). We observe that this expression properly reduces to the QO solution for $\gamma_0 \to 0$, and to the WW solution for $V \to 0$, just as in Eq. (43). In the intermediate case, however, it is clear that the terms in Eq. (43) are qualitatively different from those of Eq. (44), particularly in the appearance of the term in $\exp(-\rho\mu^2 \sin\omega' t)$. While this comparison cannot be valid for times such that $\gamma_0 t \gg 1$,[17] it does suggest a new type of modulation of the exponential decay of the upper level. This new modulation factor is of order $V^2\gamma$; presumably, it will persist when the three-level problem is done by a complete WW calculation. We note also that there is no "threshold condition" for this modulation to occur, as in the calculation by Wangsness.[18]

These remarks are speculative in lieu of a detailed treatment of the three-level $(H+V)$ by the WW method. It is clear, however, that this problem should be investigated, since experiments on this system can be done to high accuracy – in measurements of WW (Lorentzian) line shapes for the transitions involved[1,2] and of details of the modulated exponential decay.[3] If the modulation characteristics of the two-level problem persist, they might well be detected in such experiments.

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[6]H. A. Bethe, Handbuch der Physik (Julius Springer-Verlag, Berlin, 1933), 2nd ed., Vol. 24/1, pp. 452–460. An updated version of this work is available in H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One and Two Electrons Atoms (Academic Press Inc., New York, 1957), 1st ed., Sec. 67.

[7]In addition to the work in Ref. 1, the theory of the so-called "sharp transitions" among magnetic substates of the 2*s* state is developed in W. E. Lamb, Jr., Phys. Rev. 85, 259 (1952), Secs. 68–72.

[8]R. K. Wangsness, Phys. Rev. 149, 60 (1966).

[9]It should be remarked that the beam-foil experiments in Ref. 3 did not measure the effects of $(H+V)$ on the $2s \to 2p \to 1s$ system, but rather on the $(n'=3,4,5,\ldots) \to (n=2)$ transitions in hydrogen. However, the results of Ref. 8 apply as well to $(n'=2) \to (n=1)$ transitions.

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[12]This problem is discussed in the work by G. Wentzel, cited in Ref. 11; see pp. 764–768.

[13]The original WW work is cited in Ref. 11. The calculation is discussed in detail in Sec. 18 by W. Heitler, in Quantum Theory of Radiation (Oxford University Press, Cambridge, 1954), 3rd ed., pp. 181–189; and by J. J. Sakurai, in Advanced Quantum Mechanics (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1967), pp. 37–39 and p. 65.

[14]Contributions from these terms would also vanish in the phase averaging carried out in Sec. III C.

[15]See I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products (Academic Press Inc., New York, 1966), p. 488; also Handbook of Mathematical Functions, edited by M. Abramovitz and I. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Government Printing Office, Washington, D. C., 1965) p. 375.

[16]See Eq. (8) of Ref. 10, which is an exact result. The approximation in Eq. (44) above is carried out for the H and V relative strength parameters ρ and $\mu \ll 1$.

[17]For the problem of Eq. (1), probability is not conserved for the amplitudes $\mathcal{A}$ and $\mathcal{B}$, as $|\mathcal{A}|^2 + |\mathcal{B}|^2 \sim \exp(-\gamma_0 t)$.

[18]See Ref. 8. Wangsness concludes that in the three-level problem of Fig. 1, there will be no modulation of the 2*p* exponential decay unless $V > \frac{1}{2}(\hbar\gamma_0/2)$. No such condition is imposed in Eq. (43).