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Polarization of Bremsstrahlung at the Short-Wavelength Limit

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The polarization of bremsstrahlung at the short-wavelength limit is calculated by means of Sommerfeld-Maue wave functions, i.e., for low atomic numbers. The two-component spinor representation is used to obtain the correlations between the spins of the initial and final electron and the photon polarization vector. Formulas are given for the circular and linear polarization of bremsstrahlung summed over the spin directions and integrated over the angles of the final electron, the incoming electron being longitudinally or transversely polarized. The photon polarization from unpolarized electrons is derived. Finally, comparison with Born-approximation calculation is made and numerical results are shown. The agreement with experimental data is not very satisfactory; it is probably due to the fact that experiments are not performed directly at the short-wavelength limit.

I. INTRODUCTION

The polarization of bremsstrahlung has previously been investigated either using the Born approximation[1–7] or taking into account the Coulomb potential exactly. In the latter case, however, calculations are only available for nonrelativistic[8] or very high[9] energies of the incoming and outgoing electrons. At the short-wavelength limit, the Olsen-Maximon method[9] is not suited at all, and the results of Sommerfeld[8] are only valid for very low energies. Using the connection between the atomic photoeffect and the high-frequency limit of bremsstrahlung spectrum, Fano and his coworkers[10–13] have justified applying the Born approximation even for zero momentum of the final electron.[14] This relationship was extended by Pratt[15] to the next order in $Z/137$, and it can be used to evaluate cross sections and polarization correlations of bremsstrahlung.

In this paper, the polarization of bremsstrahlung at the short-wavelength limit is calculated using Sommerfeld-Maue wave functions.[16] This does not amount to a pure expansion in powers of $Z/137$, although the resulting formulas are valid only for low atomic numbers Z. This restriction is common to both the present calculation and the Born-approximation results. The criterion for the latter to be valid is[13] $Z/137 \ll q_{\min}$, where $q_{\min}$ is the minimum momentum transfer to the nucleus in units of mc (m is the electron rest mass). $q_{\min}$ is always less than unit, and is small compared to 1 for low energies. On the other hand, the formulas of Sec. III are correct for all energies provided that $Z/137 \ll 1$. Thus, the two methods of calculating the polarization of bremsstrahlung at the high-frequency limit are expected to agree on the whole for high energies of the incoming electron, whereas the Born approximation breaks down for small energies.

Previous calculations dealing with atomic photoeffect beyond the Born approximation can be compared with the present ones only to a limited extent for the following reasons: (a) Most work considered the high-energy limit, or the summation over polarization and spin directions was performed at an early stage of the calculations. (b) Polarization correlations evaluated exactly by numerical methods[17,18] are available only for high atomic numbers Z, and for relatively low photon energies where the relationship to bremsstrahlung is no more valid. So the results of Nagel,[19] correct to relative order $Z/137$, are the only ones to be compared with the polarization formulas of Sec. III. Screening is neglected throughout this paper.

II. MATRIX ELEMENT

Neglecting radiative corrections, the matrix element for bremsstrahlung is

$$M = C\int \psi_2^{\dagger}(\vec{r})(\vec{\alpha}\cdot\vec{e}^{\,*})^{-i\vec{k}\cdot\vec{r}}\psi_1(\vec{r})d\tau \quad , \qquad (2.1)$$

$$\text{where} \quad C = -e\hbar c\left(\frac{2\pi}{mc^2k}\right)^{1/2} , \qquad (2.2)$$

and $\vec{\alpha}$ is the Dirac matrix operator, $\vec{r}$ is the electron coordinate in units of $\hbar/mc$, $\vec{e}$ is the polarization vector of the photon, $\vec{k}$ is the momentum of the photon in units of mc and $k = |\vec{k}|$ its energy in units of the rest energy of the electron mc^2. The Sommerfeld-Maue wave functions, which behave asymptotically like a plane wave plus outgoing and ingoing spherical wave, are, respectively,[20,21]

$$\psi_1(\vec{r}) = N_1 e^{i\vec{p}_1\cdot\vec{r}}[1-(i/2\epsilon_1)\vec{\alpha}\cdot\vec{\nabla}]$$

$$\times F(ia_1;\ 1;\ ip_1 r - i\vec{p}_1\cdot\vec{r})\,u(\vec{p}_1), \quad (2.3a)$$

$$\psi_2^\dagger(\vec{r}) = N_2^* u^\dagger(\vec{p}_2)e^{-i\vec{p}_2\cdot\vec{r}}[1+(i/2\epsilon_2)\vec{\alpha}\cdot\vec{\nabla}]$$

$$\times F(ia_2;\ 1;\ ip_2 r + i\vec{p}_2\cdot\vec{r}), \quad (2.3b)$$

where the subscripts 1 and 2 refer to initial and final state of the electron; $\epsilon_{1,2}$ is the total energy of the electron in units of mc^2, $\vec{p}_{1,2}$ its momentum in units of mc, so that $\epsilon_{1,2}{}^2 = \vec{p}_{1,2}{}^2+1$, N_1 and N_2 are normalization factors, F is the confluent hypergeometric function with indicated arguments, $u(\vec{p}_{1,2})$ is the spinor of the free electron, and

$$a_1 = (\epsilon_1/p_1)a, \quad a_2 = (\epsilon_2/p_2)a, \quad a = \alpha Z; \quad (2.4)$$

$\alpha \approx 1/137$ is the Sommerfeld fine-structure constant.

Substituting these wave functions in (2.1), the integration yields for low atomic numbers Z [21]:

$$M = Ku^\dagger(\vec{p}_2)\,[(\vec{\alpha}\cdot\vec{e}^*)I_1 + (\vec{\alpha}\cdot\vec{e}^*)(\vec{\alpha}\cdot\vec{I}_2) + (\vec{\alpha}\cdot\vec{I}_3)(\vec{\alpha}\cdot\vec{e}^*)]\,u(\vec{p}_1), \quad (2.5)$$

with

$$I_1 = 2K_1\left\{\frac{V}{q^2}\left(\frac{\epsilon_2}{D_1}-\frac{\epsilon_1}{D_2}\right) + i\frac{W}{D_1D_2}\left[a_2\epsilon_1\left(\frac{\mu}{D_2}-1\right) - a_1\epsilon_2\left(\frac{\mu}{D_1}+1\right)\right]\right\}, \quad (2.6)$$

$$\vec{I}_2 = K_1\left\{\frac{V\vec{q}}{D_2q^2} + ia_2\frac{W}{D_1D_2}\left[\frac{\vec{P}}{p_1} - \vec{q}\left(\frac{\mu}{D_2}-1\right)\right]\right\}, \quad (2.7)$$

$$\vec{I}_3 = K_1\left\{\frac{V\vec{q}}{D_1q^2} + ia_1\frac{W}{D_1D_2}\left[\frac{\vec{P}}{p_2} - \vec{q}\left(\frac{\mu}{D_1}+1\right)\right]\right\}, \quad (2.8)$$

$$K = (\hbar/mc)^3 C N_1 N_2^*, \quad (2.9)$$

$$K_1 = 4\pi a\, e^{-\pi a_1}\left(\frac{q^2}{D_2}\right)^{ia_1}\left(\frac{q^2}{D_1}\right)^{ia_2} \quad (2.10)$$

$$D_1 = 2(\epsilon_1 k - \vec{k}\cdot\vec{p}_1), \quad D_2 = 2(\epsilon_2 k - \vec{k}\cdot\vec{p}_2), \quad (2.11)$$

$$\mu = (p_1+p_2)^2 - k^2, \quad (2.12)$$

$$\vec{P} = p_1\vec{p}_2 + p_2\vec{p}_1, \quad (2.13)$$

and $\vec{q} = \vec{p}_1 - \vec{p}_2 - \vec{k}$. (2.14)

V and W are the hypergeometric functions,

$$V = {}_2F_1(ia_1,\ ia_2;1;x) \quad (2.15)$$

and $W = {}_2F_1(1+ia_1,\ 1+ia_2;\ 2;\ x)$, (2.16)

where $x = 1 - \mu q^2/D_1D_2$. (2.17)

The quantities I_1, $\vec{I}_2$, and $\vec{I}_3$ satisfy the relation

$$\vec{I}_3 = (\epsilon_1/\epsilon_2)\vec{I}_2 + (\vec{q}/2\epsilon_2)I_1. \quad (2.18)$$

III. CALCULATION OF THE POLARIZATION

The polarization of bremsstrahlung is calculated by using the two-spinor representation (split representation).[9,22] For this purpose, the Dirac operator $\vec{\alpha}$ is expressed in terms of the Pauli spin matrices $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$, and the spinors $u(\vec{p})$ are replaced by the two component spinors v and w

$$\alpha = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0\end{pmatrix}, \quad u(\vec{p}) = N'\begin{pmatrix} v \\ w\end{pmatrix}, \quad (3.1)$$

where N' is a normalization factor. Substituting (3.1) and the representation

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} \quad (3.2)$$

in the Dirac equation for a free particle, one obtains

$$u = N'\begin{pmatrix} v \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\epsilon+1}v\end{pmatrix} = N'\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\epsilon+1}v\end{pmatrix}. \quad (3.3)$$

Assuming v to be normalized, $v^\dagger v = 1$, normalization $u^\dagger u = 1$ gives

$$N' = [(\epsilon+1)/2\epsilon]^{1/2}. \quad (3.4)$$

By introducing (3.1), (3.3), and (3.4) into (2.5)

$$M = K\left(\frac{(\epsilon_1+1)(\epsilon_2+1)}{4\epsilon_1\epsilon_2}\right)^{1/2} v_2^\dagger\left[\left(\frac{\vec{\sigma}\cdot\vec{p}_2}{\epsilon_2+1}(\vec{\sigma}\cdot\vec{e}^*) + (\vec{\sigma}\cdot\vec{e}^*)\frac{\vec{\sigma}\cdot\vec{p}_1}{\epsilon_1+1}\right)I_1 + (\vec{\sigma}\cdot\vec{e}^*)(\vec{\sigma}\cdot\vec{I}_2) + \frac{(\vec{\sigma}\cdot\vec{p}_2)(\vec{\sigma}\cdot\vec{e}^*)(\vec{\sigma}\cdot\vec{I}_2)(\vec{\sigma}\cdot\vec{p}_1)}{(\epsilon_1+1)(\epsilon_2+1)} + (\vec{\sigma}\cdot\vec{I}_3)(\vec{\sigma}\cdot\vec{e}^*) + \frac{(\vec{\sigma}\cdot\vec{p}_2)(\vec{\sigma}\cdot\vec{I}_3)(\vec{\sigma}\cdot\vec{e}^*)(\vec{\sigma}\cdot\vec{p}_1)}{(\epsilon_1+1)(\epsilon_2+1)}\right]v_1, \quad (3.5)$$

where v_1 and v_2 are the Pauli spinors describing the initial and final spin states. We use the relation (2.18) and the identity

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})=\vec{A}\cdot\vec{B}+i\vec{\sigma}\cdot(\vec{A}\times\vec{B}) \ , \tag{3.6}$$

and at the short-wavelength limit ($p_2=0$), the matrix element M is reduced to the form

$$M_0=\lim_{p_2\to 0} M=K\left(\frac{\epsilon_1+1}{2\epsilon_1}\right)^{1/2}\{v_2,\ [(\vec{e}^*\cdot\vec{I})+i\vec{\sigma}\cdot(\vec{e}^*\times\vec{J})]\ v_1\} \ , \tag{3.7}$$

where $$\vec{I}=(\epsilon_1+1)\vec{I}_{20}+[\ \vec{p}_1/(\epsilon_1+1)+\tfrac{1}{2}\vec{q}\ ]\ I_{10} \ , \tag{3.8}$$

$$\vec{J}=-k\vec{I}_{20}+[\vec{p}_1/(\epsilon_1+1)-\tfrac{1}{2}\vec{q}]I_{10} \ , \tag{3.9}$$

$$I_{10}=\lim_{p_2\to 0} I_1=\frac{K_1}{q^2}\left[\left(\frac{2}{q^2}-\frac{\epsilon_1}{k}\right)(V_0-ia_1W_0)+\epsilon_1\frac{\vec{k}\cdot\vec{p}_2}{k^2p_2}\,iaW_0\right] ,$$

$$\vec{I}_{20}=\lim_{p_2\to 0} I_2=\frac{K_1}{2kq^2}\left[\vec{q}(V_0-ia_1W_0)+\left(\frac{\vec{k}}{p_1}+\frac{\vec{p}_2}{p_2}-\frac{\vec{k}\cdot\vec{p}_2}{kp_2}\vec{q}\right)iaW_0\right] , \tag{3.10}$$

$$V_0=\lim_{p_2\to 0} V=F(ia_1;1;i\xi) \ , \qquad W_0=\lim_{p_2\to 0} W=F(1+ia_1;2;i\xi) \ , \tag{3.11}$$

and $$\xi=\lim_{p_2\to 0} a_2x=a\left(\frac{2\vec{q}\cdot\vec{p}_2}{q^2p_2}-\frac{\vec{k}\cdot\vec{p}_2}{kp_2}-\frac{p_1}{k}\right) , \tag{3.12}$$

F denoting the confluent hypergeometric function. The Pauli spinor v is chosen to be an eigenstate of the component of $\vec{\sigma}$ along a unit vector $\vec{\zeta}$

$$(\vec{\sigma}\cdot\vec{\zeta})v=v \ . \tag{3.13}$$

This equation defines the polarization direction $\vec{\zeta}$ in the rest system of the electron. The projection operator $vv^\dagger$ for the spin state v and the diagonal matrix element $v^\dagger\vec{\sigma}v$ are then given by[9,22]

$$vv^\dagger=\tfrac{1}{2}(1+\vec{\sigma}\cdot\vec{\zeta}) \ , \tag{3.14}$$

and $$v^\dagger\vec{\sigma}v=\vec{\zeta} \ . \tag{3.15}$$

Taking the absolute square of the matrix element (3.7) and using the relations (3.6), (3.14), and (3.15), we obtain

$$\begin{aligned}|M_0|^2=|K|^2\frac{\epsilon_1+1}{4\epsilon_1}\{&(|\vec{e}^*\cdot\vec{I}|^2+|\vec{e}^*\times\vec{J}|^2)(1+\vec{\zeta}_1\cdot\vec{\zeta}_2)-2\,\mathrm{Re}([(\vec{e}\times\vec{J}^*)\times\vec{\zeta}_1][(\vec{e}^*\times\vec{J})\times\vec{\zeta}_2])\\ &-2(\vec{\zeta}_1\times\vec{\zeta}_2)\cdot\mathrm{Re}[(\vec{e}^*\cdot\vec{I})(\vec{e}\times\vec{J}^*)]+2\,\mathrm{Im}[(\vec{e}^*\cdot\vec{I})(\vec{e}\times\vec{J}^*)\cdot(\vec{\zeta}_1+\vec{\zeta}_2)]\\ &+i[(\vec{e}\times\vec{J}^*)(\vec{e}^*\times\vec{J})]\cdot(\vec{\zeta}_1-\vec{\zeta}_2)\} \ .\end{aligned} \tag{3.16}$$

This equation gives the correlations between the polarization of bremsstrahlung and the initial and final spin states of the electron.

The absolute square of the matrix element summed over the polarization directions $\vec{\zeta}_2$ of the outgoing electron is

$$\sum_{\vec{\zeta}_2}|M_0|^2=|K|^2\,\frac{\epsilon_1+1}{2\epsilon_1}\{|\vec{e}^*\cdot\vec{I}|^2+|\vec{e}^*\times\vec{J}|^2+2\,\mathrm{Im}[(\vec{e}^*\cdot\vec{I})(\vec{e}\times\vec{J}^*)\cdot\vec{\zeta}_1]+i[(\vec{e}\times\vec{J}^*)\times(\vec{e}^*\times\vec{J})]\cdot\vec{\zeta}_1\}\quad. \tag{3.17}$$

In order to calculate the circular polarization of bremsstrahlung, the vector $\vec{e}$ is written

$$\vec{e}=2^{-1/2}(\vec{e}_x\pm i\vec{e}_y)\ , \tag{3.18}$$

the upper and lower signs referring to the cross section σ_r and σ_l of right and left circularly polarized radiation, respectively. $\vec{e}_x$ and $\vec{e}_y$ are unit vectors in the directions of the x and y axis of a coordinate system whose z axis points in the direction of the outgoing photon and where $\vec{p}_1$ is in the x-z plane (see Fig. 1). We then have

$$\sum_{\vec{\zeta}_2}|M_0|^2=|K|^2\,\frac{\epsilon_1+1}{4\epsilon_1}\{|\vec{I}\times\hat{k}|^2\pm i(\vec{I}\times\vec{I}^*)\cdot\hat{k}+|\vec{J}|^2+|\vec{J}\cdot\hat{k}|^2 \pm i(\vec{J}\times\vec{J}^*)\cdot\hat{k}+2\,\mathrm{Im}([\vec{I}\times\hat{k}]\cdot[(\vec{J}^*\times\zeta_1)\hat{k}]) \pm 2\,\mathrm{Re}[\vec{I}\times(\vec{J}^*\times\vec{\zeta}_1)]\cdot\hat{k}\pm 2\,\mathrm{Re}[(\vec{J}\cdot\vec{\zeta}_1)(\vec{J}^*\cdot\hat{k})]-i[(\vec{J}\times\vec{J}^*)\times\hat{k}]\cdot[\vec{\zeta}_1\times\hat{k}]\}\quad, \tag{3.19}$$

where $\hat{k}=\vec{k}/k$ (3.20)

is the unit vector in the direction of the outgoing photon.

The circular polarization is given by

$$P_z=(\sigma_r-\sigma_l)/(\sigma_r+\sigma_l)\quad. \tag{3.21}$$

From Eq. (3.19), one obtains

$$P_z(\vec{\zeta}_1)=\frac{2\,\mathrm{Re}[(\vec{J}\cdot\vec{\zeta}_1)(\vec{J}^*\cdot\hat{k})-(\vec{I}\times\hat{k})\cdot(\vec{J}^*\times\vec{\zeta}_1)]+i[(\vec{I}\times\vec{I}^*)+(\vec{J}\times\vec{J}^*)]\cdot\hat{k}}{|\vec{I}\times\hat{k}|^2+|\vec{J}|^2+|\vec{J}\cdot\hat{k}|^2+2\,\mathrm{Im}\{(\vec{I}\times\hat{k})\cdot[(\vec{J}^*\times\vec{\zeta}_1)\times\hat{k}]\}-i[(\vec{J}\times\vec{J}^*)\times\hat{k}]\cdot[\vec{\zeta}_1\times\hat{k}]}\quad. \tag{3.22}$$

The cross sections $\sigma_{\parallel}$ and $\sigma_{\perp}$, for bremsstrahlung which is, respectively, linearly polarized in the $\vec{p}_1-\vec{k}$ plane and perpendicular to that plane, are proportional to $\sum_{\vec{\zeta}_2}|M_0|^2$ [Eq. (3.17)] if one sets, respectively, $\vec{e}=\vec{e}_x$ and $\vec{e}=\vec{e}_y$:

$$\sigma_{\parallel}\propto|K|^2\,\frac{\epsilon_1+1}{2\epsilon_1}\{|I_x|^2+|\vec{J}|^2-|J_x|^2+2\,\mathrm{Im}[I_x(\vec{J}^*\times\vec{\zeta}_1)_x]-i(\vec{J}\times\vec{J}^*)_x\zeta_{1x}\}\quad, \tag{3.23}$$

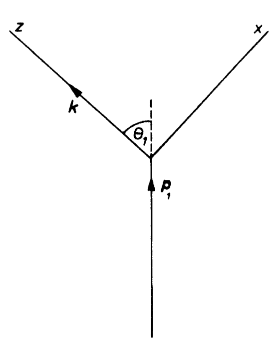

FIG. 1. Choice of coordinate axes.

$$\sigma_\perp \propto |K|^2 \frac{\epsilon_1+1}{2\epsilon_1}\{|I_y|^2+|\vec{J}|^2-|J_y|^2+2\,\mathrm{Im}[I_y(\vec{J}^*\times\vec{\zeta}_1)_y]-i(\vec{J}\times\vec{J}^*)_y\zeta_{1y}\}\quad . \tag{3.24}$$

The linear polarization

$$P_l=(\sigma_\perp-\sigma_{\parallel})/(\sigma_\perp+\sigma_{\parallel}) \tag{3.25}$$

can be obtained from Eqs. (3.23) and (3.24):

$$P_l(\vec{\zeta}_1) = \frac{|I_y|^2-|I_x|^2-|J_y|^2+|J_x|^2+2\mathrm{Im}\{I_y(\vec{J}^*\times\vec{\zeta}_1)_y-I_x(\vec{J}^*\times\vec{\zeta}_1)_x\}+i(\vec{J}\times\vec{J}^*)_x\zeta_{1x}-i(\vec{J}\times\vec{J}^*)_y\zeta_{1y}}{|\vec{I}\times\hat{k}|^2+|\vec{J}|^2+|\vec{J}\cdot\hat{k}|^2+2\,\mathrm{Im}\{(\vec{I}\times\hat{k})\cdot[(\vec{J}^*\times\vec{\zeta}_1)\times\hat{k}]\}-i[(\vec{J}\times\vec{J}^*)\times\hat{k}]\cdot[\vec{\zeta}_1\times\hat{k}]}. \tag{3.26}$$

The formulas (3.22) and (3.26) give, respectively, the circular and linear polarization of the photons for the elementary process of bremsstrahlung at the short-wavelength limit for arbitrary polarization $\vec{\zeta}_1$ of the incoming electron. If the numerators and denominators of these expressions are integrated over the angles of the final electron, we obtain the polarization of the bremsstrahlung emitted in direction $\hat{k}$. It can be shown that for definite spin directions $\vec{\zeta}_1$, some terms of the numerators and denominators of $P_z(\vec{\zeta}_1)$ and $P_l(\vec{\zeta}_1)$ vanish when these integrations are performed. In the coordinate system of Fig. 1, the unit vectors in the directions of $\vec{p}_1$, $\vec{p}_2$, and $\vec{k}$ are given by

$$\begin{aligned}\hat{p}_1&=(\sin\theta_1,\ 0,\ \cos\theta_1),\\ \hat{p}_2&=(\sin\theta_2\cos\phi,\ \sin\theta_2\sin\phi,\ \cos\theta_2)\ ,\\ \hat{k}&=(0,\ 0,\ 1).\end{aligned} \tag{3.27}$$

The functions V_0 and W_0 in $\vec{I}$ and $\vec{J}$ depend on $\cos\phi$ and are therefore even functions of ϕ:

$$V_0(-\phi)=V_0(\phi)\ ,\quad W_0(-\phi)=W_0(\phi)\ . \tag{3.28}$$

That is, integrals of the form $\int_0^{2\pi}G(\vec{k},\ \vec{p}_1)\,V_0W_0\sin\phi\,d\phi$ vanish. If Eqs. (3.10) are substituted in (3.8) and (3.9) it can be verified that the expressions $\mathrm{Im}[\vec{I}\cdot(\vec{J}^*\times\vec{\zeta}_1)]$ for $\vec{\zeta}_1=\hat{p}_1$ and for $\vec{\zeta}_1=\hat{k}$ or $i(\vec{J}\times\vec{J}^*)\cdot\hat{k}$ have the form $G(\vec{k},\ \vec{p}_1)\mathrm{Re}(V_0W_0^*)(\hat{p}_1\times\hat{p}_2)\cdot\hat{k}$, for example. Since $(\hat{p}_1\times\hat{p}_2)\cdot\hat{k}=\sin\theta_1\sin\theta_2\sin\phi$ and $d\Omega_{p_2}=\sin\theta_2 d\theta_2 d\phi$, the integrals $\int\mathrm{Im}[\vec{I}\cdot(\vec{J}^*\times\hat{p}_1)]d\Omega_{p_2}$, $\int\mathrm{Im}[\vec{I}\cdot(\vec{J}^*\times\hat{k})]\,d\Omega_{p_2}$ and $i\int(\vec{J}\times\vec{J}^*)\cdot\hat{k}\,d\Omega_{p_2}$ vanish. Using this, the following formulas for the polarization of bremsstrahlung integrated over the angles of the final electron are obtained:

A. Circular Polarization of Bremsstrahlung

1. *Electrons Longitudinally Polarized* ($\vec{\zeta}_1=\pm\hat{p}_1$)

$$P_z(\pm\hat{p}_1)=\pm\frac{2\int\mathrm{Re}[(\vec{J}\cdot\hat{p}_1)(\vec{J}^*\cdot\hat{k})-(\vec{I}\times\hat{k})\cdot(\vec{J}^*\times\hat{p}_1)]\,d\Omega_{p_2}}{\int(|\vec{I}\times\hat{k}|^2+|\vec{J}|^2+|\vec{J}\cdot\hat{k}|^2)\,d\Omega_{p_2}}\ . \tag{3.29}$$

2. *Electrons Polarized Transversely in the Plane of Emission* ($\vec{\zeta}_1=\hat{p}_1\times\vec{e}_y$)

$$P_z(\hat{p}_1\times\vec{e}_y)=\frac{2\int\mathrm{Re}\{[(\vec{I}+\vec{J})\cdot(\hat{p}_1\times\vec{e}_y)](\vec{J}^*\cdot\hat{k})-(\vec{I}\cdot\vec{J}^*)(\hat{p}_1\cdot\vec{e}_x)\}\,d\Omega_{p_2}}{\int(|\vec{I}\times\hat{k}|^2+|\vec{J}\cdot\hat{k}|^2)\,d\Omega_{p_2}}\ . \tag{3.30}$$

3. *Electrons Polarized Transversely Perpendicular to the Plane of Emission* ($\vec{\zeta}_1=\vec{e}_y$)

$$P_z(\vec{e}_y)=0\quad . \tag{3.31}$$

4. *Electrons Polarized in Photon Direction* ($\vec{\zeta}_1 = \hat{k}$)

$$P_z(\hat{k}) = \frac{2\int \mathrm{Re}[(\vec{\mathrm{I}}\cdot\hat{k})(\vec{\mathrm{J}}^*\cdot\hat{k}) - (\vec{\mathrm{I}}\times\hat{k})\cdot(\vec{\mathrm{J}}^*\times\hat{k})]\,d\Omega_{p_2}}{\int(|\vec{\mathrm{I}}\times\hat{k}|^2 + |\vec{\mathrm{J}}|^2 + |\vec{\mathrm{J}}\cdot\hat{k}|^2)\,d\Omega_{p_2}} . \tag{3.32}$$

B. Linear Polarization of Bremsstrahlung

1. *Electrons Longitudinally Polarized* ($\vec{\zeta}_1 = \pm\hat{p}_1$)

$$P_l(\pm\hat{p}_1) = \int(|I_y|^2 - |I_x|^2 - |J_y|^2 + |J_x|^2)d\Omega_{p_2} / \int(|\vec{\mathrm{I}}\times\hat{k}|^2 + |\vec{\mathrm{J}}|^2 + |\vec{\mathrm{J}}\cdot\hat{k}|^2)d\Omega_{p_2} . \tag{3.33}$$

2. *Electrons Polarized Transversely in the Plane of Emission* ($\vec{\zeta}_1 = \hat{p}_1\times\vec{\mathrm{e}}_y$)

$$P_l(\hat{p}_1\times\vec{\mathrm{e}}_y) = \int(|I_y|^2 - |I_x|^2 - |J_y|^2 + |J_x|^2)d\Omega_{p_2} / \int(|\vec{\mathrm{I}}\times\hat{k}|^2 + |\vec{\mathrm{J}}|^2 + |\vec{\mathrm{J}}\cdot\hat{k}|^2)d\Omega_{p_2} . \tag{3.34}$$

3. *Electrons Polarized Transversely Perpendicular to the Plane of Emission* ($\vec{\zeta}_1 = \vec{\mathrm{e}}_y$)

$$P_l(\vec{\mathrm{e}}_y) = \frac{\int\{|I_y|^2 - |I_x|^2 - |J_y|^2 + |J_x|^2 + 2\,\mathrm{Im}[I_x(\vec{\mathrm{J}}^*\cdot\hat{k})] - i(\vec{\mathrm{J}}\times\vec{\mathrm{J}}^*)\cdot\vec{\mathrm{e}}_y\}\,d\Omega_{p_2}}{\int\{|\vec{\mathrm{I}}\times\hat{k}|^2 + |\vec{\mathrm{J}}|^2 + |\vec{\mathrm{J}}\cdot\hat{k}|^2 - 2\,\mathrm{Im}[I_x(\vec{\mathrm{J}}^*\cdot\hat{k})] - i(\vec{\mathrm{J}}\times\vec{\mathrm{J}}^*)\cdot\vec{\mathrm{e}}_y\}d\Omega_{p_2}} . \tag{3.35}$$

Contrary to the results of Olsen and Maximon[9] for high energies, the linear polarization of the radiation is dependent on the polarization of the initial electron.

The polarization of bremsstrahlung for unpolarized electrons in the initial state is calculated by averaging $\sum_{\vec{\zeta}_2} |M_0|^2$ over all spin directions $\vec{\zeta}_1$. From Eq. (3.17)

$$\frac{1}{2}\sum_{\vec{\zeta}_1,\vec{\zeta}_2} |M_0|^2 = |K|^2 \frac{\epsilon_1 + 1}{2\epsilon_1}(|\vec{\mathrm{e}}^*\cdot\vec{\mathrm{I}}|^2 + |\vec{\mathrm{e}}^*\times\vec{\mathrm{J}}|^2) . \tag{3.36}$$

The corresponding formulas for circular and linear polarization of bremsstrahlung are, respectively,

$$P_z = -i[(\vec{\mathrm{I}}\times\vec{\mathrm{I}}^*) + (\vec{\mathrm{J}}\times\vec{\mathrm{J}}^*)]\cdot\hat{k}/(|\vec{\mathrm{I}}\times\hat{k}|^2 + |\vec{\mathrm{J}}|^2 + |\vec{\mathrm{J}}\cdot\hat{k}|^2) , \tag{3.37}$$

and $$P_l = (|I_y|^2 - |I_x|^2 + |J_x|^2 - |J_y|^2)/(|\vec{\mathrm{I}}\times\hat{k}|^2 + |\vec{\mathrm{J}}|^2 + |\vec{\mathrm{J}}\cdot\hat{k}|^2) . \tag{3.38}$$

If the numerators and the denominators of these expressions are integrated over the angles of the final electron, one obtains, because of the same symmetry reasons as above,

$$P_z = 0 , \tag{3.39}$$

and $$P_l = \int(|I_Y|^2 - |I_x|^2 - |J_y|^2 + |J_x|^2)d\Omega_{p_2} / \int(|\vec{\mathrm{I}}\times\hat{k}|^2 + |\vec{\mathrm{J}}|^2 + |\vec{\mathrm{J}}\cdot\hat{k}|^2)d\Omega_{p_2} .$$

That is, for unpolarized electrons, the circular polarization of bremsstrahlung is zero, while the linear polarization of the radiation has the same value as for an electron beam polarized longitudinally or transversely in the plane of emission.

The above formulas are valid for all initial electron energies ϵ_1 in case of low atomic numbers Z.

IV. COMPARISON WITH PREVIOUS CALCULATIONS

The Born-approximation formulas for polarization of bremsstrahlung at the short-wavelength limit can be derived from the corresponding expressions of Sec. III by taking the limit $Z\to 0$. Since $\lim V_0 = 1$ as $a\to 0$, we have

$$\lim_{a\to 0}\frac{\vec{I}}{K_1} = \frac{1}{q^2}\left[\frac{k}{p_1}\left(\frac{\epsilon_1+3}{q^2}-\frac{1}{2}\right)\hat{p}_1 - \left(\frac{k}{q^2}+\frac{1}{2}\right)\hat{k}\right], \quad (4.1)$$

and

$$\lim_{a\to 0}\frac{\vec{J}}{K_1} = \frac{1}{q^2}\left[-\frac{k}{p_1}\left(\frac{k}{q^2}+\frac{1}{2}\right)\hat{p}_1 + \left(\frac{k}{q^2}-\frac{1}{2}\right)\hat{k}\right]. \quad (4.2)$$

When these expressions are substituted in Eqs. (3.29) to (3.35), the integration over the angles of the final electron can be simply performed, yielding a factor of 4π. One gets, for instance,

$$\lim_{a\to 0} P_z(\pm\hat{p}_1) = \pm\frac{k}{p_1}\,\frac{4+(k-2/k)q^2}{4+(k-1)q^2}, \quad (4.3)$$

$$\lim_{a\to 0} P_z(\hat{k}) = \frac{k^2}{p_1^{\,2}}\,\frac{4+(\epsilon_1+2/k)q^2-q^4/2k}{4+(k-1)q^2}, \quad (4.4)$$

$$\lim_{a\to 0} P_z(\hat{p}_1\times\vec{e}_y) = \frac{4k\,|\hat{p}_1\times\hat{k}|}{4+(k-1)q^2}, \quad (4.5)$$

and for arbitrary $\vec{\zeta}_1$

$$\lim_{a\to 0} P_l = -\frac{4-q^2}{4+(k-1)q^2}, \quad (4.6)$$

where, according to (2.14), the square of the momentum transfer to the nucleus is

$$q^2 = (\vec{p}_1-\vec{k})^2 = 2(\epsilon_1 k - \vec{k}\cdot\vec{p}_1) = 2k(\epsilon_1 - p_1\cos\theta_1). \quad (4.7)$$

These formulas have been calculated by Fano, McVoy, and Albers[11] [Eqs. (4.3) and (4.4)], by Nagel[19] [Eqs. (4.3) and (4.5)], and by Gluckstern and Hull[2] [Eq. (4.6)].

It is easy to see from Eqs. (3.8)–(3.10) that in the expressions for $\vec{I}$ and $\vec{J}$, the factors of V_0 are not dependent on the unit vector $\hat{p}_2$. So the polarization obtained in Born approximation should agree with the corresponding formulas of this paper on condition that the terms proportional to $|V_0|^2$ in the numerators and the denominators of these expressions are large compared to those proportional to $\mathrm{Im}(V_0W_0^*)$ and to $|W_0|^2$. Now both the numerators and the denominators of the Born-approximation formulas originally contain the factor $(\vec{p}_1\times\vec{k})^2$ which cancels out in Eqs. (4.3)-(4.6). Likewise, in Eqs. (3.29)–(3.35), all the terms proportional to $|V_0|^2$ have this factor, and the terms proportional to $\mathrm{Im}(V_0W_0^*)$ are multiplied by $(\vec{p}_1\times\vec{k})$. Hence, only the terms proportional to $|W_0|^2$, i.e., those of relative order a^2, do not vanish for photons emitted in forward and in backward directions (cf., Appendix), and the above condition is violated for θ_1 near 0 and π. Consequently, the cross section and the polarization of bremsstrahlung should differ strongly for these angles in the two approximations considered. That does not necessarily mean that the present calculations are correct even for low atomic numbers, since the terms neglected in the approximate wave functions contribute to lowest nonvanishing order in forward and in backward directions. Nevertheless, the formulas of Sec. III are superior to the Born-approximation results. For example, they yield a zero degree of linear polarization for $\theta_1 = 0$ and $\theta_1 = \pi$, as required by symmetry. The nonvanishing terms in forward direction of relative order a^2 have been mentioned by several authors.[15,19,23] For the relativistic K-shell photoeffect, they have been calculated exactly by Weber and Mullin.[24]

Another case where the present calculations should not coincide with Born-approximation results is the low-energy limit. By neglecting terms of relative order $\beta_1^{\,4} = (p_1/\epsilon_1)^4$, Eq. (3.40) is reduced to

$$P_l = \frac{\int(|I_y|^2 - |I_x|^2)\,d\Omega_{p_2}}{\int(|I_y|^2 + |I_x|^2)\,d\Omega_{p_2}}, \quad (4.8)$$

because $|\vec{J}|/|\vec{I}|$ is of order $\beta_1^{\,2}$. For $\beta_1^{\,2}\ll 1$, this can be shown to transform into Sommerfeld's nonrelativistic expression for the linear polarization of bremsstrahlung at the short-wavelength limit, derived with the aid of the Schrödinger equation and taking into account retardation.[8]

Hence, (3.40) is correct even for high atomic numbers Z whenever $\beta_1^2 \ll 1$.[21] The Born approximation is valid for $\alpha Z \ll q_{\rm min} = p_1 - k$, $q_{\rm min}$ being the radius of convergence of the expansion of the matrix element in powers of αZ.[13] In the nonrelativistic limit, the minimum momentum transfer to the nucleus is $q_{\rm min} \approx p_1 - p_1^2/2 \ll 1$, and the condition $\alpha Z \ll p_1$ is satisfied poorly.

Polarization correlations which go beyond the Born approximation can be derived from the connection between the atomic photoeffect and the high-frequency region of bremsstrahlung. Using an improvement of the Sommerfeld-Maue final-state wave function, Nagel[19] has computed the relativistic K-shell photoeffect to relative order αZ, including all polarization effects. He has given formulas corresponding to circular polarization of bremsstrahlung at the short-wavelength limit,[25] consisting of the Born-approximation results and a first-order correction. Tables I and II show the circular polarization of bremsstrahlung from longitudinally and transversely polarized electrons, respectively. Whereas, the present calculation and the polarization of Nagel agree rather well for intermediate angles θ_1, the difference is largest in forward and in backward directions, as expected from the above remarks. It is to be noted that the Born-approximation values generally are still closer to the present ones, that is the correction of order αZ worsens the agreement. For $\sin\theta_1 \ll 1$, the corrected polarizations of Nagel are less reliable than the Born-approximation results. This can be seen especially with $P_z(\hat{p}_1 \times \vec{e}_y)$, the absolute value of which becomes larger than unity for small angles θ_1 and $\pi - \theta_1$. Hence, none of the approximations considered are valid at forward and backward angles.

V. NUMERICAL RESULTS

The equations representing the polarization of bremstrahlung have been programmed for a Siemens 2002 computer. Some results of the numerical calculations are given in Figs. 2–5. They show the polarization as a function of the photon energy $k = \epsilon_1 - 1$ for various spin directions $\vec{\zeta}_1$; the parameter is the angle of emission θ_1. In agreement with Olsen and Maximon,[9] the circular polarization $P_z(\hat{p}_1)$ tends towards unity for high-energy electrons (cf., Appendix) while all other kinds of polarization go to zero. In the nonrelativistic limit, $P_z(\vec{\zeta}_1)$ is of order p_1^2 as can easily be seen from Eq. (3.22), taking into account that $|\vec{J}|/|\vec{I}| = O(p_1^2)$. For the same reason, the linear polarization becomes independent of $\vec{\zeta}_1$. In Fig. 6, P_l is plotted against the angle θ_1 for various atomic numbers Z. As emphasized by Pratt *et al.*,[18] the polarization curves undergo very rapid changes over the region of small angles, in contrast with their comparatively slow variation through the main angular region. The linear polarization is strongly Z-dependent, especially near the forward direction ($\theta_1 = 0°$). As discussed in Sec. IV, the Born-approximation calculation differs strongly from the other curves at small angles. In particular, it results in a finite degree of polarization at $\theta_1 = 0$.

TABLE I. Circular polarization of bremsstrahlung from longitudinally polarized electrons $P_z(\hat{p}_1)$ for $Z = 13$ and photon energy $h\nu = 500$ keV.

		0°	2°	10°	20°	30°	60°	90°	150°
$P_z(\hat{p}_1)$	Present work	0.66	0.54	0.49	0.46	0.43	0.24	−0.02	−0.50
	Nagel[a]	0.48	0.48	0.47	0.45	0.41	0.23	−0.02	−0.44
	Born approximation	0.49	0.49	0.49	0.46	0.43	0.24	−0.02	−0.48

[a]Reference 19.

TABLE II. Circular polarization of bremsstrahlung from transversely polarized electrons $P_z(\hat{p}_1 \times \vec{e}_y)$ for $Z = 13$ and photon energy $h\nu = 500$ keV.

		0°	2°	10°	20°	30°	60°	90°	150°
$P_z(\hat{p}_1 \times \vec{e}_y)$	Present work	0	0.03	0.17	0.33	0.49	0.85	0.99	0.45
	Nagel[a]		−0.12	0.14	0.33	0.49	0.87	1.00	0.39
	Born approximation	0	0.03	0.17	0.34	0.49	0.86	1.00	0.51

[a]Reference 19.

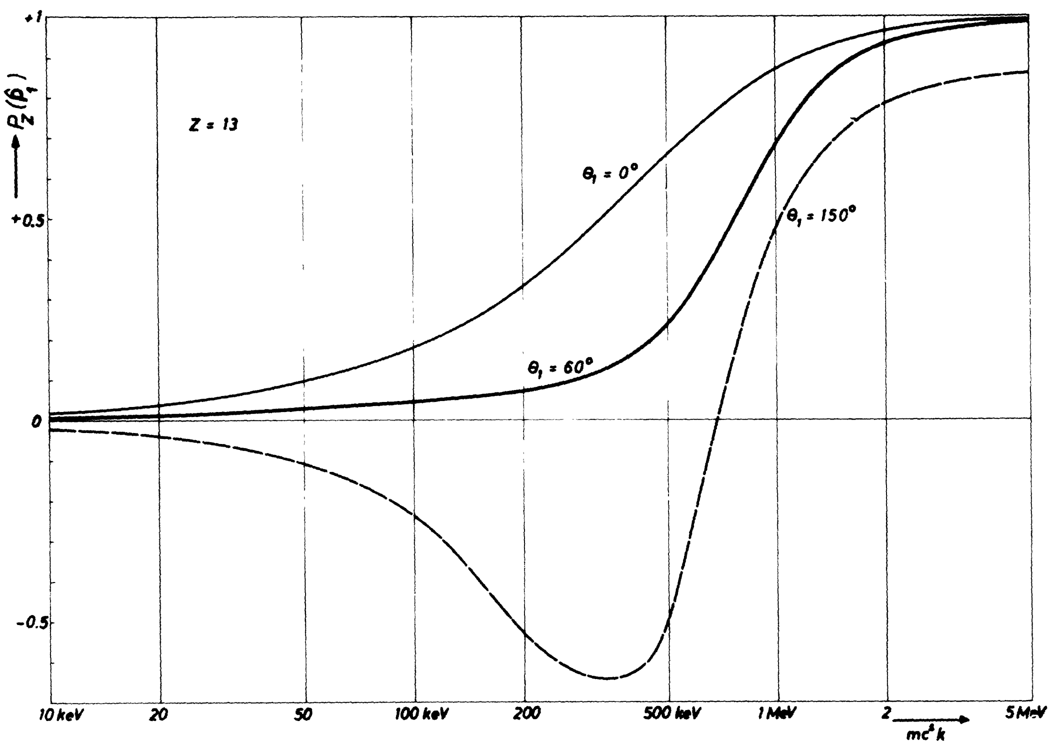

FIG. 2. Circular polarization of bremsstrahlung from longitudinally polarized electrons for $Z = 13$ and various angles of emission.

Since there have been almost no previous measurements of the polarization of the short-wavelength limit of bremsstrahlung, these calculations were compared in Figs. 7–9 with experimental data of Motz and Placious[26] for $k = 0.9(\epsilon_1 - 1)$. The theoretical values are generally too high. However, they give the form of the linear polarization (for instance, the change of sign at higher energies) correctly. It is to be expected that experiments performed directly at the short-wavelength limit will give a higher degree of polarization. From these figures, one cannot estimate how much the formulas of Sec. III are valid for high atomic numbers at the energies considered, because the agreement with the measurements is also rather unsatisfactory for $Z = 4$. Nevertheless, a single

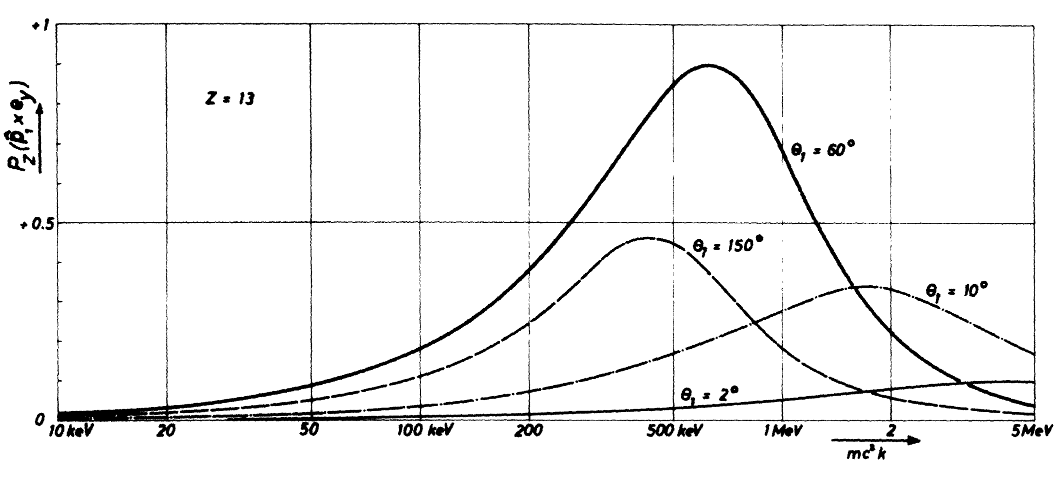

FIG. 3. Circular polarization of bremsstrahlung from electrons polarized transversely in the plane of emission.

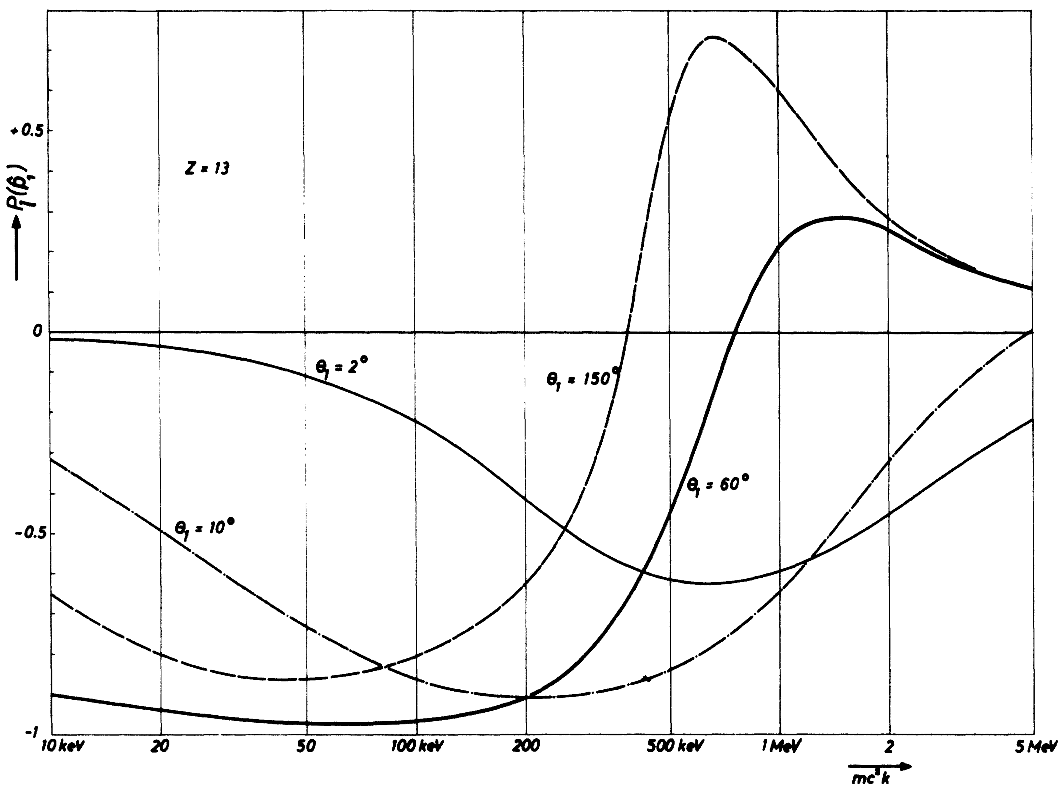

FIG. 4. Linear polarization of bremsstrahlung from longitudinally polarized electrons.

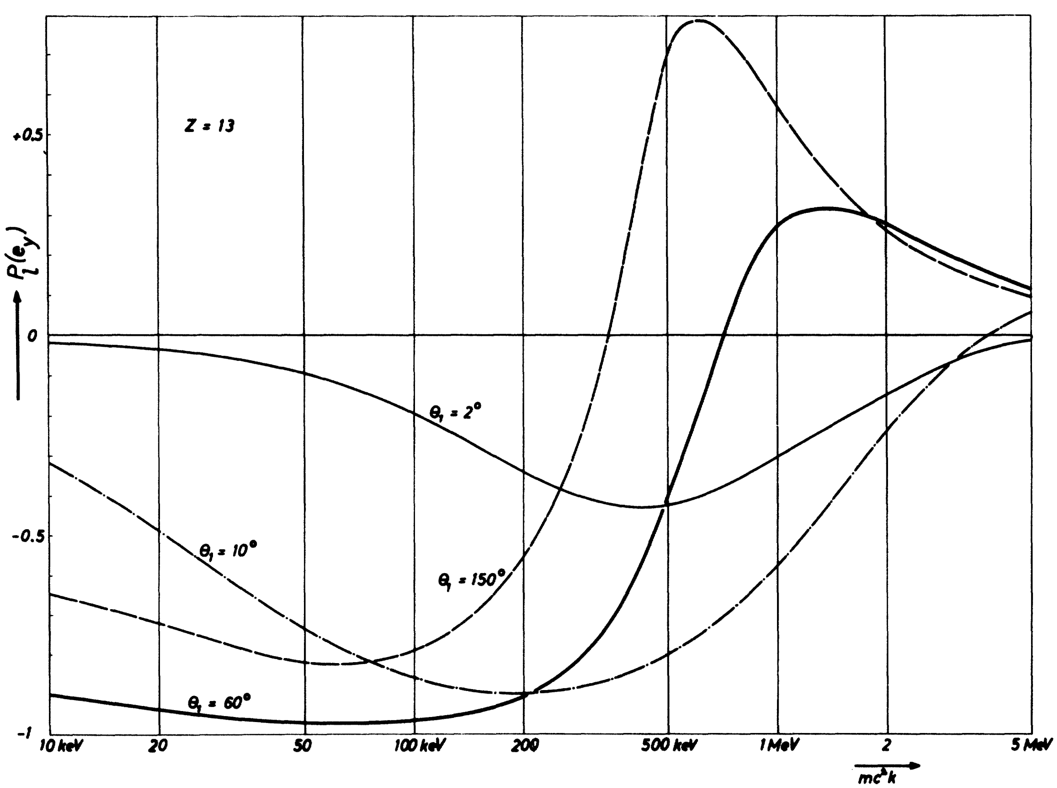

FIG. 5. Linear polarization of bremsstrahlung from electrons polarized transversely perpendicular to the plane of emission.

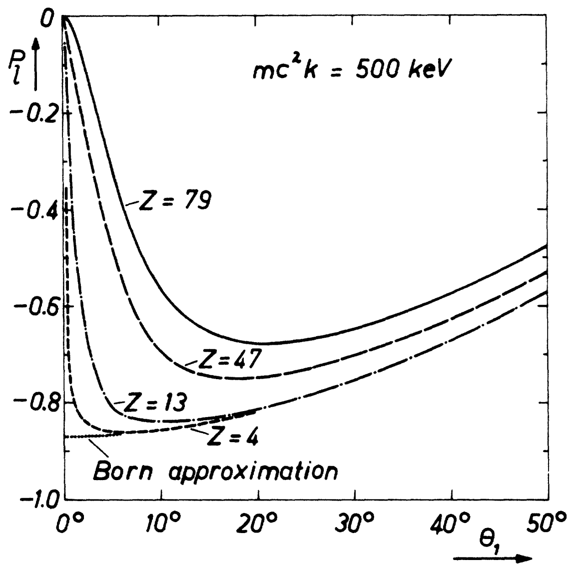

FIG. 6. Linear polarization of bremsstrahlung from unpolarized electrons for various atomic numbers Z.

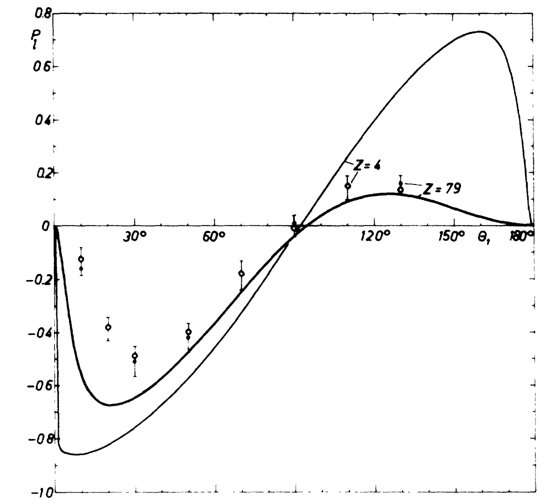

FIG. 8. Linear polarization of bremsstrahlung from unpolarized electrons in comparison with experimental data for $E_1 = 500$ keV. Theoretical curves: $mc^2k = 500$ keV; experimental points: $mc^2k = 450$ keV.

value of the linear polarization P_l may be compared with the high-frequency limit of the exact numerical calculation of bremsstrahlung polarization by Brysk, Zerby, and Penny.[27] For $Z = 79$, the photon energy $h\nu = 180\,\text{keV}$ and $\theta_1 = 47.7°$, these authors obtained $P_l \simeq -0.74$ as against $P_l \simeq -0.77$ from Eq. (3.40); the Born approximation [Eq. (4.6)] yields $P_l \simeq -0.95$.

Recently, Scheer, Trott, and Zahs[28] measured the linear polarization of bremsstrahlung for $Z = 6$ and the kinetic energy of the incoming electron $E_1 = 35\,\text{keV}$. Their results at the short-wavelength limit were compared with these calculations. The experimental degree of polarization proved lower than the computed one for all angles considered.

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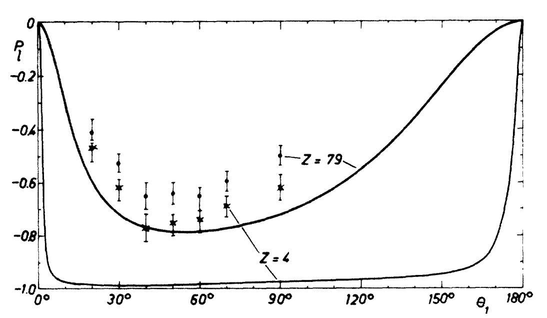

FIG. 7. Linear polarization of bremsstrahlung from unpolarized electrons in comparison with experimental data for the kinetic electron energy $E_1 = 100$ keV. Theoretical curves: $mc^2k = 100$ keV; experimental points: $mc^2k = 90$ keV.

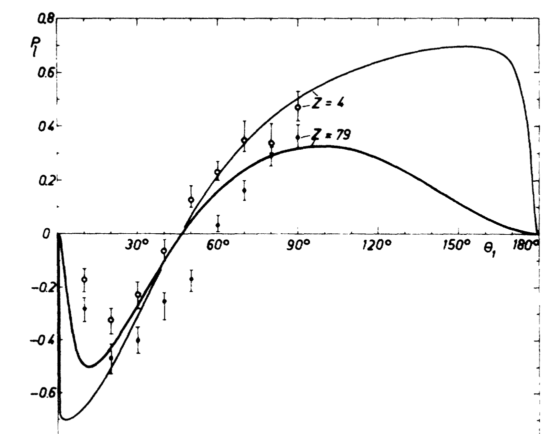

FIG. 9. Linear polarization of bremsstrahlung from unpolarized electrons in comparison with experimental data for $E_1 = 1$ MeV. Theoretical curves: $mc^2k = 1$ MeV; experimental points: $mc^2k = 900$ keV.

APPENDIX

The circular polarization of bremsstrahlung from longitudinally polarized electrons is considered for the case of radiation emitted in the forward direction ($\hat{k} = \hat{p}_1$). Then the quantities $\vec{I}$ and $\vec{J}$, [Eqs. (3.8) and (3.9)], are reduced to the form

$$\vec{I} = K_1 \frac{\epsilon_1 + p_1}{2k}\left\{\left(1 + \frac{p_1}{k}\right)\hat{p}_1 V_0 + \left[\left(\hat{p}_1\cdot\hat{p}_2\left(1 + \frac{k}{p_1}\right) - \frac{2\epsilon_1}{k} - \frac{k}{p_1}\right)\hat{p}_1 + \frac{\epsilon_1 + 1}{k}\hat{p}_2\right]\frac{ia}{2}W_0\right\} , \quad \text{(A1)}$$

$$\vec{J} = K_1 \frac{\epsilon_1 + p_1}{2k}\left\{\left[\left(1 + \frac{k}{p_1}\right)\hat{p}_1\cdot\hat{p}_2 - \frac{k}{p_1}\right]\hat{p}_1 - \hat{p}_2\right\}\frac{ia}{2}W_0 ; \quad \text{(A2)}$$

and Eq. (3.29) gives

$$P_z(\pm\hat{p}_1) = \pm\frac{\int[k^4(1-\hat{p}_1\cdot\hat{p}_2)^2 + p_1^4(\hat{p}_1\times\hat{p}_2)^2]\,|W_0|^2 d\Omega_{p_2}}{\int[k^4(1-\hat{p}_1\cdot\hat{p}_2)^2 + p_1^2(\epsilon_1^2+1)(\hat{p}_1\times\hat{p}_2)^2]\,|W_0|^2 d\Omega_{p_2}} , \quad \hat{k} = \hat{p}_1 . \quad \text{(A3)}$$

From this, one can easily derive the circular polarization for two special cases: (a) $\epsilon_1 \gg 1$: For high energies of the incoming electron, the numerator and denominator of (A3) are equal and thus

$$P_z(\pm\hat{p}_1) \simeq \pm 1, \quad \epsilon_1 \gg 1 . \quad \text{(A4)}$$

(b) $(p_1/2)^4 \ll 1$: For low energies, $k^4 \ll p_1^4$; hence,

$$P_z(\pm\hat{p}_1) \simeq \pm p_1^2/(\epsilon_1^2+1), \; (p_1/2)^4 \ll 1. \quad \text{(A5)}$$

This formula may be used for $p_1 \lesssim 0.6$, that is, up to photon energies of about 85 keV, the error being less than 1%.

For $\hat{k} = \hat{p}_1$, the confluent hypergeometric function W_0 takes the form

$$W_0 = F(1 + ia_1;\; 2;\; ia\chi) , \quad \text{(A6)}$$

where $\chi = -(p_1/k)(1 - \hat{p}_1\cdot\hat{p}_2)$. (A7)

Expanding W_0 in terms of $a = \alpha Z$, one gets

$$W_0 = 1 + i\tfrac{1}{2}\chi a - [(\epsilon_1/2p_1)\chi + \tfrac{1}{6}\chi^2]a^2 + \cdots , \quad \text{(A8)}$$

and $|W_0|^2 = 1 - [(\epsilon_1/p_1)\chi + \tfrac{1}{12}\chi^2]a^2 + O(a^4)$. (A9)

When (A9) is substituted in (A3), the integration can be performed; one obtains

$$P_z(\pm\hat{p}_1) \simeq \pm\frac{4k^4 + 2p_1^4 + (6\epsilon_1 k^3 + p_1^4 + 3.2\epsilon_1 p_1^2)a^2}{4k^4 + 2p_1^2(\epsilon_1^2+1) + [6\epsilon_1 k^3 + (\epsilon_1^2+1)(\epsilon_1^2+0.6) + 3.2\epsilon_1^3]a^2} . \quad \text{(A10)}$$

[1] R. L. Gluckstern, M. H. Hull, and G. Breit, Phys. Rev. 90, 1026 (1953).
[2] R. L. Gluckstern and M. H. Hull, Phys. Rev. 90, 1030 (1953).
[3] G. Böbel, Nuovo Cimento 6, 1241 (1957).
[4] A. Claesson, Arkiv Fysik 12, 569 (1957).
[5] K. W. McVoy, Phys. Rev. 106, 828 (L) (1957); 111, 1333 (1958).
[6] C. Fronsdal and H. Überall, Nuovo Cimento 8, 163 (1958); Phys. Rev. 111, 580 (1958).
[7] H. Banerjee, Phys. Rev. 111, 532 (1958).
[8] A. Sommerfeld, Atombau und Spektrallinien (F. Vieweg und Sohn, Braunschweig, 1939), 2nd ed., Vol. 2.
[9] H. Olsen and L. C. Maximon, Phys. Rev. 114, 887 (1959).
[10] U. Fano, H. W. Koch, and J. W. Motz, Phys. Rev. 112, 1679 (1958).
[11] U. Fano, K. W. McVoy, and J. R. Albers, Phys. Rev. 116, 1147 (1959).
[12] U. Fano, Phys. Rev. 116, 1156 (1959).
[13] K. W. McVoy and U. Fano, Phys. Rev. 116, 1168 (1959).
[14] To lowest order in $Z/137$, the bremsstrahlung cross section at the short-wavelength limit differs from the Bethe-Heitler cross section only by a normalization factor which cancels out in calculating the polarization.
[15] R. H. Pratt, Phys. Rev. 120, 1717 (1960).
[16] A. Sommerfeld and A. W. Maue, Ann. Physik 22, 629 (1935); W. H. Furry, Phys. Rev. 46, 391 (1934).
[17] S. Hultberg, B. Nagel, and P. Olsson, Arkiv Fysik 20, 555 (1961).
[18] R. H. Pratt, R. D. Levee, R. L. Pexton, and W. Aron, Phys. Rev. 134, A916 (1964).
[19] B. C. H. Nagel, Arkiv Fysik 18, 1 (1960).
[20] H. A. Bethe and L. C. Maximon, Phys. Rev. 93, 768 (1954).
[21] G. Elwert and E. Haug, Phys. Rev. 183, 90 (1969).
[22] H. A. Olsen, Application of Quantum Electrodynamics (Springer-Verlag, Berlin, 1968), Vol. 44.
[23] H. Kolbenstvedt and H. Olsen, Nuovo Cimento 22, 610 (1961).
[24] T. A. Weber and C. J. Mullin, Phys. Rev. 126, 615 (1962).
[25] Nagel's P_2 corresponds to $P_z(\hat{p}_1)$, and P_3 corresponds to $P_z(\hat{p}_1 \times \vec{e}_y)$ of this paper. The transverse polarization P_1 of photoelectrons, perpendicular to the scattering plane, from circularly polarized photons has the same value as from unpolarized photons; that implies the circular polarization of bremsstrahlung $P_z(\vec{e}_y)$ to be equal to zero, in agreement with Eq. (3.31).
[26] J. W. Motz and R. C. Placious, Nuovo Cimento 15, 571 (1960).
[27] H. Brysk, C. D. Zerby, and S. K. Penny, Phys. Rev. 180, 104 (1969).
[28] M. Scheer, E. Trott, and G. Zahs, Z. Physik 209, 68 (1968).