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S-Matrix Formulation of K_L and K_S Decays and Unitarity Relations*

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The unitarity relations for the decay of overlapping resonances as obtained by an S-matrix approach and the usual Wigner-Weisskopf approach are compared.

IN two recent letters,[1,2] McVoy and Durand discuss the use of an S-matrix approach involving overlapping resonances to discuss the decay properties of the K^0 system. In passing they mention as one of their results that the sum of the partial widths for the K_L or K_S decay is greater than the total width. In the standard Wigner-Weisskopf treatment of the problem, on the other hand, one finds the result that the sum of the partial widths is equal to the total width.[3] For the case of the K^0 system, the difference is of no practical importance since it occurs only in second order in the small CP-violating parameter $\langle K_L|K_S\rangle$; nevertheless, it is instructive to explain the difference.

Let us review first the result of the usual Wigner-Weisskopf treatment. We are given an initially prepared state at time $t=0$,

$$|I\rangle=A|K_L\rangle+B|K_S\rangle,$$

normalized to unity,

$$|A|^2+|B|^2+2\,\mathrm{Re}(A^*B\langle K_L|K_S\rangle)=1. \tag{1}$$

Here, $|K_L\rangle$ and $|K_S\rangle$ are the decaying states, which we write in the K_1-K_2 representation[4]

$$|K_S\rangle=\begin{pmatrix}1\\ \delta+\rho\end{pmatrix}N_S,\quad |K_L\rangle=\begin{pmatrix}\rho-\delta\\ 1\end{pmatrix}N_L, \tag{2}$$

normalized to unity by

$$|N_S|^2(1+|\delta+\rho|^2)=1, \tag{3a}$$

$$|N_L|^2(1+|\rho-\delta|^2)=1. \tag{3b}$$

If we assume CPT invariance, $\delta=0$; if we assume T invariance, $\rho=0$. We may define transition amplitudes from the K_1 and K_2 states to some normalized continuum state λ:

$$\langle\lambda|T|K_i\rangle=T_{\lambda i}. \tag{4}$$

The inverse transition is

$$T_{i\lambda}=\langle K_i|T|\lambda\rangle=\pm T_{\lambda i}, \tag{5}$$

where the plus sign corresponds to time-reversal invariance and the minus corresponds to time-reversal violation. For the states K_S and K_L, we have

$$\langle\lambda|T|K_S\rangle=[T_{\lambda1}+(\delta+\rho)T_{\lambda2}]N_S, \tag{6a}$$

$$\langle\lambda|T|K_L\rangle=[(\rho-\delta)T_{\lambda1}+T_{\lambda2}]N_L. \tag{6b}$$

From the initial state $|I\rangle$, we then have as a function of time the amplitude

$$A_{\lambda I}(t)=A\langle\lambda|T|K_L\rangle e^{-iM_Lt}+B\langle\lambda|T|K_S\rangle e^{-iM_St}, \tag{7}$$

where $M_i=m_i-\frac{1}{2}i\gamma_i$ are the complex masses of the decaying states. By conservation of probability, we now make the requirement

$$\sum_\lambda\int_0^\infty dt|A_{\lambda I}(t)|^2=1,$$

since the initial state was normalized. Stating this as an identity in A and B, we find

$$\sum_\lambda|\langle\lambda|T|K_S\rangle|^2=\gamma_S, \tag{8a}$$

$$\sum_\lambda|\langle\lambda|T|K_L\rangle|^2=\gamma_L, \tag{8b}$$

$$\sum_\lambda\langle\lambda|T|K_L\rangle\langle\lambda|T|K_S\rangle^*=-i(M_S^*-M_L)\langle K_S|K_L\rangle. \tag{8c}$$

Equations (8a) and (8b) are the statements that the sum of the partial widths equal the total widths, while Eq. (8c) is the Bell-Steinberger off-diagonal unitarity relation.[5]

It is important to note that in this approach the initial state $|I\rangle$ is given. In the usual experimental situation, the initial state is prepared by a strong interaction which gives the state directly in the K^0-$\bar{K}^0$ representation. For a state given as

$$\cos\theta\,|K^0\rangle+\sin\theta\,|\bar{K}^0\rangle,$$

we have

$$A=(N^2/N_L)\tfrac{1}{2}\sqrt{2}[(1-\delta-\rho)\cos\theta-(1+\delta+\rho)\sin\theta], \tag{9a}$$

$$B=(N^2/N_S)\tfrac{1}{2}\sqrt{2}[(1+\delta-\rho)\cos\theta+(1-\delta+\rho)\sin\theta], \tag{9b}$$

$$N^2(1+\delta^2-\rho^2)=1. \tag{10}$$

We turn now to the S-matrix approach. The S-matrix connecting continuum states is written,[1,2] setting the

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[1] K. W. McVoy, Phys. Rev. Letters **23**, 56 (1969).

[2] L. Durand, III, and K. W. McVoy, Phys. Rev. Letters **23**, 59 (1969).

[3] See, for example, T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. **106**, 340 (1957).

[4] Further details may be found in T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. **16**, 516 (1966). Our parameter ρ is called ϵ in this reference. Our K_1 and K_2 are $(K^0+\bar{K}^0)/\sqrt{2}$ and $(K^0-\bar{K}^0)/\sqrt{2}$, respectively.

[5] J. S. Bell and J. Steinberger, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High-Energy Laboratory, Chilton, Berkshire, England, 1966).

background S matrix equal to unity,

$$S_{\beta\lambda}=1-i\gamma_S\frac{g_{\beta S}h_{\lambda S}}{E-M_S}-i\gamma_L\frac{g_{\beta L}h_{\lambda L}}{E-M_L}. \tag{11}$$

For the case of T invariance, it is stated that $g_{\lambda T}=h_{\lambda T}$. For the case of CPT invariance, $g_{\lambda T}=\pm h_{\lambda T}$, where the plus sign corresponds to CP-conserving and the minus to CP-violating transitions. For the case of CPT invariance, the unitarity of the S matrix gives[2]

$$\sum_\beta g_{\beta S}^* g_{\beta S}=\sum_\beta g_{\beta L}^* g_{\beta L}=(1+\gamma_S\gamma_L|\alpha|^2)^{1/2}, \tag{12a}$$

$$\sum_\beta g_{\beta S}^* g_{\beta L}=-i\alpha(M_S^*-M_L), \quad \alpha \text{ real}. \tag{12b}$$

Equation (12a) could be interpreted as stating that the sum of the partial widths is greater than the total width.

To relate the two approaches, we analyze the S-matrix approach in slightly greater detail. It is important here that the states K_L and K_S enter both as initial and final states. For treating final states, it is essential to introduce the dual or adjoint state vectors.[6] We define

$$|S\rangle=\begin{pmatrix}1\\ \delta+\rho\end{pmatrix}N, \qquad |L\rangle=\begin{pmatrix}\rho-\delta\\ 1\end{pmatrix}N,$$
$$\langle S^\dagger|=(1,\delta-\rho)N, \quad \langle L^\dagger|=(-\delta-\rho,1)N, \tag{13}$$

where N is given by Eq. (10) and these satisfy the orthonormality relations $\langle T^\dagger|V\rangle=\delta_{TV}$. The states $|S\rangle$ and $|L\rangle$ are identical with $|K_S\rangle$ and $|K_L\rangle$ except for normalization. On the other hand, $\langle S^\dagger|$ and $|L^\dagger|$ are not the same as $\langle K_S|$ and $\langle K_L|$. It is possible to multiply $|S\rangle$ by a number Q and $\langle S^\dagger|$ by Q^{-1} without changing any results; our choice of normalization yields simple symmetry relations. We can define the transition amplitudes

$$\begin{aligned}
\gamma_S^{1/2}g_{\lambda S}&=\langle\lambda|T|S\rangle=[T_{\lambda 1}+(\delta+\rho)T_{\lambda 2}]N,\\
\gamma_L^{1/2}g_{\lambda L}&=\langle\lambda|T|L\rangle=[(\rho-\delta)T_{\lambda 1}+T_{\lambda 2}]N,\\
\gamma_S^{1/2}h_{\lambda S}&=\langle S^\dagger|T|\lambda\rangle=[T_{1\lambda}+(\delta-\rho)T_{2\lambda}]N,\\
\gamma_L^{1/2}h_{\lambda L}&=\langle L^\dagger|T|\lambda\rangle=[(-\delta-\rho)T_{1\lambda}+T_{2\lambda}]N.
\end{aligned} \tag{14}$$

For the case of CPT invariance, we have from Eq. (5) $T_{i\lambda}=\pm T_{\lambda i}$ with the plus sign corresponding to CP conservation and the minus sign to CP violation. Since $\delta=0$ in this case, it is easy to see that the same relations hold between g and h, as stated before. For example, if λ is CP even, $T_{1\lambda}=T_{\lambda 1}$, $T_{2\lambda}=-T_{\lambda 2}$, whence from Eq. (14) we have $h_{\lambda S}=g_{\lambda S}$ and $h_{\lambda L}=-g_{\lambda L}$.

It is now trivial to see the relation between the two versions of the unitarity relations, since

$$\begin{aligned}
\langle\lambda|T|K_S\rangle&=\gamma_S^{1/2}N_S g_{\lambda S}/N,\\
\langle\lambda|T|K_L\rangle&=\gamma_L^{1/2}N_L g_{\lambda L}/N.
\end{aligned} \tag{15}$$

Equations (8) become

$$\sum_\lambda g_{\lambda S}^* g_{\lambda S}=|N/N_S|^2, \tag{16a}$$

$$\sum_\lambda g_{\lambda L}^* g_{\lambda L}=|N/N_L|^2, \tag{16b}$$

$$\sum_\lambda g_{\lambda S}^* g_{\lambda L}=-\frac{|N|^2}{N_L N_S}i(M_S^*-M_L)\frac{\langle K_S|K_L\rangle}{\gamma_S^{1/2}\gamma_L^{1/2}}. \tag{16c}$$

Equations (16) are true quite generally, independent of CPT or T invariance. For the special cases in which it is possible to relate $h_{\alpha S}$ to $g_{\alpha S}$, they are directly derivable from the unitarity of the S matrix. For the case of CPT invariance ($\delta=0$) or T invariance ($\rho=0$), Eqs. (16a) and (16b) reduce by algebra to Eq. (12a), with

$$\gamma_S\gamma_L|\alpha|^2=|\langle L|S\rangle|^2=|N^2/N_S^2|^2|\langle K_L|K_S\rangle|^2, \tag{17}$$

while Eq. (16c) becomes

$$\sum g_{\lambda S}^* g_{\lambda L}=-i(M_S^*-M_L)\langle S|L\rangle/\gamma_S^{1/2}\gamma_L^{1/2}, \tag{18}$$

where in these cases $N_S=N_L$. Equation (18) is equivalent to Eq. (12b) provided we identify, in agreement with Eq. (17),

$$\alpha=\langle S|L\rangle/\gamma_S^{1/2}\gamma_L^{1/2}.$$

For the CPT-invariant case, $\langle S|L\rangle$ is real and proportional to $\mathrm{Re}\rho$, whereas for the T-invariant case it is imaginary and proportional to $\mathrm{Im}\delta$.

The essential difference between the two formulations is *not* that one uses time dependence and the other energy dependence. Indeed, the standard approach given by Eq. (7) can easily be reformulated in terms of the energy-dependent amplitude for the decay of state $|I\rangle$ to $|\lambda\rangle$:

$$a_{\lambda I}(E)=-\frac{iA\langle\lambda|T|K_L\rangle}{E-M_L}-\frac{iB\langle\lambda|T|K_S\rangle}{E-M_S}. \tag{19}$$

Equation (7) is regained by the relation

$$A_{\lambda I}(t)=\int_{-\infty}^{\infty}\frac{a_{\lambda I}(E)e^{-iEt}dE}{2\pi},$$

corresponding to the production of the resonance in a wave packet much broader than the widths of the K_L and K_S. Equation (19) is more appropriate for the analogous case of ρ-ω interference[7] when the energy resolution is much less than the ρ width.

The essential difference between the two approaches is that the standard formulation [Eqs. (7) or (19)] describes a *production* process whereas the S matrix describes a *formation* process for the resonances. In the case of the production process, one knows exactly how many resonant particles have been produced; for

[6] R. G. Sachs, Ann. Phys. (N. Y.) **22**, 239 (1963); C. P. Enz and R. R. Lewis, Helv. Phys. Acta **38**, 860 (1965).

[7] See the recent discussion by A. Goldhaber, G. C. Fox, and C. Quigg, Phys. Letters **30B**, 249 (1969), and references therein.

example, in the process $\pi^- + p \to K^0 + \Lambda$, one can count the number of Λ's. In the case of the formation process, in which the production and decay are inextricably linked, there is no direct way to state how many resonant particles have been produced. The analysis above gives one answer to that question.

It might be thought at first that the S-matrix approach is preferable since everyone knows that S-matrix elements are the true observables. In practice, of course, the K_L and K_S states are never observed as resonant scattering states but only in production experiments. Thus the parameters entering the standard formulation are more directly related to observation than those in the S-matrix approach.

Note added in proof. The present paper, as well as that of G. C. Wick, Phys. Letters **30B**, 126 (1969), shows that the usual Bell-Steinberger formula follows from the S-matrix approach. Therefore, the alternative formula proposed by McGlinn and Polis, Phys. Rev. Letters **22**, 908 (1969), is incorrect and should be disregarded.

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Gauge Invariance and the Born Approximation in Pion Electroproduction*

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It is shown that use of the Ward-Takahashi identity for the electromagnetic vertices of the hadrons gives a more satisfactory argument than the traditional one for the addition of subtraction terms to the Born approximation of pion electroproduction.

It has been recognized for a long time that the commonly used "renormalized Born approximation" for the pion electroproduction matrix element

$$\langle \pi(Q), N(P_2) | j_\mu | N(P_1) \rangle \qquad (1)$$

is not conserved. On the other hand, the electromagnetic current j_μ necessarily satisfies the conservation condition, and we can therefore expand the amplitude $\epsilon_\mu \langle \pi N | j_\mu | N \rangle$ as a sum

$$\sum_{i=1}^{6} A_i(s,t,K^2) \bar{u}(P_2) M_i u(P_1), \qquad (2)$$

where the M_i are a complete set of explicitly gauge-invariant quantities (such as those introduced by Fubini *et al.*[1] or by Dennery[2]). In the above, ϵ_μ represents a virtual photon of momentum K ($\equiv Q+P_2-P_1$), and s, t, and u are the standard Mandelstam variables:

$$s = -(P_1+K)^2, \quad t = -(P_2-P_1)^2, \quad u = -(P_2-K)^2.$$

Clearly, the Born approximation (BA) must be modified before it can be expanded in terms of the above set. Following the work of Fubini *et al.*,[1] it has been traditional[2,3] to add to the BA the somewhat arbitrary term

$$ig\gamma_5 [F_\pi(K^2) - F_1^V(K^2)] \tfrac{1}{2} [\tau_\alpha, \tau_3] K_\mu / K^2, \qquad (3)$$

where F_π and F_1^V are, respectively, the pion and isovector-nucleon electromagnetic form factors. The justification for such a procedure is that while the additional term restores current conservation, it does not contribute to the physical matrix element since $\epsilon \cdot K = 0$. It is the purpose of this paper to stress that there exist arguments, based upon the properties of j_μ alone, which can justify the addition of such a term. This seems preferable to an argument based upon the properties of the vector ϵ_μ into which $\langle \pi N | j_\mu | N \rangle$ happens to be contracted. For instance, the argument breaks down for the vector part of pion neutrino production (where $\epsilon \cdot K \sim m_{\text{lepton}}$) even though the conserved-vector-current (CVC) theory equates this contribution to the isovector part of pion electroproduction.

When one uses the BA simply to find the residues for the pole terms in dispersion relations, it can be shown[4] that, for consistency, additional subtraction constants are required, thus providing a satisfactory argument for the introduction of the terms in Eq. (3). However, one might also wish to use the BA as part of a model description of electroproduction,[3] and in that case one must clearly use a gauge-invariant expression. (Such an approach has recently gained in importance since

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[1] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958).

[2] See, for example, P. Dennery, Phys. Rev. **124**, 2000 (1961); N. Zagury, *ibid.* **145**, 1112 (1966); S. Adler, Ann. Phys. (N.Y.) **50**, 189 (1968); N. M. Kroll, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 75.

[3] See, for example, J. D. Walecka and P. A. Zucker, Phys. Rev. **167**, 1479 (1968). See also J. D. Walecka, Ref. 5, p. 156, and in particular the discussion on p. 195.

[4] F. A. Berends, A. Donnachie, and D. L. Weaver, Nucl. Phys. **B4**, 1 (1967), Appendix B.