Power Spectrum of Stochastic Pulse Sequences with Correlation between the Pulse Parameters

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The power spectrum $S(f)$ of pulse sequences, which belong to the class of Markov processes, is calculated for the general case of a combined distribution $\gamma(\vartheta, \tau, h)$ that permits coupling between the pulse parameters: amplitude h, duration τ, and time period ϑ preceding or following a pulse. Two special cases of coupling are considered in detail with respect to flux-transport noise in superconductors: (i) With τh = const, i.e., all pulses have the same time integral, $S(f)$ exhibits at high frequencies an asymptotic f^{-m} dependence regardless of the particular pulse shape, if the series expansion for the distribution of τ at small τ values starts with a term $\propto \tau^{m-1}$. (ii) A relaxation time ϑ proportional to the pulse size τh, and an exponential distribution for ϑ leads to an asymptotic f^2 behavior at low frequencies, again practically independent of the pulse shape.

I. INTRODUCTION

A widely used method for the study of statistical properties of more or less random physical processes employs the measurement of their power spectra or correlation functions. In many cases, these processes can be represented by a sequence of random pulses. The shot noise in vacuum tubes is a simple example, which may serve here to introduce the basic notation. Each electron traveling from cathode to anode produces an electric pulse, which may be represented by $hy(t)$. (See Fig. 1.) In this example, the pulses have constant amplitude h and shape $y(t)$. Furthermore, the occurrence of any pulse is independent of all others pulses. This leads, as in the case of radioactive decay, to an exponential distribution of the time period ϑ between two subsequent events. The resulting power spectrum has the well-known, simple form

$$S(f) = \nu S_0(f) + \nu^2 a^2 \delta(f) \quad , \tag{1}$$

where ν is the average number of pulses per unit time, and $S_0(f)$ is determined by the Fourier transform of the individual pulse[1]:

$$S_0(f) = h^2 F_0(f) F_0^*(f) = \left| \int_{-\infty}^{+\infty} hy(t) e^{-2\pi i f t} dt \right|^2 . \tag{2}$$

The second term of Eq. (1) represents the contribution of the dc component of the pulse sequence. $a = h\int_{-\infty}^{+\infty} y(t)\, dt$ is the pulse area, and $\delta(f)$ the Dirac δ function. In general the dc term has the form $\nu^2\langle a\rangle^2\delta(f)$, $\langle a\rangle$ being the average pulse area. It always occurs and therefore will be omitted in the following treatment.

Very general expressions for the power spectrum and the correlation function of random pulse sequences have been derived in recent years with different methods.[2–6] Arbitrary distributions of the period $u_1(\vartheta)$ and pulse amplitude $w(h)$, and even randomly varying pulse shapes $y(t)$ were taken into account. The results were obtained under the basic assumption of a Markov process, i.e., h_j, $y_j(t)$, and ϑ_j, describing an arbitrary jth pulse of the sequence, were considered to be independent of the quantities h_{j-1}, $y_{j-1}(t)$, and ϑ_{j-1} of the preceding pulse.

The assumption of a Markov process, however, does not prohibit a possible coupling between the three parameters ϑ_j, $y_j(t)$, and h_j of an arbitrarily chosen pulse in the series. The important case of absolute dependence $h_j = h(\vartheta_j)$ was calculated by P. Mazzetti.[3,4] It is the purpose of this paper to consider in more detail the power spectrum of pulse sequences with correlated pulse parameters.

To simplify, we assume that the shape function y remains the same for all pulses except for a time constant τ, i.e., $y_j(t) = y(t, \tau_j)$. Furthermore, since coupling is assumed to occur only between the parameters ϑ, τ, and h of one and the same pulse, and not between different pulses, the subscripts can be omitted, as long as clarity is not impaired.

There are cases of physical interest, in which a correlation between ϑ, τ and h is noticed. For instance, a strong coupling between the "size" (τh) of a random event and the following period ϑ

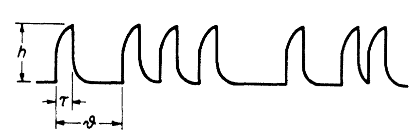

FIG. 1. Pulse sequence with constant parameters τ and h.

always occurs when an event produces an inhibiting effect proportional to the size of the event, on the following event.

One example is the retarding effect which the demagnetizing field, created by a Barkhausen jump in a ferromagnetic material, has on the following jump.[7–9] The change in external magnetic field, necessary to trigger a new event, is influenced by the size of the preceding Barkhausen jump.

Flux jumps in hard type-II superconductors seem to present another example of this type of coupling between pulse size and period. There again it appears from experimental observations[10] that the change in external magnetic field, either following or preceding a flux jump, increases with the size of the jump. These magnetic instabilities also can occur in an approximately periodic fashion,[11] i.e., the distribution for the periods $u_1(\vartheta)$ resembles a δ function for $dH/dt = \text{const}$. It is interesting to note that the experiment in this case also indicates a constant pulse size: Pulse size and period are proportional to each other.

Flux jumps in low-κ type-II superconductors showed a strong reduction of the power density at low frequencies ($f < 100$ Hz) essentially proportional to f^2, and an f^{-1} behavior for higher frequencies.[12] Again, by assuming the coupling $\vartheta \propto \tau h$ it is possible to explain the observed spectra, as will be shown later on.

Another interesting case of coupling occurs when one of the two parameters τ and h is allowed to vary randomly according to its distribution function, but the size is kept constant, i.e., $h \propto 1/\tau$. Each pulse of the sequence then represents the transport of a constant quantity, the transit time being more or less randomly distributed. This quantity may be electric charge or magnetic flux. Experiments indicate this type of coupling in the flux transport noise in type-II superconductors.[13,14] Here, we have the following noise mechanism: Under the influence of the Lorentz force, bundles of vortex lines are driven across the superconductor and overlapping pulses of amplitude φ/τ occur at the terminals of the superconductor. The transit time may vary, depending on the local pinning and the viscosity sensed by the flux bundle of size ϑ.

This coupling between τ and h tends to increase the power density at higher frequencies – pulses with short duration have higher amplitudes – and an f^{-1} behavior may be obtained under rather weak suppositions, as we shall see later on.

Cases like those mentioned above initiated the study presented in this paper. In Sec. II a rather general formula for the power spectrum of random pulse sequences with possible coupling between ϑ, τ, and h will be derived. The assumptions are: (a) The pulse sequences are considered to be stationary and ergodic; (b) Parameters of different pulses are independent; (c) All pulses have the same basic shape $y(t)$; (d) The pulse parameters ϑ, τ, h have the distribution $\gamma(\vartheta, \tau, h)$.

The combined distribution $\gamma(\vartheta, \tau, h)$ allows us to introduce coupling between the pulse parameters. This will be illustrated by several examples in Sec. III of this paper. Special emphasis will be given only to the behavior of power spectra, since one can obtain the associated correlation functions by the transformation of Wiener and Khintchine.

II. DERIVATION OF THE POWER SPECTRUM

Assumptions (a) and (b) allow us to calculate the power spectrum in the direct way, used earlier by T. Lukes.[2] We consider first a sequence of N pulses occurring in a time T, and derive the corresponding power spectrun $S_N(f)$. We then take the ensemble average $\langle S_N(f)\rangle$, which in the limit $N\to\infty$, $T\to\infty$ is assumed to be equal to the spectrum $S(f)$, obtained from the infinite pulse sequence.

A pulse sequence with N pulses can be represented by (Fig. 2):

$$Y_N(t) = h_1 y(t_1, \tau_1) + h_2 y(t-\vartheta_1, \tau_2) + h_3 y(t-\vartheta_1-\vartheta_2, \tau_3) + \cdots + h_N y(t-\vartheta_1-\vartheta_2-\cdots-\vartheta_{N-1}, \tau_N) \; . \tag{3}$$

The associated Fourier transform is

$$F_N(f) = \int_{-\infty}^{+\infty} \sum_{j=1}^{N} h_j y(t-\Theta_{j-1}, \tau_j) e^{-2\pi i f t} dt = \sum_{j=1}^{N} h_j e^{-2\pi i f \Theta_{j-1}} F_0(f, \tau_j) \; , \tag{4}$$

where we have $$\Theta_j = \sum_{k=1}^{j} \vartheta_k, \quad \Theta_0 = 0, \tag{5}$$

and $$F_0(f, \tau_j) = \int_{-\infty}^{+\infty} y(t, \tau_j) e^{-2\pi i f t} dt \; . \tag{6}$$

With Eq. (4), we obtain for the power spectrum $S_N(f)$

$$S_N(f)=\frac{1}{T}\,|F_N(f)|^2=\frac{1}{T}\Bigg\{\sum_{j=1}^{N} h_j^2|F_0(f,\tau_j)|^2$$

$$+\sum_{m=1}^{N-1}\sum_{j=1}^{N-m} h_j F_0(f,\tau_j)e^{2\pi i f\vartheta_j}h_{j+m}F_0^*(f,\tau_{j+m})\; e^{2\pi i f(\Theta_{j+m-1}-\Theta_j)}$$

$$+\sum_{m=1}^{N-1}\sum_{j=1}^{N-m} h_j F_0^*(f,\tau_j)\; e^{-2\pi i f\vartheta_j}h_{j+m}F_0(f,\tau_{j+m})e^{-2\pi i f(\Theta_{j+m-1}-\Theta_j)}\Bigg\}\ . \tag{7}$$

In order to calculate the ensemble average $\langle S_N(f)\rangle$, we need the distribution functions for the pair of parameters, τ, h, and for Θ_j. We obtain

$$\eta(\tau,h)=\int_0^\infty \gamma(\vartheta,\tau,h)\,d\vartheta\ , \tag{8}$$

and $$u_j(\Theta_j)=\int_0^\infty u_{j-1}(\Theta_j-\vartheta)u_1(\vartheta)\,d\vartheta\ , \tag{9}$$

with $$u_1(\vartheta)=\int_0^\infty\int_0^\infty \gamma(\vartheta_1,\tau_1,h)\,d\tau dh\ . \tag{10}$$

Equation (9) is valid because of assumption (b), by which also the parameters τ and h of any pulse in the series are independent of the parameters of all other pulses. The stationary nature of the pulse sequence ensures that the distribution function γ is the same for any arbitrary triple parameter, ϑ_j, τ_j, and h_j. This in turn has the consequence, that the distribution for $\Theta_{j+m-1}-\Theta_j$ is the same as for Θ_{m-1}. Furthermore, $\Theta_{j+m-1}-\Theta_j$ is independent of h_j and τ_j as well as of h_{j+m} and τ_{j+m}. So we can write for the averaging process

$$\langle S_N(f)\rangle=\frac{N}{T}\Bigg[A(f)+\sum_{m=1}^{N-1}\left(1-\frac{m}{N}\right)$$

$$\times[B^*(f)C(f)\psi_{m-1}(f)+B(f)C^*(f)\psi^*_{m-1}(f)]\Bigg]\ , \tag{11}$$

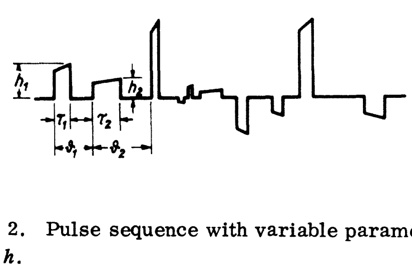

FIG. 2. Pulse sequence with variable parameters ϑ, τ, and h.

where
$$A(f)=\int_0^\infty\int_0^\infty h^2|F_0(f,\tau)|^2\eta(\tau,h)\,d\tau\,dh\ ,$$
$$B^*(f)=\int_0^\infty\int_0^\infty hF_0^*(f,\tau)\eta(\tau,h)\,d\tau\,dh\ ,$$
$$C(f)=\int_0^\infty\int_0^\infty\int_0^\infty hF_0(f,\tau)e^{2\pi i f\vartheta}\times\gamma(\vartheta,\tau,h)\,d\vartheta\,d\tau\,dh\ , \tag{12}$$
$$\psi_m(f)=\int_0^\infty u_m(\Theta_m)e^{2\pi i f\Theta_m}d\Theta_m\ .$$

With $$\psi_1(f)=\int_0^\infty u_1(\vartheta)e^{2\pi i f\vartheta}d\vartheta \tag{13}$$

it can be shown[2] that $\psi_m(f)=\psi_1{}^m(f)$, and that $\psi_1{}^m(f)$ tends to zero as $m\to\infty$. The limit as $N\to\infty$, $T\to\infty$, and $N/T\to\nu$ then gives as the final result for the power spectrum:

$$S(f)=\nu A(f)\left[1+\frac{1}{A(f)}\,\mathrm{Re}\left(\frac{B^*(f)C(f)}{1-\psi_1(f)}\right)\right]$$
$$=\nu A(f)I(f)\ . \tag{14}$$

The power spectrum in Eq. (14) appears as the product of the pulse density ν, the average spectrum $A(f)$ of the individual pulses, and an interference term $I(f)$. The pulse density ν is determined by the distribution $u_1(\vartheta)$ alone,

$$\nu=\left(\int_0^\infty \vartheta u_1(\vartheta)\,d\vartheta\right)^{-1}\ ;$$

$A(f)$ is given by the pulse shape and the combined distribution $\eta(\tau,h)$; but it is the interference term I, in which a coupling between all three parameters ϑ, τ, and h will find its expression.

Equation (14) was derived under the assumption that the correlation exists between the pulse parameters τ and h, and the period ϑ immediately following the pulse. The case of the pulse being coupled to the preceding period leads to the same result [Eq. (14)], as is obvious from the invariance of the power spectrum with respect to time reversal.

$S(f)$ is defined for the frequency range $-\infty\leq f\leq+\infty$. In measurements with conventional equipment such as amplifier, filters, and squaring devices, however, only positive frequencies occur.

Defining $S(f)$ for the interval $0 \leqslant f \leqslant +\infty$, the right-hand side of Eq. (14) has to be multiplied by a factor of 2.[1]

In the following section, Eq. (14) will be discussed with emphasis on coupling between the pulse parameters ϑ, τ, and h.

III. SPECIAL CASES

A. Independence between ϑ, τ, and h.

This condition can be expressed by

$$\gamma(\vartheta, \tau, h) = u_1(\vartheta) v(\tau) w(h) \quad . \tag{15}$$

Equations (12)-(14) then yield

$$S(f) = \nu\langle h^2\rangle\langle |F_0(f,\tau)|^2\rangle \times \left[1 + \frac{2\langle h\rangle^2 |\langle F_0(f,\tau)\rangle|^2}{\langle h^2\rangle\langle |F_0(f,\tau)|^2\rangle} \operatorname{Re}\left(\frac{\psi_1(f)}{1-\psi_1(f)}\right)\right] . \tag{16}$$

This result corresponds to other expressions published earlier for the continuous part of the power spectrum.[3-6] In the case of $\langle h\rangle = 0$, Eq. (16) reduces simply to

$$S(f) = \nu\langle h^2\rangle\langle |F_0(f,\tau)|^2\rangle \quad ,$$

no matter how ϑ and τ are distributed. A very similar relation was derived by Mazzetti[4] under the more general assumptions of a random pulse shape $y(t)$ and a possible coupling between ϑ and h. This simple result, however, is no longer true in general if τ and h are coupled, as one can easily see when calculating the expressions B^* and C in Eq. (12) with the combined distribution $\eta(\tau, h)$.

For an exponential distribution $u_1(\vartheta)$, the interference term in Eq. (16) becomes identical to unity for all frequencies, and with constant τ and h we obtain the simple case of shot noise mentioned above in Eq. (1).

B. Coupling between τ and h of the Form $\tau h = k =$ const.

In the following, sequences of pulses with constant area but variable pulse duration and amplitude, will be studied in some detail. With a pulse shape satisfying the relation $y(t,\tau) = y(t/\tau)$, pulses of constant area can be realized by the coupling $\tau h =$ const. To simplify, we assume τ and h to be independent of ϑ. Then we can write

$$\gamma(\vartheta, \tau, h) = u_1(\vartheta) v(\tau)(\tau/k)\delta(1 - \tau h/k) \quad , \tag{17}$$

and the power spectrum becomes

$$S(f) = \nu k^2 \langle |F_0(f\tau)|^2\rangle \left[1 - \frac{2|\langle F_0(f\tau)\rangle|}{\langle |F_0(f\tau)|^2\rangle} \operatorname{Re}\left(\frac{\psi_1(f)}{1-\psi_1(f)}\right)\right] , \tag{18}$$

where $F_0(f\tau) = \frac{1}{\tau}\int_{-\infty}^{+\infty} y\left(\frac{t}{\tau}\right) e^{-2\pi i f t}\, dt \quad .$

For statistically independent pulses, i.e., exponential distribution for ϑ, Eq. (18) reduces to simply:

$$S(f) = \nu k^2 \langle |F_0(f\tau)|^2\rangle = \nu k^2 \int_0^\infty |F_0(f\tau)|^2 v(\tau)\, d\tau . \tag{19}$$

With $E_0(f\tau) = 2\int_0^{f\tau} |F_0(x)|^2\, dx$, and $v'(\tau) = \dfrac{dv}{d\tau}$,

a partial integration of Eq. (19) gives

$$S(f) = \frac{\nu k^2}{f} E_0(f\tau) v(\tau)\Big|_0^\infty - \frac{\nu k^2}{f}\int_0^\infty E_0(f\tau) v'(\tau)\, d\tau \quad . \tag{20}$$

Assuming that $|F_0(f\tau)|^2$ has no singularity for $f\tau \to 0$, and that $v(\infty) = 0$. Eq. (20) reduces to

$$S(f) = -\frac{\nu k^2}{f}\int_0^\infty E_0(f\tau) v'(\tau)\, d\tau \quad . \tag{21}$$

To get a first idea of the frequency behavior of $S(f)$, we evaluate Eq. (21) for the rectangular distribution

$v(\tau) = 1/\tau_1$ for $0 \le \tau \le \tau_1$,

$v(\tau) = 0$, for $\tau > \tau_1$,

and obtain $S(f) = \frac{\nu k^2}{f\tau_1} \int_0^\infty E_0(f\tau)\delta(\tau - \tau_1)\,d\tau = \frac{\nu k^2 E_0(f\tau_1)}{f\tau_1}$. (22)

Pulses occurring in physical applications have a finite size. Thus, we can conclude from Parseval's theorem[1]:

$$2\int_0^\infty |F_0(f\tau)|^2 df\tau = \int_{-\infty}^{+\infty} y^2\left(\frac{t}{\tau}\right)\frac{dt}{\tau} = E \ ,$$

that $E_0(f\tau)$ will monotonically approach the finite value E for $f\tau \to \infty$. This behavior is schematically indicated in Fig. 3. We arrive at the important result: Independent of the particular pulse shape $y(t/\tau)$ the power spectrum given by Eq. (22) exhibits an asymptotic f^{-1} behavior at high frequencies.

It is on the other hand easy to show, that for $f \to 0$ the power density of Eq. (22) will be given by $\nu k^2 |F_0(f\tau_1)|^2$, i.e., the spectrum at low frequencies is essentially not affected by the coupling $\tau h = \text{const.}$

These results are a consequence of a more general law. Since $E_0(f\tau)$ monotonically approaches its limit E, we can find a value $f\tau = \Omega$ such that $E_0(\Omega) = (1-p)E$, p being an arbitrarily small positive number. Equation (21) then can be written

$$S(f) = -\frac{\nu k^2}{f}\int_0^{\Omega/f} E_0(f\tau)v'(\tau)\,d\tau + \frac{\nu k^2}{f}\mu_2(f)Ev\left(\frac{\Omega}{f}\right) \ ,$$

where $1 - p \le \mu_2(f) \le 1$.

Essential for the asymptotic form of $S(f)$ at high frequencies is the distribution $v(\tau)$ at small τ values. Assuming $v(\tau) = c_m\tau^m$, for $\tau \to 0$, we get for sufficiently high frequencies

$$S(f) = -\frac{\nu k^2}{f^m}c_m m\int_0^{\Omega/f} E_0(f\tau)(f\tau)^{m-1}d\tau + \frac{\nu k^2}{f^{m+1}}c_m\Omega^m\mu_2(f)E \ .$$

Since $m\int_0^{\Omega/f} E_0(f\tau)(f\tau)^{m-1}d\tau = m\mu_1(1-p)E\Omega^{m-1}\frac{\Omega}{f}$,

$$\le E_0(\Omega)\frac{\Omega^m}{f} \ ,$$

we obtain $S(f) = \frac{\nu k^2}{f^{m+1}}c_m\Omega^m E[\mu_2(f) - m\mu_1(1-p)]$, (23)

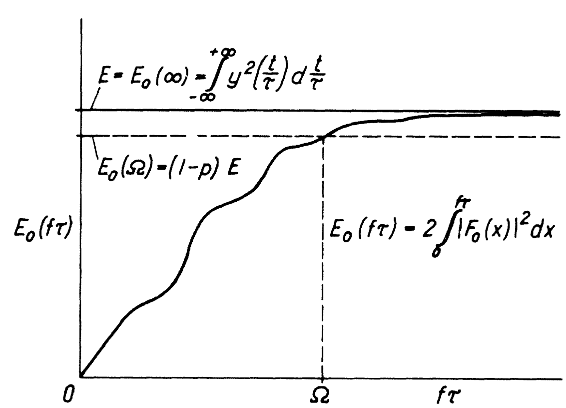

FIG. 3. Total power $E_0(f\tau)$, drawn schematically for a power density $F_0(f\tau)$ with pronounced minima, such as produced by a rectangular pulse shape.

$m\mu_1$ being a constant factor between 0 and 1.

With $v(\tau) \propto c_m \tau^m$ for $\tau \to 0$, $S(f)$ therefore asymptotically approaches the behavior

$$S(f) \propto f^{-m-1} \quad , \tag{24}$$

independent of the particular pulse shape.

Equation (24) allows us to calculate the asymptotic form of the power density for all distributions $v(\tau)$, which for $\tau \to 0$ can be expanded in power series. The f^{-1} dependence derived above [Eq. (22)] for the rectangular $v(\tau)$ is then always obtained if the series expansion contains a constant term. Regardless of the pulse shape, this spectrum therefore will occur for a wide class of distributions $v(\tau)$; the exponential, $v(\tau) = 1/\tau_0 \exp(-\tau/\tau_0)$ being one such case.

A series expansion for $\tau \to 0$ with nonzero c_0 of course is unrealistic, since pulses occurring in physical problems always have a finite smallest duration τ_0 given by the particular process in consideration. Real power spectra therefore never can exhibit an f^{-1} behavior for $f \to \infty$; instead for frequencies $f > (2\pi\tau_0)^{-1}$, the power density decreases faster than f^{-1} and essentially with the frequency dependence given by the particular pulse shape. This solves the question of infinite total power, which otherwise would arise. Nevertheless the asymptotic f^{-1} dependence will appear for frequencies $f < (2\pi\tau_0)^{-1}$ under the condition that series expansions for $v(\tau - \tau_0)$ have a nonzero constant term.

It appears from the preceding considerations that the coupling $\tau h = \text{const}$ above all tends to eliminate the influence of the pulse shape on the behavior of $S(f)$ at high frequencies. Another type of coupling, namely, $\vartheta \propto \tau h$, affects the power spectrum at low frequencies, also practically independent of the particular pulse shape.

C. Coupling between ϑ and τh

The case of absolute dependence between ϑ and τh, i.e., $\vartheta = g(\tau h)$ still allows independent distributions for two of the three variables. If we choose, for example, ϑ and h as independent variables, we can write

$$\gamma(\vartheta, \tau, h) = u_1(\vartheta) w(h) \frac{1}{\vartheta} \frac{dg(\tau h)}{d\tau} \delta\left(1 - \frac{g(\tau h)}{\vartheta}\right) . \tag{25}$$

In the following, we will concentrate on the simple case of the proportionality, $\vartheta \propto \tau h$. Also, we will keep for $u_1(\vartheta)$ the exponential distribution $1/\vartheta_0 \exp(-\vartheta/\vartheta_0)$. This has the consequence, that the product τh also has an exponential distribution.

1. Independent ϑ and h

As an example for independent ϑ and h, we consider the extreme situation $h = h_0 = \text{const}$, as in Fig. 1. Now the period ϑ is proportional to the pulse length τ, i.e., $\vartheta = \alpha\tau$. Assuming a rectangular pulse shape, we obtain for the interference term

$$I = \frac{x^2(1-\alpha)^2[\alpha^2 + (1+\alpha^2)x^2 + x^4]}{[\alpha^2 + (1-\alpha+\alpha^2)x^2]^2 + (1-\alpha)^2 x^6} , \tag{26}$$

with $x = 2\pi f \vartheta_0$.

This expression looks more complicated than its actual frequency behavior is: For $\alpha \neq 1$, $I(f)$ varies $\propto f^2$ for low frequencies, and tends to unity for high frequencies. The value of α (for $\alpha > 1$, all pulses are separated, for $\alpha < 1$, they all overlap) has no influence on this basic behavior of $I(f)$. For $\alpha = 1$, all pulses add up to a constant signal.

The interference term in this case is zero.

The average spectrum $A(f)$ in this example is given by

$$A(f) = 2h_0^2\tau_0^2/[1 + (2\pi f\tau_0)^2] \quad . \tag{27}$$

It is white for low frequencies, and decreases as f^{-2} at high frequencies. Therefore, we obtain for the total power spectrum $S(f) = \nu AI$ a behavior $\propto f^2$ for $f \ll 1/2\pi\vartheta_0$ and $\propto f^{-2}$, for $f \gg 1/2\pi\tau_0$. For $\alpha > 1$, the pulse sequence has different exponential distributions for the pulse duration τ and the time interval $\sigma = \vartheta - \tau$ between two subsequent pulses. This case, but without coupling between σ and τ, was treated earlier by Machlup[15] who obtained, similarly to Eq. (27), a white spectrum for low frequencies instead of the f^2 behavior.

2. Independent ϑ and τ

With the exponential distribution for ϑ and a normalized pulse shape $y(t/\tau)$ we get

$$I(f) = 1 - r/(1 - x^2) \quad , \tag{28}$$

where r is the ratio

$$|\langle F_0(f\tau)\rangle|^2 / \langle |F_0(f\tau)|^2\rangle \quad .$$

This ratio is unity for

(α) $\tau = \tau_0 = \text{const.}$

and (β) $f \to 0$.

(a). (α) For constant τ, $I(f)$ becomes

$$I = x^2/(1 + x^2) \quad . \tag{29}$$

As in the example of Sec. III C 1 we note that I is proportional f^2 at frequencies $f \ll 1/2\pi\vartheta_0$ and goes to unity for $f \gg 1/2\pi\vartheta_0$. Constant τ, of course, is equivalent to $h \propto \vartheta$.

(b). (β) In the limit $f \to 0$, $F(f\tau)$ approaches the constant value $\int_{-\infty}^{+\infty} y(t/\tau)dt/\tau$. The ratio r, therefore, tends to unity for all distributions $v(\tau)$ which decrease sufficiently fast as τ goes to infinity. Thus, we obtain another important result: The coupling $\vartheta \propto \tau h$ together with an exponential distribution for ϑ leads to an interference term that varies as f^2 at low frequencies independent of the particular pulse shape and for practically all distributions $v(\tau)$ playing a role in physical applications.

In contrast to the insensitivity with respect to $v(\tau)$ and $y(t/\tau)$, it appears that the noise reduction by $I(f)$ at low frequencies is rather strongly influenced by both the particular distribution $u_1(\vartheta)$ and the special type of coupling $\vartheta = g(\tau h)$. The distribution $(\vartheta/\vartheta_0^2)\exp(-\vartheta/\vartheta_0)$, for instance, which was found experimentally for Barkhausen jumps in ferromagnets,[16] reduces the power density at low frequencies only by a factor of 2. The effect of a deviation from the proportionality $\vartheta \propto \tau h$ can be seen from Fig. 4, which shows the interference term for the coupling $h = \lambda\vartheta + h_0$.

Power spectra obtained under the conditions of Sec. III C 2 b combine the essential features of case III B and case III C. The second term of Eq. (14) is now given by:

$$A(f) = \langle k^2 \rangle \langle | F_0(f\tau)|^2 \rangle \quad . \tag{30}$$

This relation is very similar to that given by Eq. (19). Both $A(f)$ and $I(f)$ approach constant values, at low and high frequencies, respectively. Consequently, the frequency dependence of $S(f)$ is determined at low frequencies by $I(f)$, and at high frequencies by $A(f)$. Thus, with a coupling $\vartheta \propto \tau h$ and exponential distributions for both the period and the pulse duration, we obtain a power spectrum, which varies independently of the particular pulse shape as f^2 for $f \ll 1/2\pi\vartheta_0$, and as f^{-1} for $f \gg 1/2\pi\tau_0$.

There are reasons to believe, that the results of Sec. III C 2 b are immediately applicable to power spectra of flux jumps in low-κ type-II superconductors. Their frequency behavior as mentioned in the Introduction is very much like that given

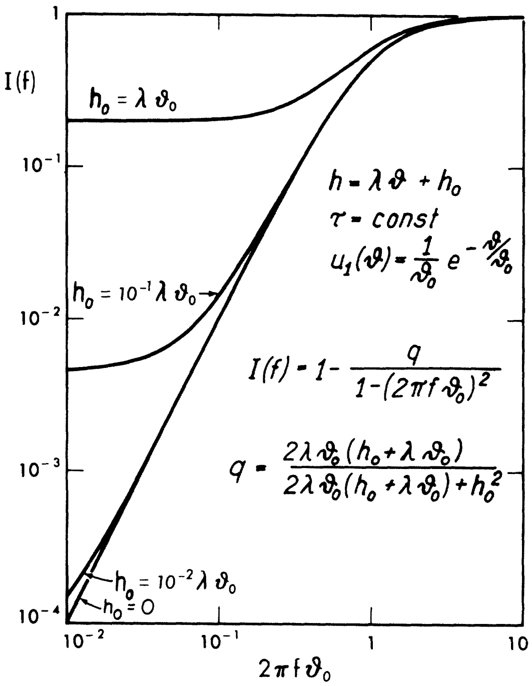

FIG. 4. Interference term $I(f)$ for a sequence of pulses with constant pulse duration τ, exponential distribution for ϑ, and the coupling $h = \lambda\vartheta + h_0$. The limit $h_0/\lambda\vartheta_0 \to \infty$ represents the case of random pulses with constant amplitude h and duration τ.

in the last example, and it is possible to describe the observed spectra rather well by choosing proper values of ϑ_0 and τ_0.[17] A more detailed discussion of the experimental results will be given in a forthcoming paper. The assumption of the coupling $\vartheta \propto \tau h$ together with an exponential distribution for ϑ, of course, implies that the size distribution of the flux jumps also should be of the exponential type. This indeed was confirmed by direct measurement.[18]

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Measurements of Diffusion in Velocity Space from Ion-Ion Collisions

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Diffusion of particles in velocity space due to ion-ion collisions is measured experimentally using the spatial ion-wave echo as a diagnostic technique. The experiment was performed in a highly ionized plasma. The ion-ion collisions are described by a Fokker-Planck term in the ion kinetic equation. The good agreement between the measurement of the echo position, phase velocity, and amplitude with the theoretical predictions verifies that the diffusion term of the Fokker-Planck equation is appropriate for describing the effect of small-angle Coulomb collisions. The magnitude of the second-order coupling coefficient relating the amplitudes of the excited waves to the echo amplitude was also measured. The agreement with theory indicates the validity of the second-order expansion of the linearized Vlasov equation.

INTRODUCTION

It is well known that particles diffuse in phase space either by self-collisions or in the presence of randomly fluctuating fields. We wish to present experimental data[1] which point towards the measurements of such diffusion coefficients for comparison with recent theoretical calculations. In this paper we report the measurement of diffusion coefficients due to particle-particle collisions. In a subsequent paper we will report the effects of diffusion due to ion-wave turbulence. As pointed out first by Gould, O'Neil, and Malmberg,[2] the ion echo, which is very sensitive to the retention of phase coherence, can be used as a diagnostic tool for the investigation of collisional and turbulent phenomena. Subsequently, several authors[3–6] have calculated the effect of collisions and turbulence on plasma-wave echoes with the use of a diffusion operator of the form

$$\left.\frac{\delta f}{\delta t}\right|_{\mathrm{F-P}} = \frac{\partial^2}{\partial v^2}\left[D(v)f(v)\right] \quad . \tag{1}$$

Our experimental results essentially support their theoretical calculations and verify the validity of the diffusion equation for describing small-angle collisions in a plasma. When compared with experimental findings of collisional effects on ion waves by Motley and Wong,[7] ion-wave echoes are shown to be more sensitive to collisions by an order of magnitude.

The method we use is the second-order ion-wave echo. The advantages of using the ion-wave echo are that the propagation speed is essentially density-independent, and furthermore it is nondispersive. One can increase the density without changing the basic characteristic of ion-wave propagation, which is a distinct advantage over the electron-wave echoes. In a finite plasma, the propagation characteristics of ion waves are essentially