Dynamical Regge Trajectories from the Multiperipheral Model*

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A multiperipheral model, based on one-pion-exchange dominance of low-momentum-transfer processes, is used to study the dynamical origin of the leading Regge trajectories which are coupled to the two-pion system. A Bethe-Salpeter technique is used to calculate the leading singularities in the angular-momentum plane from an input interaction that is determined by the observed low-energy pion-pion scattering amplitude plus the high-energy contribution of the Pomeranchon. The solution results in a partial bootstrap system in which the parameters of the Pomeranchon are determined self-consistently. The output trajectories are found to correspond to what we conjecture are effective Regge trajectories, which in reality correspond to both Regge-pole and cut contributions. The trajectories and coupling constants obtained seem to be in reasonable agreement with the values allowed by experiment.

I. INTRODUCTION

THE bootstrap description of elementary particles is certainly one of the most appealing and ambitious proposals for determining the masses and coupling constants of elementary particles. In this picture all particles are dynamically interrelated in such a way that all the parameters describing the S matrix for strongly interacting particles are determined through self-consistency requirements. At present, though a tremendous amount of effort has been expended on investigating such theories, little has been learned except that simple models of this type fail to describe the spectrum of particles that occur in nature and we are still left with the fact that the bootstrap theory may be the *correct* theory; better means of investigating its relevance to strong interactions must be developed.

Recently Chew and Pignotti[1] proposed a calculational scheme known as the "Regge bootstrap," which adds one new ingredient to the older bootstrap models. Papers[2,3] based on this general approach have yielded encouraging results. In this paper we present a careful calculation employing as much experimental information as possible, with particular emphasis on investigating the dynamical origin of the Pomeranchuk trajectory. As an introduction, let us examine what the proposal of CP has added to the usual bootstrap model of a dynamical particle. Consider, for example, the bootstrap of the ρ meson, of the type that was proposed by Chew and Mandelstam[4] in 1960, and followed by more sophisticated calculations performed in the early sixties. This is illustrated by the equation represented by the diagrams in Fig. 1(a). The left-hand side represents a dynamical pole of the pion-pion scattering amplitude, namely, the ρ meson with angular momentum $J=1$. The right-hand side of this equation is some sort of ladder representing the repeated interaction between the constituent particles, produced by the exchange of some particle. A sum of ladder graphs seems to be the only model of a composite particle which satisfies analyticity and some truncated form of unitarity. The general procedure was to guess a set of graphs to sum on the right, perform an approximate calculation, and then check to see whether the output ρ meson had the correct width and mass. This becomes the simplest bootstrap model when the interaction is taken to be the exchange of a ρ meson, which then binds two pions together to form a ρ meson, and the mass and coupling constant are determined self-consistently. More sophisticated calculations involved considering serveal resonances simultaneously, such as the ρ and f^0 on both sides of the equation, and including additional constituent particles such as K mesons and the ω meson. The general results of these calculations were that if the ρ meson was produced with the correct mass, the width of the ρ was necessarily too large, being somewhere between twice and four times experiment, depending on the particular model being considered. The general fault of this type of model is that there is no direct way to check whether the input to the right-hand side of this equation is reasonable.

The observation by CP is related to our current understanding of a dynamical particle as a pole in the angular-momentum variable whose position depends on the energy variable. In particular, if one considers the point $t=0$ and the variable s large, then the left-hand

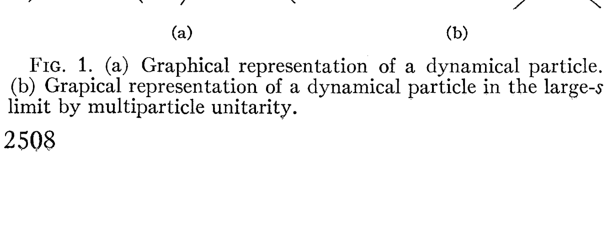

FIG. 1. (a) Graphical representation of a dynamical particle. (b) Grapical representation of a dynamical particle in the large-s limit by multiparticle unitarity.

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[1] G. F. Chew and A. Pignotti, Phys. Rev. Letters **20**, 1078 (1968); Phys. Rev. **176**, 2112 (1968); hereafter referred to as CP.
[2] G. F. Chew and W. R. Frazer, Phys. Rev. (to be published); L. Caneschi and A. Pignotti, *ibid.* **180**, 1525 (1969); *ibid.* (to be published); W. R. Frazer and C. H. Mehta, University of California, San Diego Report (unpublished).
[3] J. S. Ball and G. Marchesini, Phys. Rev. (to be published); hereafter referred to as I.
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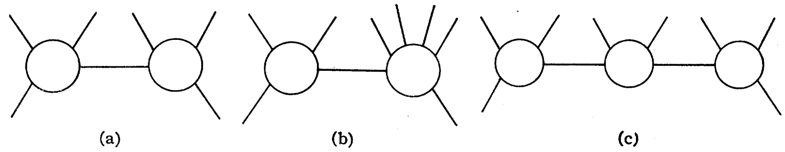

FIG. 2. (a) OPE model for $2\pi \to 4\pi$. (b) OPE model for $2\pi \to 6\pi$. (c) Multiperipheral model for $2\pi \to 6\pi$, obtained by iteration of OPE.

side of the equation in Fig. 1(a) is interpreted as the contribution of the Regge pole to the π-π scattering amplitude at high energy. Furthermore, if one takes the imaginary part of both sides of this equation, the left-hand side becomes the total cross section at high energy and the right-hand side becomes a sum over the two- to n-particle partial cross sections. This is particularly clear for the Pomeranchuk trajectory, as it is the leading singularity in the scattering amplitude, but also must be true for the ρ and other nonleading trajectories provided the appropriate partial cross sections are summed on the right. From this point of view, the dynamical equation represented in Fig. 1(a) is simply an identity and the correct ladders to sum are those that give the correct two-particle to n-particle cross sections. From experiment we know that pions are most copiously produced in high-energy reactions; thus the equation represented in Fig. 1(a) is more correctly represented by that given in Fig. 1(b), where the n particles are n pions. The requirement that we have a useful bootstrap equation is that the inelastic partial cross sections be calculable correctly by some tractable model related to the Regge pole calculated on the left.

The original proposal by CP was that the "two-to-n" amplitude could be calculated from the multi-Regge model; however, most of the experimentally observed production occurs for relatively small subenergies for particle pairs and therefore lies outside the region of phase space where the multi-Regge model is directly applicable. If the duality picture is correct, in the sense that the Regge pole which describes the scattering at high energy continues to give the correct energy-average cross section at low energies in the resonance region, one could set up a set of bootstrap equations for the leading Regge trajectories. It appears that duality seems to be a sometime thing, forcing one to search for a more generally applicable model. It has been suggested by Chew[5] that the one-pion-exchange (OPE) model, which has had considerable success in describing production processes,[6] might provide a good description of multiparticle production.

The basic assumption of the OPE model is that the pion pole which exists in a scattering amplitude at the pion mass in the appropriate momentum-transfer variable continues to dominate for small negative values of this variable. Since experimentally all processes are dominated by the small-momentum-transfer region, one might hope that this model would give good results for the total production cross sections also. Comparison of this model with production processes in which the vertices are actual physical scattering amplitudes produces surprisingly good results, and one might expect the generalizations of this model to multiparticle production to produce equally good results, at least in some average sense. The generalization or perhaps iteration of this model for pions produced from π-π collisions is that shown in Fig. 2. The two-to-four amplitude is shown in Fig. 2(a), and the two-to-six amplitude in Fig. 2(b). Note, however, the vertex that appears in the two-to-six amplitude is just the two-to-four amplitude given by Fig. 2(a), and one can conclude that the two-to-six diagram is that given in Fig. 2(c). The production of n particles is then given by a diagram obtained by an iteration of the foregoing procedure, and such an iteration we recognize will generate the multiperipheral model proposed by Amati *et al.*[7] With this picture the ladders to be summed to produce the leading Regge trajectories are those shown in Fig. 3(a), and the sum is the solution to the Bethe-Salpeter equation shown in Fig. 3(b), which has as a kernel the off-shell elastic pion-pion cross section. One should note, however, that since experimental momentum transfers are generally less than 0.5 GeV^2, and often dominated by smaller values, only a small extrapolation off shell of the physical π-π cross section is required. A detailed comparison of the OPE predictions with experimentally measured multiparticle production is being conducted by Chew, Rogers, and Snider[8] following earlier work by Berger.[6]

In a previous paper[3] we considered a multiperipheral bootstrap model of a single Regge pole in which duality was assumed to be exact. In this paper we investigate the generation of the leading trajectories in π-π scattering, namely, the ρ and P, via the model described above. In the absence of a duality assumption this results in a

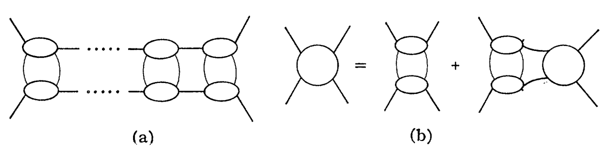

FIG. 3. (a) Multiperipheral contribution to the multiparticle production cross section. (b) Graphical representation of the Bethe-Salpeter equation for π-π scattering amplitude.

[5] G. F. Chew, Lawrence Radiation Laboratory (private communication).

[6] For comparison of this type of model with experiments, see E. L. Berger, Phys. Rev. **179**, 1567 (1969).

[7] D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento **26**, 896 (1962), hereafter AFS; see also D. Amati, S. Fubini, A. Stanghellini, and M. Tonin, *ibid.* **22**, 569 (1961); L. Bertocchi, S. Fubini, and M. Tonin, *ibid.* **25**, 626 (1962); C. Ceolin, F. Duimio, S. Fubini, and R. Stroffolini, *ibid.* **26**, 547 (1962); L. Bertocchi, E. Predazzi, A. Stanghellini, and M. Tonin, *ibid.* **27**, 913 (1963).

[8] G. F. Chew, T. W. Rogers, and D. R. Snider, Lawrence Radiation Laboratory (private communication).

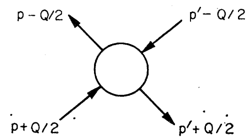

FIG. 4. Momentum assignment for the pion-pion scattering amplitude.

partial boostrap of these trajectories. A complete bootstrap model may be possible, but will of necessity be more complicated than the one we consider.

In the following two sections we generalize our previous work contained in Paper I, to treat both the ρ and P poles simultaneously and use experimental information to eliminate the duality assumption. Section IV contains the numerical solution to these equations at $t=0$ together with a discussion of the range of solutions possible. Section V deals with the generalization of this model to nonzero momentum transfers, allowing the calculation of the t dependence of the trajectories. This of necessity includes a treatment of the interaction between the poles and the cut produced by the P pole. The experimental relevance of this work is discussed in Sec. VI. It is suggested that the experimentally observed Regge poles are really effective Regge poles which represent both pole and cut contributions, and that only at extremely high energies will the actual Regge poles dominate the scattering amplitude.

II. FORMULATION OF MULTIPERIPHERAL MODEL FOR π-π SCATTERING

Our procedure is as follows: We first present the formal equations of the multiperipheral model, diagonalize the resulting Bethe-Salpeter equation for the special case of $t=0$, by using the $O(4)$ symmetry of the equation, and finally invoke several reasonable approximations which simplify the numerical calculations and make the bootstrap aspects of this equation clear. The Bethe-Salpeter equation shown in Fig. 3(b) for the π-π scattering amplitude T is

$$T(p,p',Q)=B(p,p',Q)+\int d^4q\frac{B(p,q,Q)T(q,p',Q)}{[(q-\frac{1}{2}Q)^2+\mu^2][(q+\frac{1}{2}Q)^2+\mu^2]},\quad(2.1)$$

where the momentum assignments are those shown in Fig. 4 and $s=-(p-p')^2$, $u=-(p+p')^2$, and $t=-Q^2$. Here the isospin indices have been suppressed.

For the remainder of this section we consider $t=0$ only, and drop this variable from T and B. As explained in Paper I, the discontinuity of B across the cut in the s variable is given by unitarity, and has the form

$$\text{Im}B(p,p')=\frac{\pi^3}{2pp'}\int_0^\infty dk^2\frac{|T(p,p',k)|^2}{(x^2-1)^{1/2}}\theta(x-x_0),\quad(2.2)$$

where T is again the pion-pion scattering amplitude; but in this case two of the pions are on the mass shell and

$$x=(p^2+p'^2+s)/2pp',\quad x_0=\cosh(\eta+\eta'),$$

with

$$\cosh\eta=(p^2+k^2+\mu^2)/2pk$$

and

$$\cosh\eta'=(p'^2+\mu^2+k^2)/2p'k.$$

The function B also has a similar cut in the u variable, which becomes a cut for negative s or x, which we consider later in relation to defining the signature of the Regge poles that are being calculated. In particular, for $t=0$ the imaginary part of B is given directly in terms of the pion-pion elastic cross section by the relation

$$\text{Im}B=\frac{[s(s-4\mu^2)]^{1/2}}{(2\pi)^4}\sigma_{\text{el}}(s),\quad(2.3)$$

where the cross section is for pions off the mass shell. Since both the input and output of these equations are the pion-pion scattering amplitude, the bootstrap aspects are evident but not yet explicit.

Again following Paper I, Eq. (2.1) can be diagonalized in the $O(4)$ index n; however, the resulting equation cannot be directly continued into the complex n plane due to the presence of both left- and right-hand cuts in B in the x plane. The procedure for the analytic continuation in n is similar to the usual continuation in l, and is accomplished by the introduction of signature. This is as follows: We define

$$T^{\pm}(x)=T_R(x)\pm T_L(-x),$$

where T_R and T_L are the contribution due to the right- and left-hand cut, respectively, to T. The $O(4)$ partial-wave projection of $T^{\pm}$ then takes the form

$$T_n^{\pm}=4\pi\int_{X_0}^\infty dx[\text{Im}T_R(x)\mp\text{Im}T_L(-x)]f_n(x),\quad(2.4)$$

with $f_n(x)=[x-(x^2-1)^{1/2}]^{n+1}$ and $X_0=1+s/4\mu^2$. The diagonal form of Eq. (2.1) is then

$$T_n^{\pm}(p^2,p'^2)=B_n^{\pm}(p^2,p'^2)+\frac{1}{2(n+1)}\int dq^2\frac{q^2}{(q^2+\mu^2)^2}\times B_n^{\pm}(p^2,q^2)T_n^{\pm}(q^2,p'^2).\quad(2.5)$$

The analytic continuation of the plus and minus equations is now obtained by the continuation of B_n^+ and B_n^-, since these functions have a unique definition. A pole of $T_n^{\pm}$ in the n plane at $n=\alpha_{\pm}$ contributes to the scattering amplitude the term

$$T(p,p')=\frac{\beta^{\pm}(p^2,p'^2)}{4\pi}[x+(x^2-1)^{1/2}]^{\alpha_{\pm}}\frac{e^{-i\pi\alpha_{\pm}}\pm1}{\sin\pi\alpha_{\pm}},\quad(2.6)$$

where $\beta^{\pm}(p^2,p'^2)$ is the residue of the pole in the n plane.

We now consider the isospin structure of these equations. Note that there are three relevant isospins to consider, the total isospin of the ladder I which is the isospin of the output pole, and the isospin of the four-particle vertices along $I_{\parallel}$ and across the ladder $I_{\perp}$. Since unitarity, which is expressed by Eq. (2.2), is diagonal in isospin, a definite value for I is given directly by a sum of $I_{\perp}$, and is obtained by use of the crossing relations. On the other hand, a definite value of $I_{\perp}$ is in turn given by a sum of $I_{\parallel}$, again obtained by simple application of the crossing relations. The resulting relation between I and $I_{\parallel}$ is most simply expressed by introducing the functions

$$\mathrm{Im}B_{II'}(p,p')=\frac{\pi^3}{2pp'}\int\frac{dk^2}{(x^2-1)^{1/2}}\times T^I(p,p',k)T^{I'*}(p,p',k)\theta(x-x_0)\,,\quad(2.7)$$

and the superscripts on the T's refer to the value of $I_{\parallel}$. The imaginary part of B with a definite isotopic spin along the ladder (t-channel isospin) is

$$\mathrm{Im}B^{I=0}=\mathrm{Im}B_{11}+\tfrac{1}{3}\,\mathrm{Im}B_{00}\,,$$
$$\mathrm{Im}B^{I=1}=\tfrac{1}{4}\,\mathrm{Im}B_{11}+\tfrac{1}{3}\,\mathrm{Im}(B_{10}+B_{01})\,,\qquad(2.8)$$

where we neglect the $I=2$ contribution due to the absence of Regge poles in that channel. Because of the Pauli principle, $I=0$ has only even-signature amplitudes and $I=1$ only odd. We then obtain

$$B_n^+(p^2,p'^2)=8\pi\int dx\,\mathrm{Im}B^0(p,p')f_n(x)\,,$$
$$B_n^-(p^2,p'^2)=8\pi\int dx\,\mathrm{Im}B^1(p,p')f_n(x)\,.\qquad(2.9)$$

III. DETERMINATION OF KERNEL FROM EXPERIMENTAL π-π SCATTERING

The equations which we have obtained appear to be of a bootstrap nature in that the T's, which are the solutions of Eq. (2.1), are the same T's as are used to calculate B in Eq. (2.8). This is not the case, however, since the assumptions used in deriving this set of equations restricts the energy region for the various T's. Since the T's solutions of Eq. (2.1) are calculated by a multiperipheral model, s must be large, certainly greater than 10 GeV^2, and in the region in which multiparticle production dominates the cross section. On the other hand, the input T's in Eq. (2.2) have a large contribution from the resonance region, and may well be dominated by the region of s less than 2 GeV^2. From this discussion we see that the only region of overlap is the high-energy part of the input, which must be equal to the output, and a partial bootstrap is possible only for this part of the amplitude.

At this point we consider a possible simplification of this model, based on the observation that the average subenergies are small in production processes. If one considers the pion-pion elastic cross section determined from experiment,[9] one sees that except for the high-energy tail the cross section is dominated by ρ resonance. If one approximates the input by the resonance contribution, the calculation that is actually being done is just the old-fashioned ρ bootstrap for a different range of the variables, i.e., $t=0$ rather than $t=m_\rho^2$. The results of this preliminary calculation are the following: If one wants to produce a ρ Regge pole with the experimentally observed intercept [$\alpha_\rho(0)\approx 0.5$], the input ρ width must be of the order of twice the experimental width, and because of the similarity to the older calculations this result was certainly not unexpected. It is of interest, however, that the P pole produced by this model has an intercept very close to unity, and that both trajectories have about the same slope and are in reasonable agreement with experiment.

With these results in mind we described the procedure used to approximate the input T's. Clearly some source of additional interaction is required if reasonable results are to be obtained. The two most reasonable possibilities are either including the effects of the high-energy tail which are certainly there since the P exists or making some rather drastic assumption about the off-shell dependence of the cross section. The off-shell dependence of the pion-pion scattering amplitude is already constrained to some extent by the fact that the experimental measurement of this amplitude can be accomplished only with the assumption that this dependence is weak —at least when only one pion is off the mass shell. Furthermore, the nearest singularity in the pion-mass variable is at $9\mu^2$, and, since this is a three-body branch cut, important contributions probably come from much larger values. For these reasons we restrict ourselves to rather weak dependence on these variables, allowing substantial variation only when the mass is changed by the order of a GeV.

Direct observation of the P contribution to pion-pion scattering has not yet been possible; however, the Regge-pole fits to pion-nucleon and nucleon-nucleon scattering together with the factorization imply a nearly constant high-energy tail for the elastic cross section, and because of this nearly constant behavior this contribution can be quite important in the calculation of output scattering amplitude in the limit of very high energies.

We now take as our interaction terms the contribution of the resonance region of pion-pion scattering as given by experiments including the ρ, f^0, and g contributions, plus the Pomeranchuk contribution to the high-subenergy region. Since we are looking for poles in the n plane, it is convenient to assign certain n-plane singularities to these various terms, although the exact positions of those singularities associated with the resonance terms cannot be taken very seriously. The procedure will be similar to that in Paper I, where the input T's

[9] W. Selove, in *Meson Spectroscopy*, edited by C. Baltay and A. H. Rosenfeld (W. A. Benjamin, Inc., New York, 1968).

TABLE I. Value of the solutions.

	Λ^2 (GeV^2)	α_P	α_ρ	$g_P{}^2$	$g_{\rho,\text{in}}{}^2$, $n=\alpha_\rho$	$g_{\rho,\text{in}}{}^2$, $n=\alpha_P$	$g_{\rho,\text{out}}{}^2$
$m_1=0$	4	0.95	0.45	0.1	0.119	0.126	0.049
$m_1=-1$	3	0.97	0.55	0.1	0.087	0.099	0.042

are taken to be the ρ and P Regge poles, but rather than trusting that duality will allow the ρ contribution to give an accurate representation of the resonance region, we simply adjust the ρ coupling to make duality exact. This of course means that our input ρ pole has nothing in principle in common with the output ρ pole, which is that applicable to high-energy processes. Should these terms prove to be equal, we will have shown that duality is not violated in this case. The form we assume for the input T's in Eq. (2.8) is

$$T_{\text{in}}{}^I(p,p';k)=g_{I,\text{in}}{}^2\phi_I(p^2,(p-k)^2,k^2)\phi_I(p'^2,(p'-k)^2,k^2)\times\tfrac{1}{2}\zeta^I(k^2)\left(\frac{pp'}{s_0}[x+(x^2-1)^{1/2}]\right)^{\alpha^I(-k^2)},\quad(3.1)$$

where ϕ_I is the vertex function normalized to unity at the zero value of all three arguments, and contains all the off-shell dependence. The quantities $\alpha^I(-k^2)$, ζ^I, and $g_{I,\text{in}}{}^2$ are the trajectory, the signature factor, and the coupling to the π-π system, respectively, for the Regge pole with isospin I, evaluated at $t=-k^2$. The signature factor is

$$\zeta^I(k^2)=(e^{-i\pi\alpha(-k^2)}\pm1)/\sin\pi\alpha^I.\quad(3.2)$$

We now impose the condition that

$$B_n(p^2,p'^2)=\frac{1}{(2\pi)^3}\int dx\,[s(s-4\mu^2)]^{1/2}\sigma_{\text{el}}(s)f_n(x)\quad(3.3)$$

for small values of p^2 and p'^2, i.e., near the mass shell, and allow $g_{1,\text{in}}{}^2$ to be a function of n. The quantity $g_{1,\text{in}}{}^2$ will be needed both for $n\approx\alpha_\rho(0)$ and $n\approx\alpha_P(0)$. The resulting n-plane interaction terms are computed as in Paper I, where the form of Eq. (3.1) allows analytic integration of x in Eq. (2.9). For example, the contribution of the term containing two ρ poles, ImB_{11}, to the $I=0$ amplitude is

$$B_{11,n}{}^+(p^2,p'^2)=g_{\rho,\text{in}}{}^4\frac{\pi^4}{pp'}\left(\frac{pp'}{s_0}\right)^{n+1}\times\int_0^\infty dk^2\frac{\phi_\rho{}^2(p^2,-\mu^2,k^2)\phi_\rho{}^2(p'^2,-\mu^2,k^2)}{n+1-2\alpha_\rho(-k^2)}\times\left(\frac{s_0\gamma(pk)\gamma(p'k)}{pp'}\right)^{n+1-2\alpha_\rho(-k^2)},\quad(3.4)$$

where $\gamma(p,k)=e^{-\eta}$. This function has just the AFS cut[7] arising from two ρ's, beginning at $n_c=2\alpha_\rho(0)-1$. If the P pole has an intercept near 1, the $I=0$ channel will have a cut arising from two P's near the expected position of the pole, and a more distant but perhaps stronger cut due to the two ρ's. The $I=1$ channel has a ρ-P cut near the expected position of the ρ pole, and again the two-ρ cut farther away.

A modification of the signature factors is required to eliminate the ghost poles that appear in Eq. (3.2) when $\alpha(t)$ is at a correct signature point for negative t. The procedure which we adopt is simply to replace $\sin\pi\alpha(-k^2)$ by 1 when $\alpha(-k^2)$ reaches the half-integer before the first ghost pole.

In Paper I we examined in some detail the dependence of the solution to the type of equation we are considering on the particular off-mass-shell vertex function used. In that work quite a number of different forms of ϕ were tried, and as long as the functions did not vary too rapidly, the results were more or less independent of the particular form of ϕ_I chosen. For this reason we use the simple factorizable expression

$$\phi_I{}^2(p^2,q^2,k^2)=\left(\frac{\Lambda^2}{\Lambda^2+p^2}\right)^{1/2}\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^{1/2}\left(\frac{\nu^2}{\nu^2+k^2}\right)^{m_I},\quad(3.5)$$

where Λ is a free parameter, but necessarily of the order of 1 GeV or larger, and ν for the P can be determined by the diffraction peak. For $I=0$ we take $\nu=1$ GeV and $m_0=1$.

The situation for the ρ is somewhat less clear in that we are really trying to represent an amplitude that really looks more like a p-wave resonance in the s channel, which, rather than falling off like some diffraction peak, looks more like a polynomial in the t variable $(-k^2)$. For this reason we considered two cases, $m_1=0$ and $m_1=-1$, with ν fixed to be 1 GeV. The condition given in Eq. (3.3) is imposed in both cases, and this should reduce the sensitivity of the output to the particular form assumed for the input ρ term.

IV. NUMERICAL SOLUTIONS AT $t=0$

Equation (2.5) is readily soluble by the numerical method described in Paper I, where the location of poles in the n plane is determined by obtaining the eigenfunctions of the homogeneous form of this equation. The general form of the output pole obtained is

$$T_{\text{pole}}{}^I(p,p')=\frac{\alpha^I+1}{\pi E'}\psi(p^2)\psi(p'^2)\zeta^I[x+(x^2-1)^{1/2}]^{\alpha^I}=g_{I,\text{out}}{}^2\,\tfrac{1}{2}\zeta^I\left[\frac{pp'}{s_0}[x+(x^2-1)^{1/2}]\right]^{\alpha},\quad(4.1)$$

where α^I is the position of the pole, $\psi(p^2)$ is the first eigenfunction evaluated at $n=\alpha^I$, and E' is the derivative of the first eigenvalue with respect to n.

Let us now enumerate the fixed parameters that enter into the calculation. The Regge trajectories were taken to be linear functions of $t,\alpha(t)=\alpha+\alpha' t$, with the slopes both taken to be the canonical value of 1 GeV^{-2}. We assume the values of the output α_ρ and α_p, and this determines at what values of n Eq. (3.3) must be used to obtain the input ρ coupling. For convenience we chose the input ρ intercept to be that of the output ρ. Since the values of $g_{\rho,\text{in}}$ determined by Eq. (3.3) depend on the value of $g_{p,\text{in}}$, we chose a value of $g_{p,\text{in}}$ in the range allowed by experiment (see Appendix). With α_p, α_ρ, and $g_{p,\text{in}}$ all fixed ,we vary Λ to see whether the remaining bootstrap condition on the P, $g_{p,\text{in}}=g_{p,\text{out}}$, can be satisfied. Note that $\alpha_{p,\text{in}}=\alpha_{p,\text{out}}$ has already been imposed. Once a solution has been found, the parameters of the output ρ poles are also calculated.

The results of these calculations are given in Table I for both forms of $m_1=0$, -1. We tested the effect of the condition $\alpha_{\rho,\text{in}}=\alpha_{\rho,\text{out}}$ by fixing $\alpha_{\rho,\text{in}}=0.5$, and the results are given in Table II for the case $m_1=0$; one can see that the parameters of the P are essentially unchanged. It should be noted that there is some range of acceptable solutions possible around those given in the tables; with α_p determined to about 2 and 10% variations of α_ρ and g_p are possible.

A comparison of the output ρ parameters with experiment is in principle possible, and in the Appendix we examine the various types of experiment that give some information about the ρ-π-π coupling. The values we have obtained are in reasonable agreement with those obtained from factorization and Regge-pole fits to charge-exchange processes. The experimental ρ intercept $\alpha_\rho=0.57$ is somewhat larger than ours, and leads one to favor the solution with $m_1=-1$. As far as duality is concerned, our results indicate that it is correct within a factor of 2 for the case we have considered.[10]

The solutions we have obtained of course imply definite multiparticle production cross sections, and a comparison between a model very similar to ours and experiment is being carried out by Chew, Rogers, and Snider.[8]

It was shown by AFS[7] that the average multiplicity of particles generated by a multiperipheral model at large s is quite simple, and is given by

$$\langle N\rangle = C_N \ln s/s_0\,, \tag{4.2}$$

where

$$C_N=g^2\frac{\partial\alpha_p}{\partial g^2}\,,$$

and g is the coupling constant of the three-particle vertex. For our case, with a fixed ratio of P-to-ρ coupling, $\alpha_p(0)$ is the solution of

$$E(\alpha)=g^4\,.$$

From this relation, and noting that each time a g^4 appears in our ladder two particles are produced, we obtain

$$C_N=2g^4\frac{\partial\alpha}{\partial g^4}=2E(\alpha)\frac{\partial n}{\partial E(n)}\bigg|_{n=\alpha}\,. \tag{4.3}$$

From our solutions C_N is around 0.2, to be compared with experimental estimates of 1 to 3,[11] but these depend heavily on ultra-high-energy cosmic-ray data, which have large errors.

The explanation of the small multiplicity predicted by our model is the following: The eigenvalue $E(n)$ of the homogeneous equation has a logarithmic singularity at $n=n_c$ which dominates the derivative of E if α is near n_c. This is the case for our solutions as the P intercept is close to unity. We will return to this difficulty later, since it will be seen that the proximity of this cut also causes problems with the t dependence of the trajectories.

TABLE II. Value of the solution $\alpha_{\rho,\text{in}}$ fixed to be $\alpha_{\rho,\text{in}}=0.5$.

	Λ^2	α_p	$\alpha_{\rho,\text{out}}$	g_p^2	$g_{\rho,\text{in}}^2$, $n=\alpha_{\rho,\text{in}}$	$g_{\rho,\text{in}}^2$, $n=\alpha_p$	$g_{\rho,\text{out}}^2$
$m_1=0$	3 GeV^2	0.957	0.51	0.1	0.120	0.128	0.059

V. SOLUTION FOR $t\neq 0$

We generalize our previous calculation to consider a value of $|t|$ less than 1 GeV^2. The $O(4)$ symmetry of Eq. (2.1) is for $t\neq0$, broken by the propagator and by the function $B(p,p',Q)$. The symmetry breaking in B is produced by the vertex functions $\phi_I(p\pm\frac{1}{2}Q,\,-\mu^2,\,k\pm\frac{1}{2}Q)$, by the signature factors $\zeta^I(k+\frac{1}{2}Q)\zeta^{*I'}(k-\frac{1}{2}Q)$, and by the sum $\alpha^I(k+\frac{1}{2}Q)+\alpha^{I'}(k-\frac{1}{2}Q)$. This last term does not break the symmetry if the slopes of the trajectories are equal, as in the model we are considering.

If one expands in a power series in t the terms breaking the $O(4)$ symmetry, the expansion parameter of the propagator is the quantity t/μ^2, whereas the breaking terms in B are proportional to t/s_0 and t/Λ^2 and should be negligible for small values of t. In this approximation B is diagonal in n and $B_n(p^2,p'^2,t)$ is again given by Eq. (3.4), with k^2 replaced by $k^2-\frac{1}{4}t$. Since all the symmetry breaking is now in the propagator, we apply directly the method of Chung and Snider,[12] defining

$$\tilde{T}(p,p',Q)=\frac{T(p,p',Q)}{[(p-\frac{1}{2}Q)^2+\mu^2][(p+\frac{1}{2}Q)^2+\mu^2]}\,. \tag{5.1}$$

The $O(4)$ projection of Eq. (2.1) then becomes

$$(p^2-\tfrac{1}{4}t+\mu^2)^2\tilde{T}_{n,n'}{}^l+p^2t(f_{nn}{}^l\tilde{T}_{n,n'}{}^l+f_{n,n+2}{}^l\tilde{T}_{n+2,n'}{}^l$$
$$+f_{n,n-2}{}^l\tilde{T}_{n-2,n'}{}^l)=B_n+\frac{1}{2(n+1)}\int q^2dq^2B_n\tilde{T}_{nn'}\,, \tag{5.2}$$

[10] For other comparisons see E. L. Berger and G. C. Fox, this issue, Phys. Rev. **188**, 2120 (1969).

[11] P. V. R. Murthy, Argonne National Laboratory Report (unpublished).

[12] V. Chung and D. R. Snider, Phys. Rev. **162**, 1639 (1967).

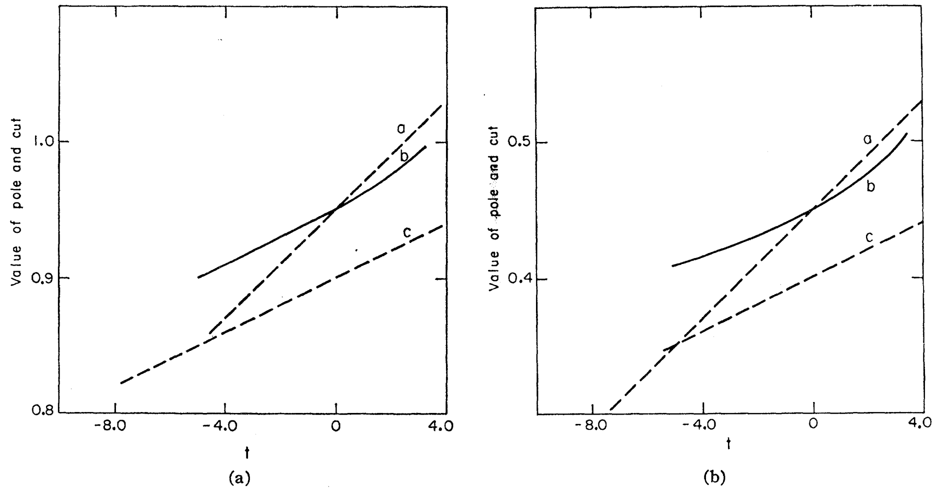

FIG. 5. (a) n-plane singularities in the $I=0$ channel. Curve a is the input P trajectory, curve b is the output P trajectory, and curve c is the input cut produced by two P's. The variable t is given in units of μ^2. (b) n-plane singularities in the $I=1$ channel. Curve a is the input ρ trajectory, curve b is the output ρ trajectory, and curve c is the input cut produced by the ρ and the P.

where

$$f_{nn}{}^l=(A_n{}^l)^2+(A_{n-1}{}^l)^2,$$
$$f_{n,n+2}{}^l=A_n{}^lA_{n+1}{}^l,$$
$$f_{n,n-2}{}^l=A_{n-1}{}^lA_{n-2}{}^l,$$

and

$$A_n{}^l=\left(\frac{(n-l+1)(n+l+2)}{4(n+1)(n+2)}\right)^{1/2}.$$

l is the usual angular momentum and n, $n'=l+$integer. The system of equations given in Eq. (5.2) is an infinitely coupled integral equation; however, the coefficients of the nondiagonal part $f_{n,n'}{}^l$ become very small as n and n' grow. In addition, the coupling terms are proportional to $p^2t/(p^2-\frac{1}{4}t+\mu^2)^2$, which is less than unity for $t<4\mu^2$. This allows one to solve Eq. (5.2) accurately with only a few coupled equations; for example, for $t>-1$ GeV^2 no more than four coupled equations are important.

We solved Eq. (5.2) for $l=n'$ for the solutions reported in Table I, and the resulting trajectories are shown in Figs. 5(a) and 5(b) for the case with $m_1=0$. The general features are the following: For positive t the trajectories appear to have about the same slope as the input trajectories, but, as the pole approaches the cut, it appears to be repelled and ends up with about the same slope as the cut, $\frac{1}{2}$ of the input slope. This effect is again due the logarithmic singularity in E, which is in turn due the singularity present in the kernel.

To clarify the question of the interaction of a pole with a cut, let us consider a simple model proposed by Frazer and Mehta[13] in which the denominator function [the function $g^2-E(n)$ in our model] is assumed to have the form

$$D(n)=n-(a+bt)+c\ln(n-n_c). \tag{5.3}$$

The trajectory is obtained from the equation $D(\alpha)=0$.

Let us consider the variation of D as a function of t for the case in which c is small. For t large positive, the zero is at $n=a+bt$; as t is reduced the pole moves to the left, approaching the cut, but never passes through the cut. Because of the logarithmic singularity at n_c the residue of the pole gets quite small when n approaches n_c, as the logarithmic term controls the derivative of D with respect to n. There are of course poles on all other sheets of the logarithmic function. For large positive t these poles simply move with the pole on the physical sheet, but as t is reduced these poles move by the point n_c and continue on to the left. One of these poles is close to the physical sheet and should have a normal residue. Thus for negative t it appears that the cut-plus-pole combination looks like a pole on the second sheet of the n plane, and this single complex pole should provide a good representation of the amplitude. It is of course true that at infinite energies the actual pole, which is to the right of n_c, dominates, but at finite energies the pole on the second sheet should be more important, due to its larger residue. From a phenomenological point of view the behavior of the pole and cut appears to be the following. For positive t the pole moves linearly with the expected slope, and for some negative t it intersects with the rather weak P cut and passes into the second sheet, becoming complex, but continuing to be a linear trajectory with the normal slope. Some of the consequences of such a model have been investigated by Ball and Zachariasen in a recent paper.[14]

Let us now see how this discussion relates to our model. Because of the large intercept of the P trajectory, the point of intersection of a linear trajectory with

[13] W. R. Frazer and C. H. Mehta, Phys. Rev. (to be published).

[14] J. S. Ball and F. Zachariasen, Phys. Rev. Letters 23, 346 (1969).

the cut is very close to $t=0$, and the calculation around $t=0$ is near this complicated region of crossover.

In our model the kernel becomes complex when evaluated for $n<n_c$, as can be seen from Eq. (3.4); however, the imaginary part of B_n has a simple factorizable form. We then write the kernel in the form

$$K=\mathrm{Re}K+i\psi\phi. \tag{5.4}$$

The function analogous to $D(n)$ for our equation is the determinant of $1-K$, and because the imaginary part is factorizable we obtain the simple result

$$\begin{aligned}\det(1-K)&=\det(1-\mathrm{Re}K)\det[1+i(1-\mathrm{Re}K)^{-1}\psi\phi]\\&=\det(1-\mathrm{Re}K)[1+i\langle(1-\mathrm{Re}K)^{-1}\psi|\phi\rangle].\end{aligned} \tag{5.5}$$

The zeros of the real part of the determinant then give the location of the poles on the second sheet of the n plane; these zeros are most easily obtained by solving the eigenvalue equation that uses the real part of the kernel. In Fig. 6 we show the inverse of the first eigenvalue plotted versus n, for the solution of Table I with $m_1=0$ in the $I=1$ channel. In this case $n_c=0.4$. Note that the function we have plotted has a behavior very similar to the function D given in Eq. (5.3). Changing the value of t corresponds roughly to shifting the curve vertically. For t positive the function has only one zero to the right of the cut, but as t moves to the left the curve falls and for some value of t develops three zeros. The rightmost is the actual pole, which is stuck to the right of the singularity at n_c. The next zero is a sort of mirror image of the pole in that $\ln|n-n_c|$ is a symmetric function around n_c. The leftmost zero is the position of the pole on the second sheet, which moves freely as one varies t. Furthermore, it appears that the curve to the right of 0.5 joins smoothly with the curve to the left of 0.3, meaning that the trajectory should be nearly linear and one should be able to extrapolate the location of the effective pole for negative t from the behavior of the actual pole for positive t.

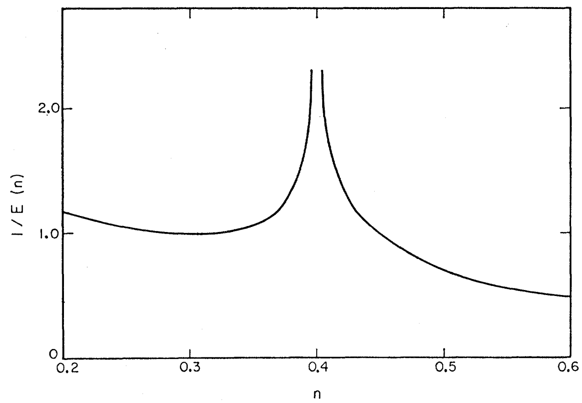

FIG. 6. Plot of the inverse of the first eigenvalue $E(n)$ as a function of n for the $I=1$ channel.

VI. DISCUSSION

In this paper we have presented a dynamical calculation based on the multiperipheral model which produced acceptable values of the intercept of the ρ and P trajectories and reasonable values of their coupling to the π-π system. The range of possible intercepts of the P is rather restrictive, and may be considered one of the predictions of this model. The value of the ρ coupling is also predicted to be relatively small and in agreement with the rather uncertain number obtained from experiment.

The apparent defects of this model are the very small value of the multiplicity predicted and the behavior of the trajectories calculated. Both of these effects are directly related to the existence of the P as a normal Regge pole. Admittedly, one needed to include this pole in order to have enough interaction to produce the P and the ρ in the output, but it seems inescapable that a P-like object does exist and plays an important part in observed high-energy processes such as πN scattering and NN scattering. There are of course several possible means to resolve the apparent conflict of this type of theory with experiment, other than discarding this model as incorrect, or claiming that some at present unknown term is important, or that some violent off-shell dependence is required.

One possibility is that the Pomeranchuk trajectory is exactly unity at $t=0$. In this case the P pole and all the cuts obtained from the iteration of this pole are no longer separated singularities. The resulting combination of singularities might have a considerably different behavior from the simple logarithmic behavior that one obtains when the intercept is less than 1.

Another possibility is the following conjecture: We accept the P cuts as correctly given by a logarithmic singularity that forces all poles to remain to the right of the cuts, and this will be the observed behavior as the energy goes to infinity. Based partly on the observation that pure cut terms have not been shown to be important at current accelerator energies, we conjecture that in some intermediate-energy region the amplitudes containing a pole plus a cut due to the P are well represented by a single complex trajectory which represents the pole on the second sheet of the n plane and that it is this pole that has been observed experimentally and has a slope of 1 GeV^{-2}. This type of pole then appears to pass through the cut without difficulty, and can easily be a linear function of t. The multiplicity implied by such a pole is still given by Eq. (4.3); however, the function α is the smooth one that would be obtained by removing the logarithmic singularity, and can be obtained in our model by simply evaluating the multiplicity for some positive value of t for which the effect of the cut is less important. A rough evaluation of this quantity gives C_N in the range of 1 to 2. Note also that the trajectories have a reasonable slope if one extrapolated the trajectory from the positive-t region. The calculation of the coupling of the poles to the π-π system at $t=0$ is probably not strongly affected by the presence of the branch cut. The use of an effective pole to represent a cut is very much like using a Breit-Wigner resonance form to represent a branch cut in the energy variable. Finally, the use of these effective poles in the input simply includes a more accurate description of the intermediate-energy amplitude used to calculate the kernel.

If such a picture is correct, the experimental implications are as follows: At $t=0$ the energy dependence should be that given by the effective pole, since there is little difference between the true pole and the effective one. However, for negative values of t, the true pole has a small residue, but dominates for sufficiently high energies because of higher intercept. At a fixed value of t and increasing energy one should see a shift in the energy dependence that marks the transition between the intermediate- and high-energy regions. The new energy dependence should be that given by a pole with half the slope of the effective pole, which controls the energy dependence at lower energies. Note, however, that the leading dependence is still given by a pole and therefore still satisfies factorization.

This conjecture about effective poles may still be valid if the P intercept is unity but the analysis of this paper is no longer applicable.

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APPENDIX: DETERMINATION OF REGGE-POLE COUPLINGS FROM EXPERIMENT

The P coupling constant as defined in Eq. (4.1) can be evaluated from the total π-π cross section. This cross section is estimated to be about 15 mb, from the factorization theorem and the observed NN and πN total cross sections.[15] This gives $g_{p,\mathrm{out}}^2 \approx 0.13$. If one assumes that the t dependence of the scattering amplitude is similar to that observed for other elastic processes, the elastic cross section can be calculated to be a few mb.

The determination of the slope of the P trajectory from fits to experimental data is somewhat confused due to the presence of the P' trajectory. The value of 1 GeV^{-2} seems to be a reasonable guess for the average effect of these two trajectories.[10]

The ρ Regge-pole coupling to the π-π system can again be obtained by factorization, and the Regge-pole fits to πN and NN charge-exchange scattering. At $t=0$ the $I=1$ pion-pion amplitude is given by

$$T(s)=\frac{1}{(2\pi)^5}\frac{m^2}{\sqrt{s}}\frac{A_{\pi N}^2}{\phi_1},$$

where ϕ_1 is the NN helicity amplitude and $A_{\pi N}$ is the usual nonflip πN amplitude. All these quantities refer to the ρ contribution only. Unfortunately the analysis of np charge exchange is complicated by uncertainties about how the pion trajectory is to be included and whether or not a conspiracy occurs. As a result, the ρ parameters for this process are not well determined, but using the analysis of Ref. 16, we obtain $g_{\rho,\mathrm{out}}^2 \approx 0.025-0.02$.

Another way of estimating $g_{\rho,\mathrm{out}}^2$ is from the coupling constant[17] $\gamma_{\rho\pi\pi}$ obtained from the e^+-$e^- \to \pi\pi$ data with the vector-dominance hypothesis.[18] The value from this source is $g_{\rho,\mathrm{out}}^2 \approx 0.07$–$0.04$, depending on the extrapolation of this quantity from $J=1$ to $J=\frac{1}{2}$. The last estimate is to use the experimental ρ width and the extrapolation from the ρ mass to $t=0$ as defined by the Veneziano formula. This result is in agreement with the value obtained above.

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