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Hydrodynamics and Third Sound in Thin He II Films*

D. Bergman†

Institute for Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037

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The linearized hydrodynamic equations of motion for a thin, flat, superfluid helium film are derived in some detail from standard two-fluid hydrodynamics. Interactions of the film with both the He vapor and the substrate which are in contract with it are included and discussed in detail. Boundary conditions for both the film-substrate and film-gas interfaces are derived. It is indicated how one may construct the equations of motion for the entire coupled system (gas-film-substrate). The equations are actually constructed and solved for the case when a certain parameter is small, which includes all the third-sound experiments on unsaturated films. A dispersion equation is found which is exact in the limit of vanishing frequency, and which is eminently suited to describe both the velocity and the attenuation of third sound in the regime of unsaturated films. No hydrodynamic instability is found. Results for the attenuation are shown to be in good agreement with preliminary experiments on unsaturated He films.

I. INTRODUCTION

As a result of increasing interest recently in the properties of third sound in thin He II films,[1-5] we have undertaken to reexamine the theory of that phenomenon with the intention of trying to resolve some difficulties that were present in its original formulation by Atkins.[6] Those difficulties had to do with the theoretical expression for the coefficient of attenuation of third sound, and were of two types: (i) The attenuation became negative for films that were too thin, i. e., it changed into amplification. If this were correct it would imply that the film was then unstable against perturbations of its shape. Such speculations have indeed been made.[4] (ii) Even for thicker films, where this did not happen, the values predicted for the attenuation coefficient were much less than those observed.[1,2] Since the attenuation effects in Atkin's theory[6] arose from the reciprocal processes of evaporation and condensation of He atoms between the film and the gas, we suspected that these processes had not been adequately dealt with in that theory. In this paper, we reformulate the hydrodynamic equations for the He film, taking care to include all hydrodynamic and thermal interactions with both the substrate and the He gas that are in contact with it. Because we do this, it is unnecessary to make any assumptions about how the evaporation depends on variables of the film; it depends, in fact, on variables of both the film and the gas. Therefore, we have to consider the equations of motion not only for the film but for the gas and substrate as well. We also need to know the boundary conditions at the film-gas interface and at the film-substrate interface.

In Sec. II, we derive the equations of motion for the film. In Sec. III, we discuss the equation of motion for the substrate and its solution, as well as the boundary condition with the film. In Sec. IV, we do likewise for the gas. In Sec. V, we use the results of Secs. III and IV to solve the film equations explicitly for the case of unsaturated films. In Sec. VI, we calculate the velocity and the attenuation of third sound from the results of Sec. V, and compare the predicted values of the attenuation with the available experimental results. In Sec. VII, we discuss and summarize the results obtained and indicate further avenues of research to be pursued. In the Appendix, we use kinetic theory to derive expressions for the mass and thermal fluxes through the boundary between a liquid and its vapor that are in mechanical but not in thermodynamic equilibrium with each other. This is required to obtain the boundary conditions at the gas-film interface.

II. EQUATIONS OF MOTION FOR THE FILM

For simplicity, we will restrict our geometry to two dimensions, as described schematically in Fig. 1. (Although it would be straightforward also to include the third dimension in all of our equations, it would make the notation cumbersome.) An imaginary planar surface $y = y_r$ has been introduced in the gas, at a distance from the film sufficiently large so that any inhomogeneities due to surface effects have died out, but sufficiently small so that the gas there is still in equilibrium (locally) with the gas at the surface of the film. This is possible because the van der Waals forces exerted by the substrate and film on atoms of the gas – the forces that bring about the inhomogeneities – become negligible at a distance much less than the mean free path in the gas, over which equilibrium is assumed to be maintained at all times.

The equations of motion for the film are derived from the usual linearized equations of the two-fluid model, with the help of the following assumptions: (a) As in Atkins's theory,[6] the liquid film is assumed to be incompressible, so as to eliminate simple compression waves. (b) The viscosity is assumed to have a negligible effect on vertical motion, but to be all important for horizontal motion, because of the thinness of the film (i.e., the film thickness is assumed to be much less than the viscous penetration depth in the film). These last effects are taken into account by assuming, also after Atkins,[6] that there is no horizontal normal motion in the film, and by otherwise ignoring the equation of motion for the momentum density. (c) We neglect thermal conduction in the horizontal direction (this is so simple to justify that we even do so explicitly), but not in the vertical direction. (d) Contrary to Atkins,[6] we do not neglect thermal conduction out of the film and into the gas and the substrate, nor do we neglect the transfer of helium atoms between the film and the gas. These processes are fully taken into account in terms of the dynamic variables of the gas and the substrate, where necessary, so that, in order to have a complete set of equations, we will eventually have to consider equations of motion for variables of the gas and the substrate, too. These equations are discussed in the following sections.

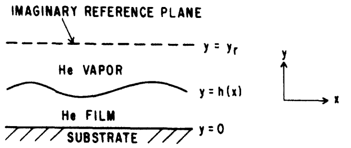

FIG. 1. Schematic drawing of a vertical section of the helium film.

Every equation for the film is obtained by integrating a two-fluid equation over the vertical coordinate (the thickness), so that we are finally left with equations of motion for an effectively two-dimensional film. In our geometry (see Fig. 1) this becomes a one-dimensional problem: only the x coordinate remains in the description of the film. Throughout the following discussion the subscript f will identify film variables, the subscript g will serve similarly for gas variables, and the subscript sub for substrate variables.

Applying the above principles to the mass conservation equation div $\vec{J}_f = 0$, where $\vec{J}_f$ is the total mass current in the film, and remembering that there is no mass flowing into the substrate, i.e., $J_{fy}(x, y=0) = 0$, we write

$$0 = \int_0^{h(x)} \operatorname{div} \vec{J}_f(x,y)\,dy = J_{fy}\big(x, h(x)\big) + h(x)\frac{\partial \bar{J}_x(x)}{\partial x} + \frac{\partial h}{\partial x}\left[\bar{J}_x(x) - J_{fx}\big(x, h(x)\big)\right], \tag{1}$$

where $h(x)$ is the film thickness at the point x, and

$$\bar{J}_x(x) \equiv \frac{1}{h(x)} \int_0^{h(x)} J_{fx}(x,y)\,dy\ .$$

The last term in (1) is of second order in small quantities (both $\partial h/\partial x$ and $\vec{J}_f$ are small) and will, therefore, be neglected, as will be all second-order terms.

To conserve mass at the film-vapor interface, the mass current in the film towards the interface must equal the mass current in the gas flowing out of the interface, if both of these currents are measured in a frame of reference that moves with the interface. In terms of quantities measured in the laboratory frame, this condition becomes

$$J_{fy}\big(x, h(x)\big) - \dot{h}(x)\rho_f\big(x, h(x)\big) = J_{gy}(x, y_r) - \dot{h}(x)\rho_g(x, y_r) \ . \tag{2}$$

Here, ρ_f and ρ_g are the total mass densities in the film and gas, and $\vec{J}_g$ is the mass current in the gas. The functions on the right-hand side refer to the gas at the reference plane. Their values are different from what they would be at $y = h(x)$, but the combination that appears on the right-hand side has the same value, due to mass conservation in the gas. Unless otherwise specified, functions of the gas will always be taken at the imaginary reference plane rather than at the interface itself.

In two-fluid hydrodynamics, the mass current is usually written as a sum of superfluid and normal contributions

$$\vec{J}_f = \rho_S \vec{v}_S + \rho_n \vec{v}_n \ ,$$

where ρ_S and ρ_n are the superfluid and normal mass densities; $\vec{v}_S$ and $\vec{v}_n$ are the superfluid and normal velocities. We are assuming $v_{nx} \equiv 0$ (no horizontal normal flow); hence

$$J_{fx} = \rho_S v_{sx} \ .$$

Combining (1) and (2) and taking this into account, we get the following equation of motion for h:

$$\dot{h}(x)\rho_f\big(x, h(x)\big) + h(x)\bar{\rho}_S(x)\frac{\partial \bar{v}_S(x)}{\partial x} + \rho_g(x, y_r)[v_{gy}(x, y_r) - \dot{h}(x)] = 0 \ . \tag{3}$$

Here we have included detailed reference to the position variables; ρ_S is the superfluid density averaged over the film thickness

$$\bar{\rho}_S(x) \equiv \frac{1}{h(x)}\int_0^{h(x)} \rho_S(x, y)dy \ ,$$

and $$\bar{v}_S(x) \equiv \frac{\bar{J}_{fx}(x)}{\bar{\rho}_S(x)} \ ,$$

$$\vec{v}_g(x, y) \equiv \frac{\vec{J}_g(x, y)}{\rho_g(x, y)} \ .$$

In defining $\bar{v}_S$, we have again used the assumption that $v_{nx} \equiv 0$.

The second-film equation is derived from the linearized entropy equation of two-fluid hydrodynamics[7]

$$\frac{\partial}{\partial t}(\rho_f s_f) + \mathrm{div}\left(\rho_f s_f \vec{v}_n - \kappa_f \frac{\vec{\nabla} T_f}{T}\right) = 0 \ ,$$

where s_f is the entropy of the film per unit mass, κ_f is the thermal-conduction coefficient for the film, T is the stationary equilibrium temperature, and T_f is the oscillating temperature of the film. When this equation is integrated over the film thickness, we get

$$0 = \frac{d}{dt}\int_0^{h(x)} \rho_f s_f dy - \dot{h}\rho_f(h)s_f(h) + \left(\rho_f s_f v_{ny} - \frac{\kappa_f}{T}\frac{\partial T_f}{\partial y}\right)_0^{h(x)} + \int_0^{h(x)} dy \frac{\partial}{\partial x}\left(\rho_f s_f v_{nx} - \frac{\kappa_f}{T}\frac{\partial T_f}{\partial x}\right) . \tag{4}$$

In order to develop this further, we must look at the consequences of energy conservation at the two film boundaries.

At the film-vapor interface, we must have equal energy currents flowing in and out of the interface, in a frame of reference moving with it. This leads to the following equation, in terms of laboratory frame variables:

$$\left(\mu(J_{fy} - \dot{h}\rho_f) + Ts_f\rho_f(v_{ny} - \dot{h}) - \kappa_f\frac{\partial T_f}{\partial y}\right)_{y=h} = \left((\mu + Ts_g)\rho_g(v_{gy} - \dot{h}) - \kappa_g\frac{\partial T_g}{\partial y}\right)_{y=y_r} \ , \tag{5}$$

where s_g is the entropy of the gas per unit mass, T_g is the oscillating temperature in the gas, κ_g is the thermal-conduction coefficient in the gas, and μ is the stationary equilibrium value of the chemical potential. In writing (5) we neglected terms that are of second order in small quantities. In this connection, temperature and chemical potential differences between the film and the gas are taken to be small in the first order. When (2) is substituted into (5) we can rewrite it as

$$\left(Ts_f\rho_f(v_{ny} - \dot{h}) - \kappa_f\frac{\partial T_f}{\partial y}\right)_{y=h}$$

$$= \left(T s_g \rho_g (v_{gy} - \dot{h}) - \kappa_g \frac{\partial T_g}{\partial y} \right)_{y = y_r} . \qquad (6)$$

At the film-substrate interface, energy conservation requires that

$$\kappa_f \frac{\partial T_f}{\partial y} \bigg|_{y=0} = \kappa_{\text{sub}} \frac{\partial T_{\text{sub}}}{\partial y} \bigg|_{y=0} ; \qquad (7)$$

where T_{sub} is the oscillating temperature of the substrate, and κ_{sub} is its thermal-condition coefficient.

We now write

$$\frac{\partial}{\partial T} \int_0^{h(x)} \rho_f s_f \, dy = \frac{\partial}{\partial T} \left(\frac{S_f}{A} \right)$$

$$= \rho_f (h) \bar{S} \dot{h} + h \bar{\rho}_f C_h \frac{\dot{T}_f}{T} , \qquad (8)$$

where A is the area of the film, S_f is its total entropy, $\bar{\rho}_f$ is the mass density of the film averaged over the thickness (as distinct from ρ_f, which is taken at the film surface $y = h$),

$$\bar{S} \equiv [\rho_f (h) A]^{-1} \left(\frac{\partial S_f}{\partial h} \right)_T$$

is the partial (as distinct from average) entropy per unit mass of the film, and

$$C_h \equiv \frac{T}{\bar{\rho}_f h A} \left(\frac{\partial S_f}{\partial T} \right)_h$$

is the average specific heat per unit mass of the film at constant thickness. Substituting (6)–(8) into (4), and remembering that $v_{nx} \equiv 0$ and that $v_{ny} = 0$ at $y = 0$, we get

$$\rho_f T \bar{S} \dot{h} + h \bar{\rho}_f C_h \dot{T}_f + \left(T s_g \rho_g (v_{gy} - \dot{h}) - \kappa_g \frac{\partial T_g}{\partial y} \right)_{y = y_r}$$

$$+ \kappa_{\text{sub}} \frac{\partial T_{\text{sub}}}{\partial y} \bigg|_{y=0} - \kappa_f \int_0^{h(x)} \frac{\partial^2 T_f}{\partial x^2} \, dy = 0 . \qquad (9)$$

In order to discuss the magnitude of the last term, which arises from horizontal thermal conduction, suppose that T_f oscillates periodically with a frequency ω and wave number k, related to each other by

$$c_3 = \omega / k ,$$

the (complex) velocity of third sound. Then the last term of (9) is smaller than the second term by a factor of the order

$$i \omega \kappa_f / \rho_f C_h c_3^2 . \qquad (9')$$

For lack of better experimental information, we approximate κ_f / C_h by η_f, the shear-viscosity coefficient of the film, whose value is known. We thus find that (9′) is very small compared to 1 in all of the experiments conducted to date on third sound[1–3]; therefore, we will ignore it. What remains of (9) may be combined with (3) to yield

$$h \bar{\rho}_f C_h \dot{T}_f - h \bar{\rho}_s T \bar{S} \frac{\partial \bar{v}_s}{\partial x}$$

$$+ \left(L \rho_g (v_{gy} - \mathring{h}) - \kappa_g \frac{\partial T_g}{\partial y} \right)_{y = y_r}$$

$$+ \kappa_{\text{sub}} \frac{\partial T_{\text{sub}}}{\partial y} \bigg|_{y=0} = 0 , \qquad (10a)$$

or

$$\dot{h} \rho_f + h \bar{\rho}_s \left(1 + \frac{T \bar{S}}{L} \right) \frac{\partial \bar{v}_s}{\partial x} - \frac{\bar{\rho}_f h C_h}{L} \dot{T}_f$$

$$+ \frac{\kappa_g}{L} \frac{\partial T_g}{\partial y} \bigg|_{y = y_r} - \frac{\kappa_{\text{sub}}}{L} \frac{\partial T_{\text{sub}}}{\partial y} \bigg|_{y=0} = 0 , \qquad (10b)$$

where $L \equiv T(s_g - \bar{S})$

is the latent heat of evaporation from the film per unit mass. This is the equation of motion for T_f.

The third equation for the film is obtained by integrating the x component of

$$\dot{\vec{v}}_s + \vec{\nabla} [\mu_f - \zeta_3 \rho_s \operatorname{div}(\vec{v}_s - \vec{v}_n) - \zeta_1 \operatorname{div} \vec{v}_n] = 0 , \qquad (11)$$

where ζ_1 and ζ_3 are coefficients of bulk viscosity,[7] and μ_f is the oscillating chemical potential of the film, to get

$$\frac{d}{dt} \int_0^{h(x)} J_{fx}(x, y) dy \equiv \frac{d}{dt} [h(x) \bar{\rho}_s(x) \bar{v}_s(x)] = - \int_0^{h(x)} dy$$

$$\times \rho_s(x, y) \left(\frac{\partial \mu_f(x, y)}{\partial x} - \zeta_3 \rho_s(x, y) \frac{\partial^2 (v_{sx} - v_{nx})}{\partial x^2} \right.$$

$$\left. - \zeta_1 \frac{\partial^2 v_{nx}}{\partial x^2} \right) + \left(\zeta_3 \rho_s^2 \frac{\partial (v_{sy} - v_{ny})}{\partial x} + \zeta_1 \rho_s \frac{\partial v_{ny}}{\partial x} \right)_0^{h(x)}$$

$$- \int_0^{h(x)} dy \left(2 \zeta_3 \rho_s \frac{\partial (v_{sy} - v_{ny})}{\partial x} + \zeta_1 \frac{\partial v_{ny}}{\partial x} \right) \frac{\partial \rho_s}{\partial y} \qquad (12)$$

We can show that all the terms in this equation that involve ζ_1 or ζ_3 are negligible: Using the symbol $\approx$ to denote equality in order of magnitude, we note that $v_{sy} \approx v_{ny} \approx \dot h$. From (3), we then find that

$$\dot h \approx -h\frac{\partial v_{sx}}{\partial x} \approx -ihkv_{sx} \ll v_{sx} , \qquad (12')$$

where k is again a typical wave number of a periodic disturbance in the film. The last inequality presupposes that the wavelength of the disturbance is always much larger than the film thickness. We assume, again for lack of better information, that $\zeta_1 \approx \rho_s \zeta_3 \approx \eta_f/\rho_f$, and also that ρ_s is nearly constant across the thickness of the film except over a distance l from the surfaces, where it varies as

$$\frac{\partial \rho_s}{\partial y} \approx \frac{\rho_s}{l} .$$

The healing or correlation length l is of the order of 1 atomic layer,[3] and this is always less than half the film thickness, $\frac{1}{2}h$. Applying these considerations, we find that all of the terms of (12) that involve either ζ_1 or ζ_3 are of order

$$\eta_f \omega/\rho_f {c_3}^2 ,$$

as compared to the left-hand side. As we have seen, this quantity is very small compared to 1, in all of the experiments conducted to date on third sound,[1-3] hence these terms will be ignored.

In order to deal with the one remaining term on the right-hand side of (12), we now turn to the other component of (11), focusing our attention on the phase relationships of the different terms. In doing this we assume that derivatives of quantities along y are in phase, while derivatives along x are out of phase with the quantities themselves. Time derivatives are, of course, out of phase. As a consequence, the equation for $\dot v_{sy}$ can be split up into two separate equations:

$$\dot v_{sy} + \frac{\partial \mu_i}{\partial y} = 0 ,$$

$$\frac{\partial \mu_0}{\partial y} - \zeta_3 \rho_s \frac{\partial^2 (v_{sx} - v_{nx})}{\partial x \partial y} - \zeta_3 \rho_s \frac{\partial^2 (v_{sy} - v_{ny})}{\partial y^2} - \zeta_1 \frac{\partial^2 v_{nx}}{\partial x \partial y} - \zeta_1 \frac{\partial^2 v_{ny}}{\partial y^2} = 0 ,$$

where we have written μ_f as a sum of an equilibrium part μ, an in-phase part μ_i (in phase with v_{sx}), and an out-of-phase part μ_0. From the first of these and from (12′), it is clear that

$$\frac{\partial \mu_i}{\partial y} \approx ihk\dot v_{sx} \approx -ihk\frac{\partial \mu_i}{\partial x} \ll \frac{\partial \mu_i}{\partial x} ,$$

and it is also clear that what is involved in the x component of (11), i.e., in (12), is essentially μ_i. Therefore, we can neglect the y dependence of μ_f and take it outside the integral in (12), thus getting the following equation of motion for $\bar v_s$:

$$\dot{\bar v}_s + \frac{\partial \mu_f(x)}{\partial x} = 0 . \qquad (13)$$

For a description of the thermodynamic properties of the film we use the variables T_f and h,[8] in terms of which we can write a differential equation of state:

$$d\mu_f = -\bar S dT_f + f dh . \qquad (13')$$

In this equation f is the van der Waals force exerted by the atoms of the substrate on atoms of the film, per unit mass of the film. Using (13′), (13) takes the following form:

$$\dot{\bar v}_s - \bar S \frac{\partial T_f}{\partial x} + f\frac{\partial h}{\partial x} = 0 , \qquad (14)$$

which is the equation of motion for $\bar v_s$.

Equations (3), (10a) [or (10b)], and (14) are the equations of motion for the film, and should be compared with Atkins's equations (8), (9), and (12), respectively, of Ref. 6. The differences that appear are: (i) the evaporation term $K(T_f - T)$ is replaced in our equations by $\rho_g(v_{gy} - \dot h)$; (ii) we have additional terms in (10a) describing thermal flux to the gas and to the substrate; (iii) the thermodynamic description of the film is different (e.g., the pressure is not an independent variable), leading to a different form for the $\dot v_s$ equation; and (iv) the dependence of film entropy on film thickness is taken into account, leading to the appearance of $\bar S$ instead of s_f.

We will be looking for wave solutions to these equations, i.e., solutions of the form

$$T_f = T + T_{f1} e^{-i\omega t + ikx} . \qquad (15)$$

This is a harmonic wave propagating through the film in the positive x direction with a (complex) velocity

$$c_3 \equiv \omega/k \ .$$

III. EQUATION OF MOTION AND SOLUTION FOR THE SUBSTRATE

The only substrate variable appearing in any of the film equations is T_{sub}, which satisfies a diffusion-type equation

$$\rho_{\text{sub}} C_{p\,\text{sub}} \dot{T}_{\text{sub}} = \kappa_{\text{sub}} \nabla^2 T_{\text{sub}} \ . \tag{16}$$

Here ρ_{sub} is the mass density of the substrate and $C_{p\,\text{sub}}$ is its specific heat at constant pressure. Since we are interested in solutions that correspond to a wave traveling through the film in the x direction, we look for a solution to (16) of the form

$$T_{\text{sub}} = T + T_{\text{sub}\,1} e^{-i\omega t + ikx + qy} \ , \tag{17}$$

where $\operatorname{Re} q > 0$,

and where k and ω are the same as in (15). The condition $\operatorname{Re} q > 0$ ensures that any disturbance in the substrate dies out as one moves away from the film. The substrate is assumed to be infinitely thick (i.e., compared to q^{-1}) so that there are no reflections to worry about. We find for q the following dispersion equation:

$$c_3^{-2} - q^2/\omega^2 = i\rho_{\text{sub}} C_{p\,\text{sub}}/\omega\kappa_{\text{sub}} \ . \tag{18}$$

The boundary condition at the film-substrate interface is

$$-\kappa_{\text{sub}} \frac{\partial T_{\text{sub}}}{\partial y}\bigg|_{y=0} = B_1 (T_{\text{sub}} - T_f)\Big|_{y=0} \ , \tag{19}$$

where B_1^{-1} is the Kapitza resistance (thermal boundary resistance) at the interface. Substituting (17) and (15) in (19), and using (18), we can solve for $T_{\text{sub}1}$ in terms of T_{f1}. We can then write, instead of (19),

$$-\kappa_{\text{sub}} \frac{\partial T_{\text{sub}}}{\partial y}\bigg|_{y=0} = -B(T_f - T) \ , \tag{20}$$

where

$$\frac{1}{B} \equiv \frac{1}{B_1} + \left[\left(\frac{\omega\kappa_{\text{sub}}}{c_3}\right)^2 - i\omega\kappa_{\text{sub}}\rho_{\text{sub}}C_{p\,\text{sub}}\right]^{-1/2} , \tag{21}$$

and the real part of the square root is taken to be positive.

The question of whether an equation of the type (16) is applicable, is answered by comparing the mean free path for thermal conduction,

$$l_f \equiv 3\kappa_{\text{sub}}/\rho_{\text{sub}}C_{p\,\text{sub}}c_{\text{sub}} \ ,$$

with the thermal diffusion length,

$$l_d \equiv (\kappa_{\text{sub}}/\omega\rho_{\text{sub}}C_{p\,\text{sub}})^{1/2} \ .$$

Here c_{sub} is the phonon velocity in the substrate. Putting in the numbers for a glass substrate at 1.5 °K, we find that even at a frequency of 20 kHz (the highest used in experiments to date[2]) l_d is 50 times greater than l_f, which means that (16) can safely be used. However, for the low frequency (100 Hz) used in some experiments,[1] l_d becomes a few millimeters, in which case it may be comparable to the thickness of the substrate. In that case the detailed geometry of the substrate would have to be considered, as well as boundary conditions on its other surfaces, and one would not be able to assume the simple form (17) for the solution of (16).

IV. EQUATIONS OF MOTION AND SOLUTION FOR THE GAS

The equations of the gas that are required are the linearized equations of hydrodynamics including thermal conduction and viscosity; namely,

$$\frac{\partial \rho_g}{\partial t} + \rho_g \operatorname{div}\vec{v}_g = 0 \ ,$$

$$\rho_g \frac{\partial \vec{v}_g}{\partial t} + \vec{\nabla} P_g - \eta_g \nabla^2 \vec{v}_g - (\zeta_g + \tfrac{1}{3}\eta_g)\vec{\nabla}\operatorname{div}\vec{v}_g = 0 \ , \tag{22}$$

$$\rho_g T \frac{\partial s_g}{\partial t} - \kappa_g \nabla^2 T_g = 0 \ ;$$

and the equation of state, which we take to be that of an ideal gas,

$$P_g = \rho_g k_B T_g / m \ . \tag{23}$$

Here P_g is the pressure of the gas, m is the mass of one He atom, k_B is Boltzmann's constant, and η_g and ζ_g are the shear and bulk viscosity coefficients of the gas. From considerations similar to those made in Sec III, we look for a solution to

(22) where all the variables have the form

$$T_g = T + T_{g1} e^{-i\omega t + ikx - qy} \;, \quad \text{where } \mathrm{Re}\, q > 0 \;. \tag{24}$$

Substituting this form in (22) leads, of course, to the general dispersion equation for small amplitude waves in the gas:

$$0 = (x - i\rho_g/\omega\eta_g)\Big\{x^2[1 - (i\omega/P_g)(\zeta_g + \tfrac{4}{3}\eta_g)] - x\left[\frac{i\rho_g C_{pg}}{\kappa_g \omega} + \frac{C_{pg}}{\kappa_g c^2}\left(\frac{\gamma\kappa_g}{C_{pg}} + \zeta_g + \tfrac{4}{3}\eta_g\right)\right] + \frac{i\rho_g C_{pg}}{\kappa_g \omega c^2}\Big\} \;, \tag{25}$$

where $x \equiv c_3^{-2} - q^2/\omega^2$,

c is the velocity of ordinary sound in the gas, C_{pg} and C_{vg} are the specific heats of the gas at constant pressure and constant volume, and

$$\gamma \equiv C_{pg}/C_{vg} \;.$$

In view of (23), c is given by

$$c^2 = \gamma k_B T/m \;.$$

Equation (25) implies, besides ordinary acoustic waves, a viscous wave and a thermal-conduction wave. Which of these are excited in any given situation depends, of course, on the specific details of the situation. In our case, a general linear combination of the three modes will be assumed to begin with, since it is not clear *a priori* that any of them can be ignored in the final solution. The three modes have the following dispersion equations:

$$M_1\text{: } x_1 = i\rho_g/\omega\eta_g \;\text{(viscous mode)} \;,$$

$$M_2\text{: } x_2 = (i\rho_g C_{pg}/\kappa_g\omega)[1 + O(\kappa_g\omega/\rho_g C_{pg} c^2)] \;\text{(thermal-conduction mode)} \;, \tag{26}$$

$$M_3\text{: } x_3 = 1/c^2 + (i\omega/\rho_g c^4)[(\gamma - 1)\kappa_g/C_{pg} + \zeta_g + \tfrac{4}{3}\eta_g]\,[1 + O(\kappa_g\omega/\rho_g C_{pg} c^2)] \;\text{(acoustic mode)} \;.$$

Rather than write the exact solutions of (25), we have expanded the second- and third-mode solutions in powers of the parameter

$$\kappa_g\omega/\rho_g C_{pg} c^2 \;, \tag{27}$$

which is of the same order as

$$\eta_g\omega/\rho_g c^2 \;. \tag{28}$$

Both are very small for all the frequencies that have ever been used in thin-film experiments,[1-3] and we will henceforth always ignore terms of this order as compared to unity.

The hydrodynamic equations are applicable whenever the wavelengths involved in the various modes are large compared to the mean free path for thermal conduction in the gas:

$$l_g \equiv (3\kappa_g/\rho_g C_{pg})(m/3k_B T)^{1/2} \;.$$

The wavelengths involved are the viscous penetration depth $(\eta_g/2\omega\rho_g)^{1/2}$ for the mode M_1, the thermal-conduction length $(\kappa_g/2\omega\rho_g C_{pg})^{1/2}$ for M_2, and k^{-1} for M_3. All three are much greater than l_g.

In each of the three gas modes the amplitudes of the various oscillating quantities are related to each other in a definite way, so that one can express any variable of the gas in terms of three independent amplitudes. We will denote by suffixes 1, 2, and 3 amplitudes that belong to one of these modes, while primed variables will denote the sum of all three contributions. In terms of the amplitudes v'_{gx}, T'_g, and T_{g3}, some of the gas variables which we shall need are written as follows:

$$\rho'_g = -\frac{\rho_g}{T} T'_g + \frac{\rho_g}{T}\frac{C_{pg}}{k_B/m} T_{g3} \;, \tag{29}$$

$$-\kappa_g \frac{\partial T'_g}{\partial y} = \frac{\kappa_g \omega}{c_{02}} T'_g + \kappa_g \omega\left(\frac{1}{c_{03}} - \frac{1}{c_{02}}\right) T_{g3} \;, \tag{30}$$

$$v'_{gy} = \frac{ic_{01}}{c_3} v'_{gx} + \frac{T'_g}{T}\left(ic_{02} + \frac{(c_{01} - c_{02})\omega}{\rho_g c_3^{\,2}} \times \frac{\kappa_g T/c^2 + \zeta_g + \frac{1}{3}\eta_g - (\zeta_g + \frac{4}{3}\eta_g)/(\gamma - 1)}{1 - \eta_g C_{pg}/\kappa_g}\right) + \frac{T_{g3}}{T}\left(ic_{03} T C_{pg} u - ic_{02} - \frac{ic_{01} T C_{pg}}{c_3^{\,2}}\right), \tag{31}$$

where $u^{1/2} \equiv (1/c_3^{\,3} - 1/c^2)^{1/2}$; $\mathrm{Re}\, u^{1/2} > 0$, (32) and $\quad 1/c_{0i} \equiv (1/c_3{}^2 - x_i)^{1/2}\ ;\quad \mathrm{Re}\, c_{0i} > 0\ ;$

$$i = 1, 2, 3\ . \tag{33}$$

When not only (27) and (28), but also the parameters

$$k_g\omega/\rho_g C_{pg} c_3{}^2,\quad \eta_g\omega/\rho_g c_3{}^2\ , \tag{34}$$

are small, we can write more explicit expressions for some of the gas variables. In this way, instead of (30) and (31), we get

$$-\kappa_g \frac{\partial T_g'}{\partial y} = -ie^{i\pi/4}(\kappa_g\omega\rho_g C_{pg})^{1/2} T_g' + [\kappa_g\omega u^{1/2} + ie^{i\pi/4}(\kappa_g\omega\rho_g C_{pg})^{1/2}]T_{g3}\ , \tag{35}$$

$$v_{gy}' = \frac{ic_{01}}{c_3} v_{gx}' + ie^{i\pi/4}\left(\frac{\kappa_g\omega}{\rho_g C_{pg}}\right)^{1/2} \frac{T_g'}{T} + \left[iu^{1/2} TC_{pg} - ie^{i\pi/4}\left(\frac{\kappa_g\omega}{\rho_g C_{pg}}\right)^{1/2} - ie^{i\pi/4}\left(\frac{\eta_g\omega}{\rho_g}\right)^{1/2} \frac{TC_{pg}}{c_3{}^2}\right] \frac{T_{g3}}{T}\ , \tag{36}$$

where terms, like (34), have been neglected as compared to 1.

Since the gas variables depend on three independent amplitudes, we need three boundary conditions at the film-gas interface in order to determine them. One of these is a condition on the horizontal velocity in the gas v_{gx}'. The horizontal superfluid flow in the film is assumed to take place without any impediment from either the substrate or the gas. Consequently, there is no restriction on $v_{sx} - v_{gx}$. But the horizontal normal velocity in the film must join continuously with the horizontal velocity in the gas. Since we have assumed in Sec. II that $v_{nx} \equiv 0$, we must now also require

$$v_{gx}' = 0 \tag{37}$$

at the interface.

The other two boundary conditions arise from conservation of mass and energy at the interface. In the Appendix, we show from simple kinetic-theory arguments that the net mass flux from film to gas, J_M, is given by

$$J_M(1 - \rho_g/\rho_f) = 2\rho_g(m/2\pi k_B T)^{1/2} \times [\mu_f - \mu_g + (s_g - k_B/2m)(T_f - T_g)]\ , \tag{38}$$

while the net thermal flux from film to gas J_Q is given by

$$J_Q = -\tfrac{1}{2}\rho_g\left(\frac{k_B T}{2\pi m}\right)^{1/2}\left[\mu_f - \mu_g + \left(s_g - \frac{9}{2}\frac{k_B}{m}\right) \times (T_f - T_g)\right]\ . \tag{39}$$

Here μ_g is the oscillating chemical potential of the gas. The two boundary conditions are obtained by setting $J_M = \rho_g(v_{gy}' - \dot{h})$ and $J_Q = -\kappa_g \partial T_g'/\partial y$ at the film-gas interface. Using the differential equation of state for the gas,

$$d\mu_g = -s_g dT_g + \frac{1}{\rho_g} dP_g = \left(-s_g + \frac{k_B}{m}\right) dT_g + \frac{k_B T}{m}\frac{d\rho_g}{\rho_g}\ , \tag{39$'$}$$

and the analogous equation of state for the film Eq. (13′), we find that for small deviations from equilibrium the following equations must hold at the gas-film boundary:

$$\left(1 - \frac{\rho_g}{\rho_f}\right)\rho_g(v_{gy}' - \dot{h}) = \frac{4m}{k_B T} A\left[\left(\frac{L}{T} - \frac{k_B}{2m}\right) T_{f1} - \frac{k_B}{2m} T_g' - \frac{k_B T}{m}\frac{\rho_g'}{\rho_g} + fh_1\right]\ , \tag{40}$$

$$-\kappa_g \frac{\partial T_g'}{\partial y} = -A\left[\left(\frac{L}{T} - \frac{9}{2}\frac{k_B}{m}\right) T_{f1} + \frac{7}{2}\frac{k_B}{m} T_g' - \frac{k_B T}{m}\frac{\rho_g'}{\rho_g} + fh_1\right]\ , \tag{41}$$

where $\quad A \equiv \tfrac{1}{2}\rho_g(k_B T/2\pi m)^{1/2}\ . \tag{42}$

V. THIRD-SOUND SOLUTION IN UNSATURATED FILMS

In order to look for the third-sound solution, we assume the form (15) for variables of the film and use the expressions (29)–(31) for variables of the gas. We must consider the equations of motion for the film (3), (10b), and (14), as well as all of the boundary conditions with the substrate and the gas. The boundary condition at the substrate (20), as well as one of the boundary conditions with the gas, i.e., (37), can be incorporated in the other equations very easily to determine T_{sub} and v'_{gx}. We are thus left with five nontrivial homogeneous linear equations for the variables h_1, v_{s1}, T_{f1}, T'_g, and T_{g3}. In general these equations are too complicated to warrant the effort needed to get an explicit analytic solution. Moreover, once obtained, that solution would in general be so complicated as to lack any practical value. But when c_3 is large enough so that

$$\kappa_g\omega/\rho_g C_{pg} c_3^2 \ll 1 \quad \text{and} \quad \eta_g\omega/\rho_g c_3^2 \ll 1 \ , \tag{43}$$

the equations can be sufficiently simplified to make an explicit solution worthwhile. This condition is satisfied when ω/c_3^2 is small enough, which it always happens to be in the unsaturated film experiments of Rudnick's group.[2,3]

Under these conditions the equations become

$$(1-\rho_g/\rho_f)h_1-(h\bar{\rho}_s/c_3\rho_f)v_{s1}-(e^{i\pi/4}/TC_{pg}\rho_f)(\rho_g C_{pg}\kappa_g/\omega)^{1/2}T'_g-(u^{1/2}C_{pg}/\omega)(\rho_g/\rho_f)J_1 T_{g3}=0 \ , \tag{44}$$

$$h_1-(h\bar{\rho}_s/c_3\rho_f)(1+T\bar{S}/L)v_{s1}-(1/\rho_f L)(iB/\omega+\bar{\rho}_f hC_h)T_f-(e^{i\pi/4}/\rho_f L)(\rho_g C_{pg}\kappa_g/\omega)^{1/2}(T'_g-J_2 T_{g3})=0 \ , \tag{45}$$

$$v_{s1}+(\bar{S}/c_3)T_{f1}-(f/c_3)h_1=0 \ , \tag{46}$$

$$\begin{aligned}&[1-(i\omega P_g/4Af)(1-\rho_g/\rho_f)]fh_1+(L/T-k_B/2m)T_{f1}+(k_B/2m)[1-(i\omega/2AC_{pg})e^{i\pi/4}(\rho_g C_{pg}\kappa_g/\omega)^{1/2}\\&\quad\times(1-\rho_g/\rho_f)]T'_g-(iu^{1/2}P_g C_{pg}/4A)[J_1(1-\rho_g/\rho_f)-4iA/P_g u^{1/2}]T_{g3}=0 \ ,\end{aligned} \tag{47}$$

$$\begin{aligned}&fh_1+(L/T-\tfrac{9}{2}k_B/m)T_{f1}+\tfrac{9}{2}k_B/m[1-\tfrac{2}{9}(i\omega m/Ak_B)e^{i\pi/4}(\rho_g C_{pg}\kappa_g/\omega)^{1/2}]T'_g\\&\quad-C_{pg}[1-(i\omega J_2/AC_{pg})e^{i\pi/4}(\rho_g C_{pg}\kappa_g/\omega)^{1/2}]T_{g3}=0,\end{aligned} \tag{48}$$

where
$$J_1\equiv 1-(1/c_3^2)e^{i\pi/4}(\omega\eta_g/\rho_g u)^{1/2}-(TC_{pg})^{-1}e^{i\pi/4}(\omega\kappa_g/\rho_g C_{pg}u)^{1/2}+O(\kappa_g\omega/\rho_g C_{pg}c_3^2) \ , \tag{49}$$

$$J_2\equiv 1-e^{i\pi/4}(\omega\kappa_g u/\rho_g C_{pg})^{1/2}+O(\kappa_g\omega/\rho_g C_{pg}c_3^2) \ . \tag{50}$$

The Eqs. (44)–(48) were derived from Eqs. (3), (10b), (14), (40), and (41), respectively, by straightforward substitution from (15), (20), (29), and (35)–(37). When their determinant is set equal to zero, the following dispersion equation results for c_3:

$$\begin{aligned}&1-\frac{hf\bar{\rho}_s}{c_3^2\rho_f}\left(1+\frac{T\bar{S}}{L}\right)^2-\left(\frac{T\bar{S}}{L}\right)^2+\left(1-\frac{J_3}{J_4}\right)\left(\frac{T\bar{S}}{L}-\frac{\rho_g}{\rho_f}\right)\left(1+\frac{T\bar{S}}{L}\right)+\frac{J_3}{J_4}\frac{\rho_g}{\rho_f}\frac{T\bar{S}}{L}\left(1+\frac{T\bar{S}}{L}\right)\\&+\frac{hf\bar{\rho}_s}{c_3^2\rho_f}\frac{T\bar{S}}{L}\left(1+\frac{T\bar{S}}{L}\right)\frac{9}{2}\frac{k_BT/m}{L}(J_5-1)\left(1+\frac{9}{2}\frac{k_BT/m}{L}(J_5-1)\right)^{-1}+\frac{J_3}{J_4}\left(1+\frac{T\bar{S}}{L}\right)\left(1+\frac{9}{2}\frac{k_BT/m}{L}(J_5-1)\right)^{-1}\end{aligned}$$

$$\times \frac{Tf}{\rho_f L^2}\left\{\frac{iB}{\omega}+\bar{\rho}_f hC_h+\frac{J_5}{J_6}e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\left[1-\frac{L}{TC_{pg}}\left(1+\frac{T\bar{S}}{L}-\frac{J_4}{J_3}\right)\right]\right\}=0\ , \tag{51}$$

where $J_3 \equiv J_1-(\lambda\omega J_7/u^{1/2}f)\rho_f/\rho_g$,

$$J_4 \equiv J_8+(\omega J_7/u^{1/2}f)(-(\rho_f/\rho_g)T\bar{S}/L+1+T\bar{S}/L)\ ,$$

$$J_5 \equiv J_6\left\{1+\left(1-\frac{9}{2}\,\frac{k_B T/m}{L}\right)\frac{i\omega\rho_g L}{16Af}\left[\frac{J_9}{J_4}\frac{\rho_f}{\rho_g}\frac{T\bar{S}}{L}\left(1-\frac{\rho_g}{\rho_f}\right)-\frac{J_9}{J_4}+1-\frac{\rho_g}{\rho_f}\right]-\frac{i\omega T}{16AL}\frac{J_9}{J_4}\left(\frac{iB}{\omega}+\bar{\rho}_f hC_h\right)\right\}$$

$$\times\left\{1-\frac{9}{32}\,\frac{k_B T/m}{L}\,\frac{i\omega\rho_g LJ_6}{Af}\left[\frac{J_9}{J_4}\frac{\rho_f}{\rho_g}\frac{T\bar{S}}{L}\left(1-\frac{\rho_g}{\rho_f}\right)-\frac{J_9}{J_4}+1-\frac{\rho_g}{\rho_f}\right]+\frac{i\omega T}{16AL}e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\right.$$

$$\left.\times\left[\left(1-\frac{L}{TC_{pg}}\left(1+\frac{T\bar{S}}{L}\right)\right)\frac{J_9}{J_4}+\frac{L}{TC_{pg}}\left(1-\frac{\rho_g}{\rho_f}-\frac{4C_{pg}}{k_B/m}\right)\right]\right\}^{-1}\ ,$$

$$J_6 \equiv 1-\tfrac{2}{9}\,(i\omega m/Ak_B)e^{i\pi/4}(\rho_g C_{pg}\kappa_g/\omega)^{1/2}\ ,$$

$$J_7 \equiv 1-(i\omega J_2/AC_{pg})e^{i\pi/4}(\rho_g C_{pg}\kappa_g/\omega)^{1/2}\ ,$$

$$J_8 \equiv J_1(1+T\bar{S}/L)+(\omega J_2/\rho_g C_{pg}u^{1/2}L)e^{i\pi/4}(\rho_g C_{pg}\kappa_g/\omega)^{1/2}\ ,$$

$$J_9 \equiv \left(1-\frac{\rho_g}{\rho_f}\right)\left\{J_1+\frac{\omega}{u^{1/2}f}\left[1-\frac{i\omega J_2}{AC_{pg}}e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\right]\right\}+\frac{4\omega J_2}{P_g C_{pg}u^{1/2}}e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\ ,$$

$$\lambda \equiv 1-hf\bar{\rho}_s/c_3{}^2\rho_f-\rho_g/\rho_f\ . \tag{52}$$

Over the whole region of unsaturated films investigated by Rudnick's group[2, 3] all the J's are very nearly 1, while λ is of the order 1. Consequently, an excellent approximation to (51) is then

$$(hf\bar{\rho}_s/c_3{}^2\rho_f)(1+T\bar{S}/L)^2=1+(Tf/\rho_f L^2)[iB/\omega+\bar{\rho}_f hC_h+e^{i\pi/4}(\rho_g C_{pg}\kappa_g/\omega)^{1/2}]\ . \tag{53}$$

It is not difficult to convince oneself that this is also the correct dispersion equation in the limit $\omega \to 0$. No inconsistency is involved in taking this limit, because even though $c_3 \to 0$, the ratio $\omega/c_3{}^2$ also tends to zero, [according to (53) and (21)] so that (43) is satisfied in the limit.

Because $\mathrm{Re}B > 0$ [see Eq. (21)], it is clear that $\mathrm{Im}(1/c_3{}^2) > 0$. Consequently, since $\mathrm{Re}(1/c_3)$ has to be positive, we find that wherever (53) is valid,

$$\mathrm{Im}(1/c_3) > 0\ ,$$

which means that the third-sound wave is attenuated. This is in contrast with the expression obtained by Atkins,[6] and modified slightly by Goodstein,[4] which predicts that $\mathrm{Im}(1/c_3) < 0$ for sufficiently thin films. There have been speculations that the latter result may indicate an inherent hydrodynamic instability of the film when it is thin enough, thus explaining the abrupt disappearance of third sound or superfluid flow that has been found to occur in such films as soon as their thickness is reduced below some critical value.[2,3,9] What we have shown here is that hydrodynamically the film is perfectly stable against small, low-frequency perturbations of its shape.

The amplitudes of the various oscillating variables in the third-sound wave are related to each other as follows:

$$v_{s1}/c_3=(hf/c_3^2)(1+T\bar{S}/L)h_1/h\ , \tag{54}$$

$$T_{f1}/T=-(hf/L)h_1/h\ , \tag{55}$$

$$T_g'-T_{f1}=T_{f1}\frac{i\omega\rho_f L}{16Af}\left(\frac{1-\rho_g/\rho_f}{1+T\bar{S}/L}\right)\left\{\frac{T\bar{S}}{L}-\frac{Tf}{\rho_f L^2}\left[\frac{iB}{\omega}+\bar{\rho}_f hC_h+e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\left(1-\frac{4L}{k_B T/m}\frac{1+T\bar{S}/L}{1-\rho_g/\rho_f}\right)\right]\right\}, \tag{56}$$

$$T_{g3}=T_{f1}\frac{\omega}{u^{1/2}f}\frac{L}{TC_{pg}}\frac{\rho_f}{\rho_g}\frac{1}{1+T\bar{S}/L}\left\{-\frac{T\bar{S}}{L}+\frac{\rho_g}{\rho_f}\left(1+\frac{T\bar{S}}{L}\right)+\frac{Tf}{\rho_f L^2}\left[\frac{iB}{\omega}+\bar{\rho}_f hC_h+e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\right.\right.$$

$$\left.\left.\times\left(1-\frac{L}{TC_{pg}}\left(1+\frac{T\bar{S}}{L}\right)\right)\right]\right\}. \tag{57}$$

The first of these is nearly the same as the relationship found by Atkins,[6] since it depends mainly on the mass conservation equation. The second one is quite different from Atkins's[6] result, however, and it means that the amplitude of the temperature oscillations inherent in a third-sound wave will be much larger than his expression would lead us to believe. The third and fourth relationships are of interest mainly in order to show that

$$T_g'-T_{f1}\ll T_{f1}\quad\text{and}\quad T_{g3}\ll T_{f1}\ . \tag{58}$$

Let us now calculate $\mu_f-\mu_g$, using (13′), (39′), and (29)

$$\mu_f-\mu_g=fh_1+(L/T)T_g'-C_{pg}T_{g3}-\bar{S}(T_{f1}-T_g')\ . \tag{59}$$

Keeping only the leading terms in the coefficients of (48) and using the fact that $T_f\cong T_g$, we now find that $\mu_f\cong\mu_g$, too. With this hindsight we now know that instead of (38) and (39) we could have used

$$T_f=T_g,\quad \mu_f=\mu_g\ , \tag{60}$$

as boundary conditions to get the solution that we got.

The second inequality in (58) serves to motivate a reexamination of the roles of the various external modes in the makeup of the third-sound wave. Making use of (57) we find that, *a posteriori*, all of the T_{g3} terms in Eqs. (44) and (45) are very small. Likewise, M_1 only makes a contribution that is implicit in the coefficient J_1, where it only amounts to a small correction. If we completely ignore all of the T_{g3} terms and if we set $T_g=T_f$, Eqs. (44)–(46) become a complete set of equations for h_1, v_{s1}, and T_{f1}, whose solution again leads to the result (53), albeit in a less tortuous way. From the final result (53), it is also clear that the only external modes that play a significant role in determining the dispersion equation for third sound are M_2 in the gas and the thermal conduction mode in the substrate. We wish to stress very strongly that this does not imply that M_1 and M_3 are not excited in a third-sound wave. They have to be excited in order to satisfy the various boundary conditions. In Table I, we list the principal contributing modes to amplitudes of some of the gas variables in the third-sound solution presented here. This listing describes conditions near the gas-film interface. Further away, the relative importance of different contributions changes of course, since M_1 and M_2 decay over a much shorter distance than M_3.

VI. COMPARISON OF THEORY WITH EXPERIMENTS

If we substitute some numbers into (53), we find first that, in the experiments by Atkins's group on nearly saturated helium films,[1] all of the terms on the right-hand side are very small except for the unity. Since $T\bar{S}/L$ is small, we find for the velocity of third sound nearly the same result as Atkins.[6] The result for attenuation is entirely different, however, even in its frequency dependence. In absolute value it is about two orders of magnitude less than the experimentally observed attenuation,[1] and, besides that, it increases for decreasing film thickness, also in contradiction with experiment.[1] In view of some of the approximations we have made to get (53), it is probably not applicable to most of the saturated-film experiments, so that the disagreement may not be significant.

When (53) is applied to the thin unsaturated films that were used by Rudnick's group,[2] the situation looks much better. Since the frequencies used are higher and the films thinner, we calculate larger values for the attenuation. Also, for the thinnest films observed, the calculated velocity is considerably different from what one would expect if one neglected the interactions of the film with the gas and substrate.

Since we do not wish to discuss here the problem of what $\bar{\rho}_S$ should be for thin films,[3] we present our results in a way independent of $\bar{\rho}_S$. We neglect the term $(\kappa_{\mathrm{sub}}\omega/c_3)^2$ in (21) [it gives a rather small contribution in the experiments we are citing], as well as the term $hC_h\bar{\rho}_f(Tf/L^2\bar{\rho}_f)$ in (53) [it is negligible], and write (53) as follows:

$$\frac{hf\bar{\rho}_S}{c_3{}^2\rho_f}\left(1+\frac{T\bar{S}}{L}\right)^2 = 1+\frac{Tf}{L^2\rho_f}\,e^{i\pi/4}\left\{\left(\frac{\kappa_g\rho_g C_{pg}}{\omega}\right)^{1/2}+\left(\frac{\kappa_{\mathrm{sub}}\rho_{\mathrm{sub}}C_{p\mathrm{sub}}}{\omega}\right)^{1/2}\right.$$
$$\left.\times B_1\left[B_1 - i\omega e^{i\pi/4}\left(\frac{\kappa_{\mathrm{sub}}\rho_{\mathrm{sub}}C_{p\mathrm{sub}}}{\omega}\right)^{1/2}\right]^{-1}\right\}. \qquad (61)$$

We define c_{30} by

$$(hf\bar{\rho}_S/c_{30}{}^2\rho_f)(1+T\bar{S}/L)^2 = 1 \quad . \qquad (62)$$

Then the right-hand side of (61) is enough to determine the ratio c_{30}/c_3, while c_{30} itself depends on $\bar{\rho}_S$.

Since the (real) velocity of third-sound u_3 is given by

$$u_3 = [\mathrm{Re}(1/c_3)]^{-1} = \frac{c_{30}}{\mathrm{Re}(c_{30}/c_3)} \quad ,$$

while the coefficient of attenuation α is given by

$$\alpha \equiv 2\omega\,\mathrm{Im}(c_3{}^{-1}) = c_{30}{}^{-1}\times 2\omega\,\mathrm{Im}(c_{30}/c_3) \quad ,$$

we present our results by plotting the two quantities $u_3/c_{30} \equiv 1/\mathrm{Re}(c_{30}/c_3)$ and $\alpha c_{30} \equiv 2\omega\,\mathrm{Im}(c_{30}/c_3)$ as functions of the film thickness h. This has been done in Fig. 2 for $T = 1.51$ °K and for two frequencies 1 kHz and 5 kHz. Despite the appearance of B_1 in (61), which cannot be neglected, α is still approximately proportional to $\omega^{1/2}$, except for the thinnest films.

We can compare our calculations to measurements published in Ref. 2 if we take into consideration the fact that the results given there are for the coefficient of attenuation of the amplitude of third sound,[10] rather than the energy or amplitude squared, which is the form of our results. In order to compare with our results, therefore, we double the results from Ref. 2 to get

$$\alpha_{\mathrm{expt}} = 0.6\ \mathrm{cm}^{-1} \quad \text{for} \quad \nu = 1\ \mathrm{kHz} \;, \qquad (\nu \equiv \omega/2\pi)$$

$$\alpha_{\mathrm{expt}} = 3.6\ \mathrm{cm}^{-1} \quad \text{for} \quad \nu = 20\ \mathrm{kHz} \;,$$

when the film is six atomic layers thick and $T = 1.5$ °K. Since $u_3 \cong 3000$ cm/sec under these conditions,[2,3] we find from Fig. 2(b) that our calculation gives, under the same conditions,

$$\alpha_{\mathrm{theoret}} = 1\ \mathrm{cm}^{-1} \quad \text{for} \quad \nu = 1\ \mathrm{kHz} \;.$$

From the approximate proportionality to $\nu^{1/2}$ we get also

$$\alpha_{\text{theoret}} = 4.5\ \text{cm}^{-1} \quad \text{for} \quad \nu = 20\ \text{kHz} \ .$$

These are in good agreement with the experimental results.

VII. SUMMARY AND DISCUSSION

We have derived the linearized hydrodynamic quations of motion for a thin superfluid (i.e., atisfying the equations of two-fluid hydrodynamics) ilm in contact with its own vapor on one side and ith a flat solid substrate on the other side. These quations were solved explicitly for the case of a eriodic wave propagating horizontally in an unaturated film. Inclusion of the interactions with he substrate and the gas was found to be absolutely ssential in order to get a stable solution. A imple dispersion equation [Eq. (53)] was derived hat is valid in the regime of unsaturated He films, nd over a wide range of frequencies, including hose used in all the unsaturated-film experiments to date.[2] This equation becomes exact in the imit of zero frequency. Some numerical results btained from this equation were presented and ound to be in agreement with preliminary measurements of the attenuation in unsaturated films.[1] Equation (53) was found to disagree with experimental results on the attenuation in saturated films.[1] However, in the experiments described in Ref. 1, the assumptions of (43) are not always satisfied and, even when they are, some of the other assumptions made to get (53) are not (e.g., not all the J's are equal to 1). As a consequence, the algebra becomes much more complicated, and it is probably unprofitable and unilluminating to seek an explicit solution. Rather than do that, one should go back to the original equations (3), (10b), (14), (40), and (41), and use (30) and (31), rather than (35) and (36), to substitute for the gas variables. The resulting equations should then be solved numerically in the regime of saturated films.

TABLE I. Principal contributors to various gas variables.

Gas variable	Principal contributing modes
$T,\ \rho,\ \frac{\partial T}{\partial y}$	M_2
v_y	$M_2,\ M_3$
v_x	$M_1,\ M_3$

ACKNOWLEDGMENTS

This work was motivated by the very beautiful experiments performed by Dr. I. Rudnick and his associates, Dr. R. S. Kagiwada, J. C. Fraser, and Dr. E. Guyon. I am indebted to all of them for providing the experimental background for it.

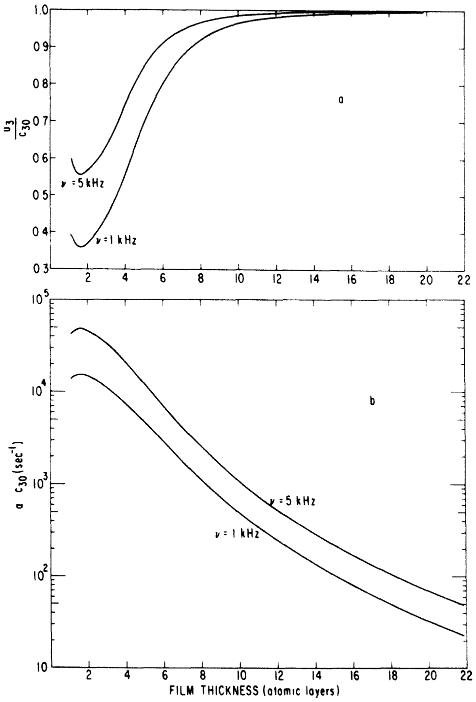

FIG. 2. (a) Plot of u_3/c_{30} versus film thickness at $T = 1.51\,^\circ\text{K}$ and frequencies of 1 kHz and 5 kHz. (b) Plot of αc_{30} versus film thickness under the same conditions. The right-hand side of (61) was used to calculate these functions. Most of the physical constants required to evaluate them are standard published data. B_1 was taken from K. Fokkens, K. W. Taconis, and R. DeBruyn Ouboter [Physica 32, 2129 (1966)], where it is given for thin helium films on a copper rather than a glass substrate as used in Ref. 2. For f we used [see Ref. 2 and W. D. McCormick, D. L. Goodstein, and J. G. Dash, Phys. Rev. 168, 249 (1968)] $f \equiv \partial\mu_f/\partial h$, where $\mu_f = \mu_{\text{bulk}} - \sigma/h^3$, and $\sigma = 87\ k_B/m$, when h is measured in atomic layers.

But most of all, I am indebted to Dr. Rudnick for his constant encouragement, and for many helpful discussions that we had while it was in process.

APPENDIX: KINETIC THEORY OF MASS AND ENERGY TRANSFER BETWEEN LIQUID AND GAS

Consider the case of a one-component system which includes a liquid and a gas phase separated by a stationary interface. The liquid and the gas both move perpendicularly to that interface with the velocities u_l and u_g, respectively (see Fig. 3). Suppose further that the mechanical pressure is the same on both sides of the interface (otherwise a shock wave would develop), but that there is a discontinuity in the temperature and the chemical potential. Our problem is to calculate the net mass and energy fluxes, J_M and J_E, respectively, through the interface.

To that end we will calculate separately the outgoing currents from each phase. The currents flowing out of the gas, $J_M{}^g$ and $J_E{}^g$, are given by simple kinetic theory as

$$J_M{}^g = P_g(m/2\pi k_B T_g)^{1/2} - \tfrac{1}{2}\rho_g u_g\,, \tag{A1}$$

$$J_E{}^g = 2P_g(k_B T_g/2\pi m)^{1/2} - \tfrac{5}{4} P_g u_g\,, \tag{A2}$$

when u_g is small. Here P_g, ρ_g, and T_g are the actual pressure, density, and temperature of the gas. For the liquid, simple kinetic theory is not directly applicable. We will assume, however, that the currents out of the liquid are unaffected by variations in the pressure, and are the same as they would be if the liquid were in equilibrium with its vapor at the same temperature. This leads us to expect the following forms for $J_M{}^l$ and $J_E{}^l$, the mass and energy currents out of the liquid, when u_l is small:

$$J_M{}^l = P_v(T_l)(m/2\pi k_B T_l)^{1/2} + \tfrac{1}{2}\rho_g[T_l, P_v(T_l)]u_l\,, \tag{A3}$$

$$J_E{}^l = 2P_v(T_l)(k_B T_l/2\pi m)^{1/2} + \tfrac{5}{4} P_v(T_l)u_l\,. \tag{A4}$$

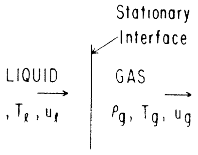

FIG. 3. Schematic drawing of a single component gas-liquid system not in thermodynamic equilibrium. Both mass and energy are transferred across the phase boundary, and both phases are drifting in a direction perpendicular to the interface. The interface itself is assumed to be stationary.

Here the subscript l refers to the liquid, T_l is the temperature of the liquid, $P_v(T_l)$ denotes the saturated vapor pressure at the temperature T_l, and $\rho_g[T_l, P_v(T_l)]$ is the gas density at temperature T_l and pressure $P_v(T_l)$.

We now calculate the net mass and energy fluxes from liquid to gas:

$$J_M \equiv J_M{}^l - J_M{}^g\,,$$

$$J_E \equiv J_E{}^l - J_E{}^g\,,$$

noting at the same time that u_l and u_g are given by

$$u_l = J_M/\rho_l\,,$$

$$u_g = J_M/\rho_g\,,$$

where ρ_l is the mass density of the liquid. Using these expressions and expanding to lowest order in $(T_l - T_g)$ and $P_v(T_l) - P_g$, we find

$$\left(1 - \frac{\rho_g}{\rho_l}\right)J_M = 2\left(\frac{m}{2\pi k_B T}\right)^{1/2}\left(P_v(\mathrm{T}_l) - P_g - \frac{P_g}{2T}(T_l - T_g)\right)\,, \tag{A5}$$

$$J_Q \equiv J_E - \frac{5}{2}\frac{k_B T}{m} J_M = -\frac{1}{2}\left(\frac{k_B T}{2\pi m}\right)^{1/2} \times \left(P_v(T_l) - P_g + \frac{9}{2}\frac{P_g}{T}(T_l - T_g)\right)\,, \tag{A6}$$

where the subscriptless variable T refers to either T_g or T_l, which are assumed to be very close to each other. As defined by (A6), J_Q is just the thermal flux from liquid to gas.

When we replace the liquid phase by a thin adsorbed film, the only difference, besides replacing T_l and ρ_l with T_f and ρ_f, is that instead of $P_v(T_l)$ we have $P_g(T_f, \mu_f)$ appearing in the equations [for a given T_f, a film can exist in equilibrium with its vapor at any pressure up to $P_v(T_f)$]. It is convenient then to rewrite (A5) and (A6) in terms of the chemical potentials μ_g and μ_f of the gas and the film, rather than in terms of the pressures, by using the differential equation of state of the gas

$$d\mu_g = -s_g dT_g + (\rho_g{}^{-1})dP_g\,.$$

In this way we get

$$\left(1-\frac{\rho_g}{\rho_f}\right)J_M = 2\rho_g\left(\frac{m}{2\pi k_B T}\right)^{1/2}\left[\mu_f - \mu_g + \left(s_g - \frac{k_B}{2m}\right)(T_f - T_g)\right] , \qquad \text{(A7)}$$

$$J_Q = -\frac{\rho_g}{2}\left(\frac{k_B T}{2\pi m}\right)^{1/2}\left[\mu_f - \mu_g + \left(s_g - \frac{9}{2}\frac{k_B}{m}\right)(T_f - T_g)\right] , \qquad \text{(A8)}$$

to first order in $(T_f - T_g)$ and $(\mu_f - \mu_g)$.

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[†] Permanent address: Physics Department; Tel Aviv University, Tel Aviv, Israel.

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Low-Temperature Density and Solubility of He^3 in Liquid He^4 under Pressure*

G. E. Watson,[†] J. D. Reppy, and R. C. Richardson
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14850

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The molar volumes of liquid mixtures of He^3 in He^4 up to 10 molar % have been measured to 22 atm pressure and to 50 mK using a dielectric technique. The single-phase measurements provide a determination of the Bardeen-Baym-Pines (BBP) parameter α under pressure. A weaker pressure dependence is found than in the measurements of Boghosian and Meyer. The ground-state kinetic energy for pure He^4 is deduced from α. The two-phase measurements indicate that the solubility at 50 mK of He^3 in He^4 rises from $(6.6 \pm 0.1)\%$ at $P = 0$ to a maximum of $(9.5 \pm 0.12)\%$ at 10 atm; it then drops to $(8.3 \pm 0.14)\%$ at 22.5 atm. This behavior is found to be consistent with the BBP effective interaction theory using the Ebner potential.

I. INTRODUCTION

In this experiment, we have measured the molar volumes of dilute mixtures of He^3 in He^4 and of pure He^4 at pressures above the saturated vapor pressure and at temperatures down to 50 mK. The molar volumes are determined by measuring the dielectric constant of the helium with a capacitor which is part of an oscillator tank circuit. We then use these molar volumes to determine the solubility of He^3 in He^4 at 50 mK and the relative excess volume of He^3 in He^4 at pressures up to 22 atm. The latter differs considerably from the measurement of Boghosian and Meyer.[1]

Dilute mixtures of He^3 in He^4 have been the subject of very intense study, both experimentally and