Vector Currents and Current Algebra. II. An N-Point Beta-Function Model*

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A simple N-point beta-function model (generalized Veneziano model) of the hadron bootstrap is assumed, and the properties of vector currents consistent with it are investigated. We find that this hadron bootstrap admits conserved vector currents satisfying the Gell-Mann current algebra in a first approximation which assumes single vector-meson poles in the form factors and requires factorization only at resonances on leading Regge trajectories. We believe the techniques employed in this simplified model will be useful in constructing current amplitudes in more general dual zero-width models. In addition, a model for a Pomeranchon contribution which does not fall off at large q^2 is proposed. Throughout we treat amplitudes for one or two vector currents and an arbitrary number N of spinless hadrons.

I. INTRODUCTION

IN this paper we make the first step in a study of currents consistent with the N-point beta-function model (generalized Veneziano model) of the meson bootstrap.[1,2] A number of general properties of currents in such Reggeized zero-width models with duality have been discussed in the preceding paper.[3] Here we explicitly consider the question of the existence of vector current amplitudes that are compatible with current algebra and consistent with this particular hadron bootstrap.

We shall show that the N-point beta-function model[4–6] to first approximation admits current amplitudes for one or two conserved vector currents (CVC) and N mesons, where the two-current amplitudes satisfy the constraints given by the time-time and time-space current-density commutation relations of the Gell-Mann algebra.[7] Our results are a first approximation, since we assume *single* vector-meson poles in the "masses" q_i^2 of the currents and satisfy the factorization (unitarity) constraints in *all* channels for *leading* trajectories only.

We believe that these two restrictions are intimately related and that the lack of factorization on nonleading trajectories can be remedied only by including more vector-meson poles in the q_i^2. Factorization will no doubt give constraints on the form factors (i.e., the vector-meson–current coupling constants). The factorization of nonleading trajectories in the hadron problem[8] determines the vector-meson spectrum and thus will have important consequences for current amplitudes. Indeed, we feel that the factorization of lower trajectories and the introduction of further poles in the q_i^2 represent a qualitatively more intricate problem than the present work. For example, one can see in Ref. 1 how involved the parametrization of arbitrary form factors becomes.

Most of this paper is devoted to a study of the "orbital factor" of the amplitudes, i.e., the space-time part which contains the poles and Regge behavior. We also investigate the "internal symmetry factor" of the amplitudes, assuming the Chan-Paton[9] internal symmetry factors for the hadron amplitudes. We note here that recently a very interesting model has been proposed by Mandelstam[10] which includes also a "spin factor." In his model the lowest-mass vector mesons have orbital angular momentum zero and spin angular momentum 1, whereas, in the simple model we discuss, they have orbital angular momentum 1 and spin angular momentum zero. We remark that, of course, this simple model has a spin-zero ghost with imaginary mass on the leading (vector-meson) trajectory, since its intercept is positive, and ghosts on lower trajectories with imaginary coupling constants. Mandelstam's model removes the spin-zero ghost with imaginary mass at the expense of having leading trajectories with imaginary coupling constants ("repulsive trajectories") and equal masses for the ρ meson and the pion. However, in general, Mandelstam's model has a better particle spectrum. For example, the simple model has no nonzero three-particle vertices with an odd number of unnatural spin-parity particles (i.e., $\omega\rightarrow\rho\pi$, $A_2\rightarrow\rho\pi$, etc., are excluded). Clearly our current amplitudes must inherit all these bad features of the hadron amplitudes, but we feel that our general approach to the consistency problem will apply to more realistic models for the hadrons.

As a very useful technical aid in our construction, we expand the single-current amplitudes V^μ and the two-

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[1] R. C. Brower and M. B. Halpern, Phys. Rev. **182**, 1779 (1969). Our amplitudes are for the most part a generalization of the specific four-body amplitudes of this paper to arbitrary numbers of hadrons, arbitrary masses q^2, and general internal symmetry states of the currents.

[2] M. Bander [Weizmann Report, 1969 (unpublished)] and H. Sugawara [Tokyo Report, 1969 (unpublished)] have also considered the double-helicity-flip two-current amplitude for $N=2$.

[3] R. C. Brower and J. H. Weis, preceding paper, Phys. Rev. **188**, 2486 (1969), hereafter referred to as I.

[4] C. J. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969).

[5] Chan Hong-Mo and Tsou Sheung Tsun, Phys. Letters **28B**, 485 (1969); Z. Koba and H. B. Nielsen, Nucl. Phys. **B10**, 633 (1969).

[6] K. Bardakci and H. Ruegg, Phys. Rev. **181**, 1884 (1969).

[7] M. Gell-Mann, Physics **1**, 63 (1964), and other papers.

[8] K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969); S. Fubini and G. Veneziano, Nuovo Cimento (to be published).

[9] Chan Hong-Mo and J. Paton, CERN Report No. Th. 994, 1969 (unpublished).

[10] S. Mandelstam, Phys. Rev. **184**, 1625 (1969).

current amplitudes $M^{\mu\nu}$ in terms of essentially all the available momenta. When there are more than five external lines, this will be a dependent set. However, in the construction of amplitudes this causes no problem and allows one to make the invariant amplitudes free of kinematic singularities. This is analogous to the use of a dependent set of invariants in the construction of the hadronic amplitudes.[4] As discussed in I, we always deal with amplitudes with N spinless hadrons and the covariant tensor amplitudes for currents. Throughout we attempt to present the basic kinematics and techniques in a manner that might be naturally extended to treat the important problems of (i) arbitrary form factors and (ii) axial-vector currents.

In Sec. II we construct single-current amplitudes V^μ with N hadrons which satisfy CVC and have single vector-meson (V) poles in the mass q^2 of the current. The orbital factor is first discussed in Sec. II A. Factorization for it directly follows from factorization of the corresponding vector-meson N-hadron amplitude. Moreover, if further vector mesons V_n are similarly included, it is evident that factorization will not determine their couplings f_{V_n} to the current. In Sec. II B internal symmetries are easily incorporated following Chan and Paton.[9] The result is a factorizable single-current amplitude with no exotic resonances *or* currents.

In Sec. III we construct the two-current amplitudes $M^{\mu\nu}$ with single vector-meson poles in q_i^2 and with divergences given exactly by the single-current amplitude V^u of Sec. II, as demanded by current algebra. Factorization at poles in subenergies that overlap both currents is again a trivial consequence of factorization for the hadronic-amplitude $VV\to N$ mesons, but factorization at poles in subenergies overlapping only one current is satisfied only for leading trajectories. In Sec. III B the isospin symmetry factors of Chan and Paton are again employed to obtain a factorizable internal symmetry factor with no exotic resonances or currents.

In Sec. IV we present an interesting parametrization for the Pomeranchuk trajectory which cannot have form factors (poles in q_i^2) and requires exotic resonances. Such a Pomeranchon with little damping for $q^2\to -\infty$ has been suggested on the basis of electroproduction data.[11]

In Sec. V we discuss possible modifications of the solution of Secs. II and III within the single-vector-dominance approximation. We shall give terms which allow one to modify the space-space commutators[12] without affecting the others. We also show how to construct amplitudes that violate CVC and current algebra. Although such flexibility may be useful in a more complete implementation of factorization, it may also indicate a lack of uniqueness of the consistent currents in our model without considerable input from current algebra.

II. SINGLE-CURRENT AMPLITUDES

In this section, we give an explicit construction of the single-current amplitude $V^\mu(q)$ with N hadrons consistent with the N-point beta-function meson bootstrap. This provides a simple solution to the full set of properties discussed in I for a single vector current in the zero-width approximation:

(*i*) *Divergence condition.* $q_\mu V^\mu=0$, i.e., CVC.

(*ii*) *Generalized vector-meson dominance.* The only singularities in q^2 are simple poles that completely determine V^μ (no subtractions in q^2 dispersion relations). The residues of the poles at $q^2=m_{V_n}{}^2$ are products of the vector-meson (V_n) scattering amplitudes and current–vector-meson coupling constants (f_{V_n}).

(*iii*) *Regge asymptotics.* V^μ has Regge behavior in all subenergies $s_{ij\cdots k}=(p_i+p_j+\cdots+p_k)^2$.

(*iv*) *Particle spectrum.* The only singularities in $s_{ij\cdots k}$ are simple poles with polynomial residues in overlapping variables. They occur at fixed positions (masses) in particular channels (with given quantum numbers), as determined by the hadron amplitudes.

(*v*) *Factorization.* At any pole in V^μ the residue factorizes into a current amplitude and a purely hadronic amplitude.

As discussed in I, we can always project out the conserved part of a tensor $T^\mu(q)$ with the projection operator $\mathcal{P}^{\mu\nu}(q)=g^{\mu\nu}-q^\mu q^\nu/q^2$ to satisfy condition (i). However, condition (ii) demands that V^μ have fixed singularities only at the masses of the vector mesons m_{V_n}, and not at $q^2=0$. Indeed, the central problem is to introduce a vector-meson singularity at $q^2=m_V{}^2$ and to continue off the mass shell at fixed spin, $J=1$, without introducing unwanted singularities in q^2. In our model (and probably in general), once condition (ii) is satisfied, the remaining conditions (iii)–(v) follow trivially from the corresponding properties of the hadron amplitudes.

Through condition (ii), our current amplitude inherits the pathologies of the N-point beta-function meson bootstrap. These include ghosts on the leading vector-meson trajectory at $\alpha=0$ (imaginary mass states) and on lower trajectories[8] (imaginary coupling constants), as well as numerous difficulties with the quantum numbers of the particle spectrum. However, we are optimistic that many of the methods presented here can be adapted to more realistic dual, zero-width hadron models.

Our present discussion is based on the simple meson bootstrap which consists of products of the orbital factors $B(p_1,p_2,\ldots,p_N)$ (N-point functions) and the internal symmetry factors[9] $\frac{1}{2}\,\mathrm{Tr}(\lambda_1\lambda_2\cdots\lambda_N)$ which are *summed* over all permutations (except cyclic and anti-

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[11] H. Harari, Phys. Rev. Letters **22**, 1078 (1969).

[12] R. C. Brower, A. Rabl, and J. H. Weis, Nuovo Cimento (to be published). In this paper the $N=2$ case is studied in detail. The space-space commutators are investigated through the Bjorken limit; Pomeranchuk exchange in Compton scattering, electroproduction phenomenology, and electromagnetic mass differences are investigated. The reader may find this paper helpful in understanding the results given here because of its simpler kinematics.

cyclic) of the particles. The single beta function for each term in the sum yields a nondegenerate factorized spectrum on the leading trajectory and the isospin factor achieves the exclusion of all exotic resonances. It is sufficient to consider one particular term with given ordering of the hadrons, which we choose to be $p_1, p_2, \ldots, p_N$ for definiteness. Corresponding to this ordering of the hadrons, there will be N terms in the single-current amplitude. The orbital factors for these terms are designated by $V_i^\mu(q)$ for the ordering $p_1, p_2, \ldots, p_{i-1}, q, p_i, \ldots, p_N$, and have their external-line insertions (ELI), i.e., poles which dominate for $q_\mu \to 0$, normalized as described in I [Eq. (3.3)]. After constructing a single term V_i^μ in Sec. II A, we show in Sec. II B how to take the appropriate sum over i for the Chan-Paton internal symmetry scheme. Some general features of such sums are discussed in I (see Sec. III). The resulting amplitudes satisfy all the conditions (i)–(v), with the exception of some violations of (iv) due to the pathologies of the purely hadronic bootstrap.

In Appendix B we show that, for $N=3$ and physical $(q^2=0)$ photons, the amplitude given here is the same as the photoproduction amplitude given in Ref. 1. Hence our results may be considered as a generalization of the results of Ref. 1 to arbitrary q^2 and N although the techniques used are different.

A. Orbital Factor

The first step is to calculate the amplitude for a vector meson and N spinless hadrons. To do this, we start from the $(N+2)$-point beta-function amplitude.[4–6] For the particular ordering of the particles $a, b, 1, 2, \ldots, N$, it is convenient to choose the integration variables appropriate to the multi-Regge diagram of Fig. 1. Hence, we find[6]

$$B_{N+2}=\int_0^1 du_0\cdots du_{N-2}I_{N+2}(u_0,\ldots,u_{N-2})\,, \quad (2.1)$$

where the integrand is defined recursively by

$$\begin{aligned}I_{N+2}&(u_0,\ldots,u_{N-2})\\&=u_0^{-\alpha_{ab}-1}(1-u_0)^{-\alpha_{b1}-1}(1-u_0u_1)^{-\Delta_{b2}}\cdots\\&\quad\times(1-u_0\cdots u_{N-2})^{-\Delta_{b,N-1}}I_{N+1}(u_1,\ldots,u_{N-2})\,,\end{aligned} \quad (2.2)$$

and where

$$\alpha_{ij}=a_{ij}+b(p_i+p_{i+1}+\cdots+p_j)^2=a_{ij}+bs_{ij}\,, \quad (2.3)$$

$$\Delta_{ij}=(\alpha_{i,j}-\alpha_{i+1,j})-(\alpha_{i,j-1}-\alpha_{i+1,j-1})\,. \quad (2.4)$$

In (2.4) the relation

$$0=\alpha_{ii}=a_{ii}+bm_i^2 \quad (2.5)$$

is to be understood. From now on we choose our units so that $b=1$.

We now take $\alpha_{ab}=1+(q^2-m_V^2)$, where $q=p_a+p_b$, go to the pole at $\alpha_{ab}=1$, and extract the coefficient of the relative momentum $(r=p_a+p_b)$ in the residue. Thus

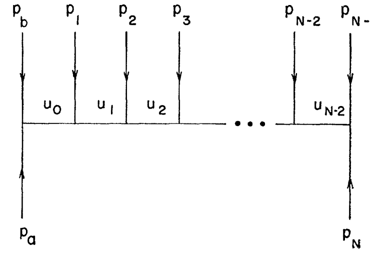

FIG. 1. Choice of variables for B_{N+2}.

we consider[6]

$$\begin{aligned}B_1^\mu(q)=\int_0^1& du_1\cdots du_{N-2}\\&\times[q^\mu+2p_1^\mu+2p_2^\mu u_1+\cdots+2p_{N-1}^\mu(u_1\cdots u_{N-2})]\\&\qquad\times I_{N+1}(u_1,\ldots,u_{N-2})\,,\end{aligned} \quad (2.6)$$

corresponding to the vector-meson amplitude of Fig. 2. In I_{N+1}, α_{ai} now becomes $\alpha_{Vi}=a_{Vi}+(q+p_1+\cdots+p_i)^2$. We shall sometimes write (2.6) in the alternative form[13]

$$B_1^\mu(q)=q^\mu B_{N+1}+2\sum_{m=1}^{N-1}{}_R\, p_m^\mu B_{N+1}(\alpha_{V,l<m}-1)\,. \quad (2.7)$$

Only the trajectories which are displaced relative to their usual values have been explicitly indicated. The symbol $\alpha_{V,l<m}$ means all $\alpha_{V,l}$ for $1\le l<m$, and the subscript R is explained in Appendix A. The trajectory displacements are just those required to compensate for the momentum factors so as to yield the correct asymptotic behavior. The correct asymptotic behavior is assured, since we started with a B_{N+2} with the correct behavior.

The obvious way to construct the amplitude for a single conserved vector current from the purely hadronic amplitude B^μ is to take

$$V_1^\mu(q)=C(q^2)m_V^2/(m_V^2-q^2)[g^\mu{}_\nu-q^\mu q_\nu/q^2]B_1^\nu(q)\,, \quad (2.8)$$

where $C(0)=1$. However, as seen in I, B^μ must be a function of q^2 if property (iv) is to be satisfied. Equation (2.6) clearly has a natural continuation in q^2 satisfying this property; it can be regarded as a function of all the p_i with q determined by energy-momentum conservation.

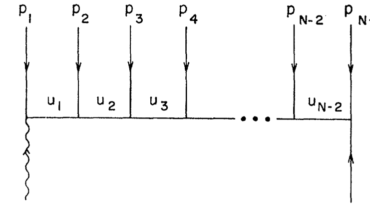

FIG. 2. Choice of variables for B^μ.

[13] Further expressions are given in Appendix A.

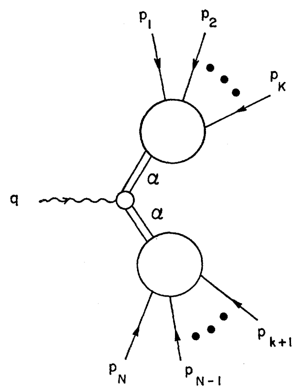

FIG. 3. Reggeized internal-line insertion. The double lines indicate *any* resonances in the family generated by the trajectory α.

In addition, we must assure that the apparent singularity at $q^2=0$ in (2.8) does not occur in V^μ. Since $C(q^2)=O(q^2)$ would eliminate the ELI poles, it is not permitted, and we must have

$$q_\mu B_1{}^\mu(q)=O(q^2). \quad (2.9)$$

Fortunately this condition can be satisfied if the trajectories are restricted so that

$$a_{Vk}=a_{1k} \quad (2.10)$$

or, equivalently,

$$a_{k+1,N}=a_{k+1,V}.$$

These restrictions mean that the trajectory corresponding to the current and k adjacent hadrons must be the same as the trajectory for the k hadrons alone. We call the soft-current poles due to resonances on such trajectories "Reggeized internal-line insertions" (see Fig. 3). These are a natural generalization of the insertion of a soft current into an internal line of a tree graph. When (2.10) holds, we find that not only does (2.9) hold, but also $q_\mu B^\mu(q)\equiv 0$ (the proof is given in Appendix A).[14] We note that in our simple model with its restricted spectrum (2.10) always holds.

The continuation in q^2 described above and the restriction $C(q^2)=1$ introduce minimal q^2 dependence into (2.8). In fact, we have explicitly verified that this corresponds to no subtractions in the q^2 dispersion relation for the *single* term $V_i{}^\mu$, as required by assumption (ii).[15] Therefore our final result is

$$V_i{}^\mu(q)=F(q^2)B_i{}^\mu(q), \quad (2.11)$$

where

$$F(q^2)=m_V{}^2/(m_V{}^2-q^2), \quad (2.12)$$

and the "off-mass-shell" vector-meson amplitude $B_i{}^\mu$ is given by (2.6) with the appropriate cyclic permutation of the p_i.[16]

We note that the factorization property (v) follows from the factorization of the N-point beta functions. This is obvious for leading trajectories, and one can verify that the continuation in q^2 does not affect the spectrum of the lower trajectories either.[17]

We have not yet investigated the divergence properties of the $B_{V_n}{}^\mu$ for V_n lying on nonleading trajectories. In considering these vector mesons the degeneracy of nonleading trajectories[8] must be taken into account. We believe that further vector-meson poles can be included in a manner analogous to the above which satisfies factorization and leads to no constraints on the couplings f_{V_n} (except $\sum_n f_{V_n}=1$).

B. Internal Symmetries

We now show how to incorporate $SU(3)$ symmetry without obtaining exotic resonances [$SU(n)$ for $n\neq 3$ can be treated in the same manner]. In I we noted that the absence of exotic resonances implies that only one quark line (δ-function contraction) is permitted between each adjacent pair of external momenta.[9,10,18] When octets and singlets of external particles are projected out, one obtains the internal symmetry factor[9]

$$\tfrac{1}{2}\,\mathrm{Tr}(\lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_N}) \quad (2.13)$$

for the ordering of particles p_1, p_2, ..., p_N in the hadronic amplitude. The matrices λ_a are the usual $SU(3)$ matrices, $a_i=0, 1, \ldots, 8$. The factorization of the internal symmetry factor is clear from the δ-function construction and is explicitly exhibited by the identity

$$\tfrac{1}{2}\,\mathrm{Tr}(\lambda_{a_1}\cdots\lambda_{a_N})=\sum_{a=0}^{8}[\tfrac{1}{2}\,\mathrm{Tr}(\lambda_{a_1}\cdots\lambda_{a_k}\lambda_a)] \times[\tfrac{1}{2}\,\mathrm{Tr}(\lambda_a\lambda_{a_{k+1}}\cdots\lambda_{a_N})]. \quad (2.14)$$

Specifically, we choose the external particles to be members of the pseudoscalar nonet $P(\eta,\pi,K,\ldots)$. (Roughly speaking, this is a spinless quark model.) There are two different internal trajectories: the exchange-degenerate vector nonet $V(\omega,\rho,K^*,\phi)$ and tensor nonet $T(\ldots,A_2,K_N{}^*,f_0)$, and the exchange-degenerate pseudoscalar nonet P and nonexistent axial-vector nonet.

With this elegant, but very approximate, model of the hadron bootstrap, the symmetry factors for $V_i{}^\mu$ are

[14] This result has been obtained independently in Refs. 8 in another context. We note that here we are in fact using the divergence condition very much like a Ward identity.

[15] This can be seen by examining the large-q^2 behavior at fixed values of the BCP group variables for Fig. 2: N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967). One finds $s_{1k}\propto q^2$, whereas the other invariants in (2.6) are fixed. Hence for sufficiently small momentum transfers the amplitude will decrease as $q^2\to\infty$.

[16] A factor g^{N-2}, where g is the strength of the strong-interaction vertex, is to be understood in all such equations.

[17] As one can see from Refs. 8, the factorization properties (spectrum) of lower trajectories depend upon the internal trajectory intercepts. Since these of course do not vary with q^2, the spectrum is independent of q^2. Thus the only nonkinematical q^2 dependence is in the current-hadron-hadron vertex. We neglect here the complications of the linear dependencies between states which give a reduction in the spectrum.

[18] H. Harari, Phys. Rev. Letters **22**, 562 (1969); J. L. Rosner, *ibid.* **22**, 689 (1969).

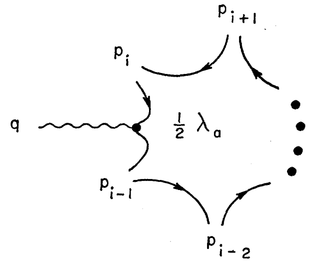

FIG. 4. Duality diagram for currents. The current (no-quark state) couples to vector mesons (two-quark states).

obviously

$$V_i{}^\mu{}_{aa_1\cdots a_N}(q)=\tfrac{1}{2}\,\mathrm{Tr}[\lambda_{a_1}\cdots\lambda_{a_{i-1}}(\tfrac{1}{2}\lambda_a)\lambda_{a_i}\cdots\lambda_{a_N}]\times F(q^2)B_i{}^\mu(q) \quad (2.15)$$

and

$$V^\mu{}_{aa_1\cdots a_N}(q)=\sum_i V_i{}^\mu{}_{aa_1\cdots a_N}(q) \quad (2.16)$$

for the single ordering $p_1, \ldots, p_N$ of the hadrons and a, $a_i=0, 1, \ldots, 8$.[19] Thus the absence of exotic resonances has allowed us to introduce precisely the nonet of conserved currents whose charges can generate $SU(3)$.

As in I, the internal symmetry solution can be represented by a duality diagram (Fig. 4). Moreover, one can give a further interpretation of the diagram for the orbital part which has considerable heuristic appeal, particularly for the two-current amplitudes. The current is regarded as a no-quark object that couples to the two-quark system that has vector-meson (bound-state) poles. Hence the current-quark-quark vertex is always to be thought of as a form factor in q^2.

III. TWO-CURRENT AMPLITUDES AND CURRENT ALGEBRA

In this section we discuss the construction of amplitudes for two vector currents (covariant correlation functions),

$$M_{(\pm)}{}^{\mu\nu}(q_1,q_2)=\tfrac{1}{2}[M_{ab}{}^{\mu\nu}(q_1,q_2)\pm M_{ba}{}^{\mu\nu}(q_1,q_2)],$$

with the following properties (see I for normalization conventions):

(*i*) *Divergence conditions.* (a) Charge-current density algebra:

$$q_{1\mu}M_{(\pm)}{}^{\mu\nu}(q_1,q_2)\rightarrow\tfrac{1}{2}(1\mp1)V^\nu(q_1+q_2)$$

for $q_{1\mu}\rightarrow0$; (b) photon correspondence:

$$q_{1\mu}M_{\gamma\gamma}{}^{\mu\nu}(q_1,q_2)=O(q_1{}^2),$$

and similarly for $q_{2\nu}$.

(*ii*) *Generalized vector dominance.* There are only simple poles in $q_1{}^2$ and $q_2{}^2$, and the residues of the poles at $q_1{}^2=m_{V_n}{}^2$ (or $q_2{}^2=m_{V_n}{}^2$) are products of single-current amplitudes for the production of a vector meson of mass m_{V_n} and coupling constants f_{V_n}.

(*iii*) *Regge asymptotics.* $M_{(\pm)}{}^{\mu\nu}$ has Regge behavior in all subenergies *except* those subenergies $(q_1\cdot p_k)$ that overlap the two-current channel $[(q_1+q_2)^2=t]$.

(*iv*) *Particle spectrum.* The only singularities in the subenergies are simple poles with polynomial residues in overlapping variables. The locations (masses) and quantum numbers of the poles are determined by the hadronic and single-current amplitudes.

(*v*) *Factorization.* (See Fig. 6 of I.) (a) "Linear factorization" at poles in subenergies *not* overlapping t; (b) "quadratic factorization" at poles in subenergies overlapping t.

As these conditions indicate, the single-current amplitude V^μ will be a basic input in the construction of the two-current amplitudes $M_{(\pm)}{}^{\mu\nu}$, just as the hadronic amplitude was the input for the construction of the single-current amplitude. However, the new features presented by the nonvanishing divergence (ia) and the "quadratic factorization" (vb) make the connections of $M_{(\pm)}{}^{\mu\nu}$ with V^μ far less trivial to satisfy. As demonstrated in I, these two conditions [(ia) and (vb)] require the existence of fixed poles and the extension of the divergence condition (1a) to the kinematical region $q_1{}^2=0$ and $q_2{}^2=t$ (to within terms that vanish at $q_1{}^\mu=0$).

In spite of the strength of the above assumptions, to obtain a unique solution it may be necessary to impose the full strength of the current-density commutation relations:

(*i'*) *Divergence conditions of current algebra.*

$$q_{1\mu}M_{(\pm)}{}^{\mu\nu}(q_1,q_2)=\tfrac{1}{2}(1\mp1)V^\nu(q_1+q_2),$$
$$M_{(\pm)}{}^{\mu\nu}(q_1,q_2)q_{2\nu}=\tfrac{1}{2}(1\mp1)V^\mu(q_1+q_2)$$

for all $q_1{}^2$, $q_2{}^2$. Our construction procedure will avoid the direct use of (i'), so that it may be regarded as a heuristic argument for the power of the conditions (i)–(v) in a possible eventual proof of the current-algebra condition (i').

Our construction of $M_{(\pm)}{}^{\mu\nu}$ is limited to the single-vector-meson (ω,ρ,K^*,etc.) approximation for the form factor $F(q^2)$ [Eq. (2.12)], and the resultant single-current amplitudes $V^\mu(q)$ [Eq. (2.16)], which we constructed in Sec. II. We find it encouraging that in this approximation we can satisfy linear factorization completely (va) and quadratic factorization (vb) on all leading trajectories with *exact* current algebra (i'). The isospin factor for this first-order current-algebra solution is presented in Sec. III B. In Appendix B we show that for physical Compton scattering ($N=2$) the amplitudes obtained here are in general the same as those given in Ref. 1.

A. Orbital Factor

The first step is to calculate the amplitude for two vector mesons and N hadrons. We can construct this amplitude from the amplitude B^μ for one vector meson

[19] Our normalization is such that, for example, $\pi^\pm=2^{-1/2}\times(\lambda_1\mp i\lambda_2)$. For physical photons one uses the Gell-Mann–Nishijima formula $Q=\frac{1}{2}\lambda_3+\frac{1}{6}\sqrt{3}\lambda_8$.

and $N+2$ additional particles. As before, we go to a pole at $\alpha=1$ and extract the coefficient of the relative momentum in the residue. The expression for the resulting tensor amplitude $B^{\mu\nu}$ is more symmetrical if we sum over momenta to the right for one meson (q_1,V_1) and momenta to the left for the other (q_2,V_2). We then find (for the fixed ordering of the hadrons 1, 2, ..., N)

$$\begin{aligned} B_{ij}^{\mu\nu}(q_1,q_2) &= -q_1^\mu q_2^\nu B_{N+2} - q_1^\mu\Big[2 \sum_{n=j-1}^{j+1}{}_L\, p_n^\nu B_{N+2}(\alpha_{n<l,V_2}-1)\Big] \\ &\quad -\Big[2 \sum_{m=i}^{i-2}{}_R\, p_m^\mu B_{N+2}(\alpha_{V_1,l<m}-1)\Big] q_2^\nu \\ &\quad -4 \sum_{m=i}^{i-2}{}_R \sum_{n=j-1}^{j+1}{}_L\, p_m^\mu p_n^\nu B_{N+2}(\alpha_{V_1,l<m}-1;\, \alpha_{n<l,V_2}-1) \\ &\quad -2g^{\mu\nu}\big[B_{N+2}(\alpha_{V_1,l<j}-1;\, \alpha_{i-1<l,V_2}-1) \\ &\qquad -B_{N+2}(\alpha_{V_1,l<j}-1;\, \alpha_{i-1<l,V_2}-1;\, \alpha_{V_1V_2}-1)\big]. \quad (3.1) \end{aligned}$$

Comparing (3.1) with (A1) and (A2), one sees that the only really new feature is the $g^{\mu\nu}$ term.

We now discuss a few features of this lengthy expression in order to clarify its structure. The indices i and j indicate that V_1 is just to the left of i and V_2 is just to the left of j. To avoid ambiguities for adjacent mesons $(i=j)$, we adopt the convention that $B_{ii}^{\mu\nu}(q_1,q_2)$ and $B_{ii}^{\nu\mu}(q_2,q_1)$ refer to the ordering with q_1 to the left and to the right of q_2, respectively. If the same trajectory—which can only be $\alpha_{V_1V_2}$—occurs in both sets of arguments in the third term, it is lowered by two units. The summations and the inequalities for the lowered trajectories are understood to include the momenta q_1 and q_2 in the appropriate position. The reader may find it helpful to draw diagrams such as Fig. 5 in order to keep track of the lowered trajectories.

The divergences $q_{1\mu}B_{ij}^{\mu\nu}$ and $B_{ij}^{\mu\nu}q_{2\nu}$ behave rather differently for nonadjacent $(i\neq j)$ and adjacent $(i=j)$ vector mesons. For nonadjacent mesons, the conditions (2.10) for the Reggeized internal-line insertions can be satisfied independently for both mesons, and the presence of a second meson does not affect the vanishing of the divergence for the first. The reader should thus find

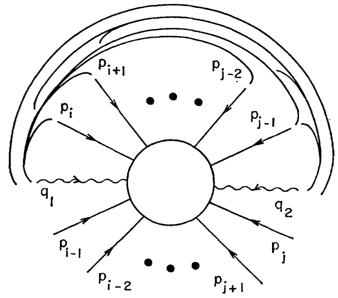

FIG. 5. The last B_{N+2} in Eq. (3.1). The trajectories lowered by one unit have their corresponding subenergies indicated.

very plausible the identities

$$q_{1\mu}B_{ij}^{\mu\nu}\equiv 0, \quad B_{ij}^{\mu\nu}q_{2\nu}\equiv 0 \quad (\text{for } i\neq j), \qquad (3.2)$$

which follow directly from (3.1), (A6), and identities similar to (A4).

As pointed out in I, the nonadjacent-current terms cannot contribute to the divergence (i′) due to its pole structure. Hence the orbital factors for nonadjacent-current terms can be represented by the divergenceless tensors

$$M_{ij}^{\mu\nu}(q_1,q_2)=F(q_1^2)F(q_2^2)B_{ij}^{\mu\nu}(q_1,q_2) \quad (\text{for } i\neq j). \quad (3.3)$$

The justification of this construction is essentially the same as for the single-current amplitude $V_i^\mu(q)$. The adjacent-current terms pose the only essentially new problem and the remainder of this subsection is devoted to them.

For adjacent vector mesons, condition (2.10) can also be satisfied *except* when the two-vector-meson channel (designated t channel) is involved. The difficulty arises because the spins of the mesons are fixed at 1, while the q_i^2 vary in the off-shell continuation (see Sec. II A). Therefore $a_{V_1V_1}$ and $a_{V_2V_2}$ depend upon q_1^2 and q_2^2, but are related by (2.10) to $a_{V_1V_2}$, which should be independent of q_1^2 and q_2^2. This gives a nonvanishing divergence which we may calculate by using (A9) and (A10) and assuming $\alpha_{V_1V_2}(t)=\alpha_t=1+t-m_V^2$:

$$\begin{aligned} q_{1\mu}B_{ii}^{\mu\nu}(q_1,q_2) &= -q_2^\nu[B_{N+2}(\alpha_t-1)-B_{N+2}]+(m_V^2-q_2^2)C_L^\nu, \quad (3.4) \end{aligned}$$

where

$$\begin{aligned} C_L^\nu=(q_2^\nu+2q_1^\nu)[B_{N+2}(\alpha_t-1)-B_{N+2}] \\ +2\sum_n{}_L\, p_n^\nu[B_{N+2}(\alpha_{n<l,V_2}-1;\,\alpha_t-2) \\ -B_{N+2}(\alpha_{n<l,V_2}-1;\,\alpha_t-1)]. \quad (3.5) \end{aligned}$$

Similar expressions hold for $B_{ii}^{\nu\mu}(q_2,q_1)$ and q_2 divergences.

We note that for $q_{1\mu}\to 0$ one may explicitly verify that (3.4) reduces to the (correctly normalized) contribution from the ELI of V_1 on p_{i-1}. This is expected, since the $B_{ij}^{\mu\nu}$ have the correct soft poles even for $i=j$.

The $B_{ii}^{\mu\nu}$ are useful as basic building blocks even though they are not divergenceless and hence not pure spin-1 even for $q_1^2=q_2^2=m_V^2$. In a moment we shall show how to construct appropriate spin-1 tensors for q_1^2 or q_2^2 equal to m_V^2, but we first discuss the divergence conditions for the adjacent-current terms.

As we demonstrated in I, the divergence condition can be applied to a single adjacent-current term. The divergence condition on $M_{ii}^{\mu\nu}(q_1,q_2)$ is

$$q_{1\mu}M_{ii}^{\mu\nu}(q_1,q_2)=F(t)B_i^\nu(q_1+q_2), \qquad (3.6)$$

$$M_{ii}^{\mu\nu}(q_1,q_2)q_{2\nu}=-F(t)B_i^\mu(q_1+q_2), \qquad (3.7)$$

which, by the theorem of I, hold for $q_1^2=0$, $q_2^2=t$ and $q_2^2=0$, $q_1^2=t$, respectively, or, by current algebra, hold for all q_1^2 and q_2^2. From now on we choose for definiteness $i=1$, corresponding to the ordering $q_1, q_2, p_1, p_2, \ldots, p_N$ for $M^{\mu\nu}(q_1,q_2)$, and drop the subscript labels.

The decomposition of the adjacent-current term into two "signature" amplitudes is a great simplification in constructing the Chan-Paton-type solution of this section. In such a solution one has degeneracy between singlet and octet trajectories (for example, f_0 and ρ) and $M_{(\pm)}{}^{\mu\nu}$ are the even and odd parts of the *same* solution to (3.6) and (3.7). However, more general solutions can be found by adding any function $D_{(\pm)}{}^{\nu}(q_1+q_2, p_1, \ldots, p_N)$ to the right-hand side of (3.6) and $\mp D_{(\pm)}{}^{\mu}$ to (3.7) and then finding *separate* solutions to these equations for $M_{(+)}{}^{\mu\nu}$ and $M_{(-)}{}^{\mu\nu}$. The Pomeranchuk solution in Sec. IV is an example of such a procedure and it necessarily lies outside Chan-Paton models.

1. Hadronic Part

We now discuss the part of $M_{ii}{}^{\mu\nu}(q_1,q_2)$ which contains the vector-meson poles (hadronic part). As remarked above, $F(q_1^2)F(q_2^2)B_{11}{}^{\mu\nu}$ has the correct divergences at $q_{1\mu}\to 0$ and $q_{2\nu}\to 0$. However, the residue of this tensor at $q_1^2=m_V^2$ (or $q_2^2=m_V^2$) should yield a suitable single-current amplitude for a vector meson and N spinless hadrons by condition (ii). Consequently, the divergence with respect to $q_{1\mu}$ (pure spin-1 vector meson) and $q_{2\nu}$ (CVC) should be zero at $q_1^2=m_V^2$ for all q_2^2, and similarly at $q_2^2=m_V^2$ for all q_1^2. Clearly we can get zero divergences with the use of the projection operator $\mathcal{P}^{\mu\mu'}=g^{\mu\mu'}-q^\mu q^{\mu'}/q^2$, but this destroys the good divergence $q_{1\mu}\to 0$ and $q_{2\nu}\to 0$ and introduces unwanted singularities at $q_1^2=0$ and $q_2^2=0$, violating (ii). Fortunately the nonsingular projection operator $\bar{\mathcal{P}}^{\mu\mu'}(q)=g^{\mu\mu'}-q^\mu q^{\mu'}/m_V^2$ yields all these divergence conditions.

It is easy to show that the divergence of the tensor

$$F(q_1^2)F(q_2^2)\bar{B}^{\mu\nu}(q_1,q_2)=F(q_1^2)F(q_2^2)\bar{\mathcal{P}}(q_1)^{\mu}{}_{\mu'}B^{\mu'\nu'}\bar{\mathcal{P}}(q_2)_{\nu'}{}^{\nu} \quad (3.8)$$

with respect to $q_{1\mu}$ (or $q_{2\nu}$) is unchanged at $q_{1\mu}\to 0$ (or $q_{2\nu}\to 0$). For example, at $q_{1\mu}\to 0$, $\bar{\mathcal{P}}(q_1)_{\mu'}{}^{\mu}$ becomes $g_{\mu'}{}^{\mu}$ and the divergence of $B^{\mu\nu}$ is V^ν, which is unaffected by $\mathcal{P}(q_2)_{\nu'}{}^{\nu}$ because of CVC. Moreover, the conditions

$$q_{1\mu}\bar{B}^{\mu\nu}(q_1q_2)=0 \quad \text{for} \quad q_2^2=m_V^2,$$
$$\bar{B}^{\mu\nu}(q_1,q_2)q_{2\nu}=0 \quad \text{for} \quad q_1^2=m_V^2,$$

required by assumption (ii) and CVC, follow immediately from the divergence formulas for $B^{\mu\nu}$ [see (3.4) and (3.5)],

$$\begin{aligned} q_{1\mu}B^{\mu\nu}&=-q_2{}^{\nu}[B(\alpha_t-1)-B]+(m_V^2-q_2^2)C_L{}^{\nu},\\ B^{\mu\nu}q_{2\nu}&=-q_1{}^{\mu}[B(\alpha_t-1)-B]+(m_V^2-q_1^2)C_R{}^{\mu}.\end{aligned} \quad (3.9)$$

To simplify the general divergence equation for $\bar{B}^{\mu\nu}$, we add two terms that give no contributions at the vector-meson poles [or to the right-hand sides of (3.6) and (3.7) for $q_{1\mu}\to 0$ and $q_{2\nu}\to 0$]. We give an explicit expression for this new function $M_H{}^{\mu\nu}$, which we call the hadronic part because it has the correct vector-meson poles at $q_i^2=m_V^2$ and is purely Regge-behaved:

$$\begin{aligned} M_H{}^{\mu\nu}&(q_1,q_2)\\ &\equiv F(q_1^2)F(q_2^2)B_H{}^{\mu\nu}\\ &=F(q_1^2)F(q_2^2)\bar{B}^{\mu\nu}\\ &\quad+[2m_V^2g^{\mu\nu}+q_1{}^{\mu}q_2{}^{\nu}][B(\alpha_t-2)-2B(\alpha_t-1)+B]\\ &=F(q_1)F(q_2)B^{\mu\nu}-F(q_1^2)q_1{}^{\mu}C_L{}^{\nu}-F(q_2^2)C_R{}^{\mu}q_2{}^{\nu}\\ &\quad+(q_1{}^{\mu}q_2{}^{\nu}/m_V^2)F(q_1^2)F(q_2^2)[B(\alpha_t-1)-B]\\ &\quad\quad+2m_V^2g^{\mu\nu}[B(\alpha_t-2)-2B(\alpha_t-1)+B]. \end{aligned} \quad (3.10)$$

In addition to the divergence condition (3.9), we have used the double divergence

$$\begin{aligned} q_{1\mu}B^{\mu\nu}q_{2\nu}&=-m_V^2[B(\alpha_t-1)-B(\alpha_t)]-(m_V^2-q_1^2)(m_V^2-q_2^2)\\ &\quad\times[B(\alpha_t-2)-2B(\alpha_t-1)+B(\alpha_t)] \end{aligned} \quad (3.11)$$

to expand $\bar{\mathcal{P}}B\bar{\mathcal{P}}$. Note that the $q_1{}^{\mu}q_2{}^{\nu}$ term added to $\bar{B}^{\mu\nu}$ precisely cancels the second term in (3.11). This hadronic amplitude has the simple divergence

$$\begin{aligned} q_{1\mu}M_H{}^{\mu\nu}&(q_1,q_2)\\ &=m_V^2(q_2{}^{\nu}+2q_1{}^{\nu})[B(\alpha_t-2)-B(\alpha_t-1)]\\ &\quad+2m_V^2\sum_n{}_L\, p_n{}^{\nu}[B(\alpha_t-2,\, \alpha_{n<l,V_2}-1)\\ &\quad\quad-B(\alpha_t-1,\, \alpha_{n<l,V_2}-1)], \end{aligned} \quad (3.12)$$

as computed by using the identities in Appendix A.

What are the pathologies of this function? It satisfies conditions (i)–(v) with only two important exceptions: (1) The symmetric part of $M_H{}^{\mu\nu}$ is unsuitable for $M_{\gamma\gamma}{}^{\mu\nu}$, since $q_{1\mu}M_{(+)}{}^{\mu\nu}\neq O(q_1^2)$, and (2) the function does not satisfy quadratic factorization on the trajectories below the leading trajectory. In view of the theorem in I it is a little surprising that such a function exists, but we notice that the divergence does have poles in the overlapping variables α_{V_1k} (or α_{V_2k}) on the nonleading trajectories. Clearly the absence of these poles in the divergence, which is a consequence of CVC and quadratic factorization for all trajectories, plays the crucial role in forcing the fixed-power behavior into $M_{(-)}{}^{\mu\nu}$.

2. Current-Algebra Construction

Aside from the immediate interest in obtaining a solution consistent with current algebra (i′), we find that such a solution gives the simplest and most elegant means of satisfying properties (i)–(v). We must introduce a fixed-pole term $M_{\mathrm{FP}}{}^{\mu\nu}$ to satisfy the divergence conditions (3.6) and (3.7), which follow from these properties, so we try an amplitude of the form

$$M^{\mu\nu}(q_1,q_2)=M_H{}^{\mu\nu}+M_C{}^{\mu\nu}+M_{\mathrm{FP}}{}^{\mu\nu},$$

where the correction term $M_C{}^{\mu\nu}$ cancels the divergence introduced by $M_H{}^{\mu\nu}$, but does not affect the poles at $q_i^2=m_V^2$. Remarkably, we shall discover a correlation between $M_C^{\mu\nu}$ and $M_{\mathrm{FP}}^{\mu\nu}$ that is necessary to cancel nonsense poles in α_t. There are two equivalent approaches. Either one is led to the correlation by insisting that $M_{\mathrm{FP}}^{\mu\nu}$ obey current algebra, or, by demanding the correlation, one is led to the current-algebra fixed pole. Although the latter suggests a derivation of the current-algebra condition, we remind the reader that it is possible that the correction $M_C^{\mu\nu}$ could be made in an entirely different manner. Also, additional terms as discussed in Sec. V can be added to the divergence. Only the imposition of factorization can remove these ambiguities.

The reader who wishes to follow closely our construction procedure should expand each of our tensors as follows:

$$\begin{aligned}M^{\mu\nu}=&p_m^\mu p_n^\nu M_{mn}+p_m^\mu q_1^\nu M_{m(1)}+q_2^\mu p_n^\nu M_{(2)n}\\&+q_2^\mu q_1^\nu M_{(2)(1)}+g^{\mu\nu}M_0+q_1^\mu p_n^\nu M_{(1)n}+p_m^\mu q_2^\nu M_{m(2)}\\&+q_1^\mu q_2^\nu M_{(1)(2)}+q_1^\mu q_1^\nu M_{(1)(1)}+q_2^\mu q_2^\nu M_{(2)(2)}\,,\end{aligned}\tag{3.13}$$

where m is summed over $1, 2, \ldots, N-1$ and n is summed over $N, N-1, \ldots, 2$. By equating the coefficients of each tensor ($p_n^\mu p_n^\nu$, $q_1^\mu p_n^\nu$, etc.) one will discover that our equations reduce to the identities proved in Appendix A. In the discussion below, for each divergence condition we refer only to appropriate identity for the coefficient of p_n^ν.

Let us consider constructing the solution to the current-algebra condition. In terms of the "physical" amplitudes (the first five terms of the expansion), these conditions become

$$q_1\cdot p_m M_{nm}+q_1\cdot q_2 M_{(2)n}=-F(t)B_{N+1}(\alpha_{n<l,V}-1)+O(q_1^2)\,,\tag{3.14a}$$

$$q_2\cdot p_n M_{mn}+q_1\cdot q_2 M_{m(1)}=-F(t)B_{N+1}(\alpha_{V,l<m}-1)+O(q_2^2)\,,\tag{3.14b}$$

$$\begin{aligned}\tilde{M}_0=&M_0+q_1\cdot q_2M_{(2)(1)}\\=&-q_1\cdot p_m M_{m(1)}-F(t)B_{N+1}+O(q_1^2)\end{aligned}\tag{3.14c}$$

$$=-q_2\cdot p_n M_{(2)n}-F(t)B_{N+1}+O(q_2^2)\,,\tag{3.14d}$$

$$M_{(2)(1)}\text{ is arbitrary}.\tag{3.14e}$$

The fundamental difficulty is to find a form for the double-flip amplitudes[20] M_{mn} that does not introduce a kinematical singularity $(q_1\cdot q_2)^{-1}$ into the single-flip amplitudes $M_{(2)n}$ and $M_{m(1)}$ through (3.14a) and (3.14b). The unphysical amplitudes are of no help in this problem, since they are of order q_1^2 (or q_2^2). We first consider the divergenceless Regge part $(M_H^{\mu\nu}+M_C^{\mu\nu})$ that satisfies (3.14a)–(3.14e) with $F(t)$ set to zero. We take the hint from Ref. 1 that the double-flip amplitudes might be proportional to $F(q_1^2)F(q_2^2)-F(t)$. For the single-vector-dominated form factor (2.12) we have the expansion

$$F(q_1^2)F(q_2^2)-F(t)=-\left[\frac{2q_1\cdot q_2}{m_V^2}+\frac{q_1^2q_2^2}{m_V^4}\right]F(t)F(q_1^2)F(q_2^2)\,.\tag{3.15}$$

The idea is to let the term in this expansion proportional to $q_1\cdot q_2$ satisfy (3.14b) to $O(q_1^2)$ with a nonsingular single-flip amplitude and match the remainder with the "unphysical" amplitudes to give precisely zero divergence. To carry out these manipulations in a compact manner we employ the identities (A9)–(A11).

From the divergence (3.12) of the hadronic term $M_H^{\mu\nu}$ we obtain, from the coefficient of p_n^ν,

$$\begin{aligned}&\tfrac{1}{2}q_1^2\{F(q_1^2)F(q_2^2)B(\alpha_t-1)\\&\quad+F(q_1^2)[B(\alpha_t-2)-B(\alpha_t-1)]\}\\&\quad+q_1\cdot q_2F(q_1^2)F(q_2^2)B(\alpha_t-1)\\&\quad+\sum_m{}_R\,F(q_1^2)F(q_2^2)q_1\cdot p_mB(\alpha_t-2;\alpha_{V_1,l<m}-1)\\&=-\tfrac{1}{2}m_V^2[B(\alpha_t-2)-B(\alpha_t-1)]\,,\end{aligned}\tag{3.16}$$

where any number of trajectories α_{lV_2} may be displaced. This formula can also be derived from (A11) with the aid of the identity

$$F(q_1^2)=1+(q_1^2/m_V^2)F(q_1^2)\,.$$

From another form of (A11), obtained by adding $(\frac{1}{2}q_1^2+q_1\cdot q_2)[B(\alpha_t-2)-B(\alpha_t-1)]$ to both sides, we have (A10):

$$\begin{aligned}&(\tfrac{1}{2}q_1^2+q_1\cdot q_2)B(\alpha_t-2)+\sum_m{}_R\,q_1\cdot p_mB(\alpha_t-2;\alpha_{V_1,l<m}-1)\\&=-\tfrac{1}{2}m_V^2F^{-1}(t)[B(\alpha_t-2)-B(\alpha_t-1)]\,.\end{aligned}\tag{3.17}$$

In this form, we can see that $F(t)B_C^{\mu\nu}(\alpha_t-2)$ has the *same* divergence as $M_H^{\mu\nu}$, where $B_C^{\mu\nu}(\alpha_t-2)$ is identical to $B^{\mu\nu}$ except that all the $\alpha_{V_1V_2}$ arguments are set to α_t-2 and the $g^{\mu\nu}$ term is omitted. Consequently the difference of these two tensors,

$$F(q_1^2)F(q_2^2)B_H^{\mu\nu}-F(t)B_C^{\mu\nu}(\alpha_t-2)\,,\tag{3.18}$$

is a pure Regge function with no divergence.

The only difficulty with the correction piece is that it introduces vector-meson poles at $t=m_V^2$ into nonsense amplitudes M_{mn}.[20] An obvious way to cancel the poles in the nonsense amplitudes is to add a fixed-pole term,[21] $F(t)B_C^{\mu\nu}(-1)$. With the additional term, $-F(t)B_{N+1}g^{\mu\nu}$, this fixed-power contribution yields precisely the divergences required by current algebra [(3.6) and (3.7) for arbitrary q_i^2]. The reader may verify this by the use of identity (A12). Consequently a solution to the

[20] One can easily verify that the M_{mn} are double-helicity-flip amplitudes in the t-channel c.m., whereas $M_{m(1)}$ and $M_{(2)n}$ are single-flip and $M_{(2)(1)}$ and M_0 are nonflip.

[21] This is just an inversion of the reasoning used by J. B. Bonzan *et al.* [Phys. Rev. Letters **18**, 32 (1967)] and V. Singh [*ibid.* **18**, 36 (1967)] to deduce the existence of singularities in nonsense Regge residues from current algebra.

current-algebra problem (i′) is the symmetric and antisymmetric part of

$$M_{ii}{}^{\mu\nu}(q_1,q_2)=F(q_1{}^2)F(q_2{}^2)B_{H,ii}{}^{\mu\nu} -F(t)[B_{C,ii}{}^{\mu\nu}(\alpha_t-2)-B_{\mathrm{FP},ii}{}^{\mu\nu}(-1)], \quad (3.19)$$

where $B_{\mathrm{FP},ii}{}^{\mu\nu}(-1)=B_{C,ii}{}^{\mu\nu}(-1)-B_{N+1}g^{\mu\nu}$. The external momenta in B_{N+1} are $p_1, \ldots, p_{i-1}, q_1+q_2, p_i, \ldots, p_N$, just as in $B_i{}^\mu(q_1+q_2)$.

In addition to (i′) this solution satisfies conditions (ii)–(v), except quadratic factorization (vb) for nonleading trajectories. That (vb) is satisfied for all resonances on leading trajectories is most easily seen by examining (3.19) as $t\to\infty$ for fixed s_{V1k}. In this Regge limit, the part proportional to $F(t)$ has one less power of t than normal and hence contributes only to nonleading trajectories. Further, from (3.10), one sees that all terms in $B_H{}^{\mu\nu}$ except $B^{\mu\nu}$ contribute only to nonleading trajectories. Since $B^{\mu\nu}$ factorizes,[8] $M_{ii}{}^{\mu\nu}$ factorizes for leading trajectories α_{V1k}.

B. Internal Symmetries

From I and Sec. II B it is clear that the proper internal symmetry factor for $M_{ij}{}^{\mu\nu}(q_1,q_2)$ with the absence of exotic resonances is

$$\tfrac{1}{2}\,\mathrm{Tr}[\lambda_{a_1}\cdots\lambda_{a_{i-1}}(\tfrac{1}{2}\lambda_a)\lambda_{a_i}\cdots\lambda_{a_{j-1}}(\tfrac{1}{2}\lambda_b)\lambda_{a_j}\cdots\lambda_{a_N}]. \quad (3.20)$$

To illustrate some of the properties of this solution, we consider the contribution of the adjacent-current terms to $M_{ab}{}^{\mu\nu}(q_1,q_2)$:

$$M_{ab}{}^{\mu\nu}(q_1,q_2) =\sum_i \tfrac{1}{2}\,\mathrm{Tr}[\lambda_{a_1}\cdots\lambda_{a_{i-1}}(\tfrac{1}{2}\lambda_a)(\tfrac{1}{2}\lambda_b)\lambda_{a_i}\cdots\lambda_{a_N}]M_{ii}{}^{\mu\nu}(q_1,q_2) +\sum_i \tfrac{1}{2}\,\mathrm{Tr}[\lambda_{a_1}\cdots\lambda_{a_{i-1}}(\tfrac{1}{2}\lambda_b)(\tfrac{1}{2}\lambda_a)\lambda_{a_i}\cdots\lambda_{a_N}] \times M_{ii}{}^{\nu\mu}(q_2,q_1).$$

Using the relation[22]

$$[(\tfrac{1}{2}\lambda_a),(\tfrac{1}{2}\lambda_b)]_\pm=\begin{Bmatrix} d_{abc} \\ if_{abc}\end{Bmatrix}(\tfrac{1}{2}\lambda_c),$$

we easily obtain

$$M_{(\pm)ab}{}^{\mu\nu}=\tfrac{1}{2}(M_{ab}{}^{\mu\nu}\pm M_{ba}{}^{\mu\nu}) =\begin{Bmatrix} d_{abc} \\ if_{abc}\end{Bmatrix}\sum_i \tfrac{1}{2}\,\mathrm{Tr}[\lambda_{a_1}\cdots\lambda_{a_{i-1}}(\tfrac{1}{2}\lambda_c)\lambda_{a_i}\cdots\lambda_{a_N}] \times\tfrac{1}{2}[M_{ii}{}^{\mu\nu}(q_1,q_2)\pm M_{ii}{}^{\nu\mu}(q_2,q_1)].$$

From the divergence conditions (3.6) and (3.7) and Eqs. (2.15) and (2.16) we obtain

$$q_{1\mu}M_{(\pm)ab}{}^{\mu\nu}=\tfrac{1}{2}(1\mp1)if_{abc}V_c{}^\nu,$$

which are precisely the required divergence conditions of current algebra (i′).

The degeneracy of the V and T nonets (for example ω, ρ, f_0, A_2) is crucial for the success of this construction. In the construction of the orbital factor in Sec. III A it was necessary to have $1-a_t=m_V{}^2$, the mass occurring in $F(q^2)$. Since from two currents in V one can obtain some configurations with V leading trajectories and some with T leading trajectories, V and T must be degenerate. In $SU(2)$ this corresponds to

$$0^-(\omega)\otimes0^-(\omega)=0^+(f_0),$$
$$0^-(\omega)\otimes1^+(\rho)=1^-(A_2),$$
$$1^-(\rho)\otimes1^-(\rho)=0^+(f_0)+1^-(\rho)+2^+$$

(the numbers are I^G). In other words, a consistent nonet of currents can be constructed by our methods (and perhaps generally) if and only if the V and T nonets are degenerate.

Nonvanishing current commutators imply nonvanishing divergences for the $M_{(-)}{}^{\mu\nu}$. This in turn implies fixed poles with residues singular in t and the necessity of Regge trajectories with singular residues to eliminate these singularities in nonsense amplitudes. Now, if the solution of the hadron bootstrap has no exotic trajectories, we see immediately that the commutator of two currents can only be another current of the same type (i.e., there is no 10 or $\overline{10}$ part in the octet-octet commutator). The foregoing simple observations point out some effects of the hadron solution on the currents and current algebra.

Finally, we mention the duality diagrams for the adjacent-current amplitudes. As before, the current-quark-quark vertex is to be regarded as a form factor $F(q^2)$. In Fig. 6 we have represented the current-algebra solution of the adjacent-current amplitude as the sum of Regge exchange with form factors $[F(q_1{}^2)F(q_2{}^2)]$ and a fixed-pole piece which has an "exchanged" current. As the diagram indicates, there are no form factors in $q_i{}^2$ [i.e., $F(q_1{}^2)F(q_2{}^2)$] for the current-exchange piece, but there is a form factor $F(t)$ where the "exchanged" current attaches to the quark line.

IV. POMERANCHON SOLUTION

There is a solution for the symmetric amplitude $M_{(+)}{}^{\mu\nu}$ that cannot have form factors $[F(q_1{}^2)F(q_2{}^2)]$ and therefore has no counter part in purely hadronic or single-current processes. For the Pomeranchuk trajectory such a solution is particularly interesting for several

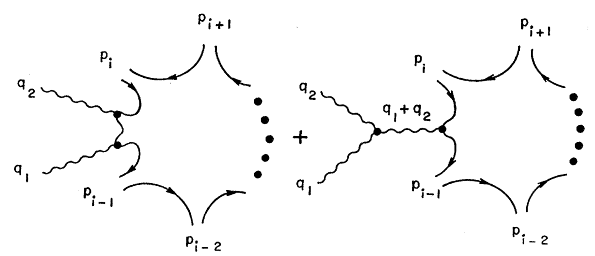

FIG. 6. Vector-meson exchange with form factors $[F(q_1{}^2)F(q_2{}^2)]$ plus a current-algebra fixed ($J=1$) singularity with $F(t)$ and no $F(q_1{}^2)F(q_2{}^2)$.

[22] M. Gell-Mann, Phys. Rev. **125**, 1069 (1962).

reasons. (1) It allows a Pomeranchon with $\alpha_P(0)=1$ to couple in double-helicity-flip (nonsense) amplitudes at the forward direction ($t=0$), as required to yield a constant total photoproduction cross section. (2) The existence of a Pomeranchon contribution that does not fall rapidly for $q^2\to-\infty$ (i.e., has no form factors) has some experimental support in recent electroproduction data.[12] (3) For $\alpha_P(0)=1$, this solution gives no *right*-signature fixed poles (e.g., $J=0, -2, \ldots$) for physical ($q_i^2=0$) Compton scattering. We remark, however, that the solution given below can be used for any trajectory in a symmetric amplitude.

In this solution, the possible kinematical singularity $[(q_1\cdot q_2)^{-1}]$ in single-flip amplitudes is avoided by directly introducing the factor

$$-\frac{2q_1\cdot q_2}{\alpha_t-1}=1-\frac{q_1^2+q_2^2-m_P^2}{t-m_P^2}, \tag{4.1}$$

where $\alpha_t=t-m_P^2+1$. This is just the term proportional to $q_1\cdot q_2$ in the factor

$$\frac{F(q_1^2)F(q_2^2)-F(t)}{F(q_1^2)F(q_2^2)} =-\frac{2q_1\cdot q_2}{\alpha_t-1}-\frac{q_1^2q_2^2}{m_V^2(\alpha_t-1)} =1-\frac{q_1^2+q_2^2-m_V^2-q_1^2q_2^2/m_V^2}{t-m_V^2}, \tag{4.2}$$

which we introduced in Sec. III. In both cases, the leading asymptotic behavior is unaffected. In the symmetric amplitudes $(\alpha_t-1)^{-1}$ is canceled by the signature factor in leading order, and fixed poles at $J=0, -2, \ldots$, are introduced to cancel the singularity in lower orders. By comparing the two expansions, one can see how similar this problem is to the current-algebra problem.

This time, we use the identity (A9) in the form

$$\tfrac{1}{2}q_1^2B(\alpha_t-2)+q_1\cdot q_2B(\alpha_t-1)+\sum_{m}{}_R\, q_1\cdot p_mB(\alpha_t-2,\, \alpha_{V1,l<m}-1) =\tfrac{1}{2}(q_1^2+q_2^2-m_P^2)[B(\alpha_t-2)-B(\alpha_t-1)]. \tag{4.3}$$

The resulting parametrization for $M_{(+)}{}^{\mu\nu}$ is the symmetric part of $M_{\text{Pom}}{}^{\mu\nu}$:

$$M_{\text{Pom}}{}^{\mu\nu}(q_1,q_2)=B_{\text{Pom}}{}^{\mu\nu}-\frac{q_1^2+q_2^2-m_P^2}{t-m_P^2}\times[B_C{}^{\mu\nu}(\alpha_t-2)-B_{\text{FP}}{}^{\mu\nu}(-1)], \tag{4.4}$$

where

$$B_{\text{Pom}}{}^{\mu\nu}=B^{\mu\nu}(q_1,q_2)-q_1{}^\mu C_L{}^\nu-C_R{}^\mu q_2{}^\nu -[q_1{}^\mu q_2{}^\nu+2g^{\mu\nu}(q_1^2+q_2^2-m_P^2)] \times[B(\alpha_t-2)-2B(\alpha_t-1)+B], \tag{4.5}$$

and as in Sec. III,

$$B_{\text{FP}}{}^{\mu\nu}(-1)=B_C{}^{\mu\nu}(-1)-B_{N+1}g^{\mu\nu}.$$

The first two terms in $M_{\text{Pom}}{}^{\mu\nu}$ cancel in the divergence, and the fixed-pole piece gives a divergence which cancels in the symmetric amplitude.

From (4.4) and (4.5) one sees that $M_{\text{Pom}}{}^{\mu\nu}$ has the same ELI poles as $B^{\mu\nu}$. Therefore, as discussed in I, the symmetric function

$$M_{\text{Pom}}{}^{\mu\nu(\Sigma)}=\sum_{i\neq j}B_{ij}{}^{\mu\nu}(q_1,q_2) +\sum_i[M_{\text{Pom},ii}{}^{\mu\nu}(q_1,q_2)+M_{\text{Pom},ii}{}^{\nu\mu}(q_2,q_1)] \tag{4.6}$$

has no ELI poles. Hence the Pomeranchon can be introduced with arbitrary coupling strength C_0 into $I=0$ symmetric amplitudes. We note that the above amplitude cannot be multiplied by the form factors $F(q_1^2)\times F(q_2^2)$ because this would introduce an unpermitted $J=0$ fixed pole (and Kronecker-δ singularities) at a right-signature point in the purely hadronic process $VV\to N$ hadrons.

For $q_1^2=q_2^2=m_P^2=0$, the fixed-pole contribution to (4.4) vanishes. Further, from (4.5) one sees that only the "unphysical" terms in $B^{\mu\nu}$ are modified. Therefore in this case $B^{\mu\nu}$ leads to perfectly acceptable photon amplitudes. This can also be seen directly from (3.4). We also note that for $N=2$, (4.6) is precisely the form of the Pomeranchon contribution suggested in Ref. 1.

Since one cannot have an $I=1$ trajectory degenerate with the Pomeranchon, this solution necessarily lies outside the Chan-Paton scheme and involves exotic resonances in cross channels. However, unlike the Pomeranchon for purely hadronic amplitudes, we do not require exotic trajectories entering into the same term as the Pomeranchon. Such a "hadronic" Pomeranchon contribution may also be present in the two-current amplitude, with form factors $F(q_1^2)F(q_2^2)$, but it will not contribute to forward elastic Compton scattering if $\alpha_P(0)=1$.

A diagrammatic representation of the above Pomeranchon solution is given in Fig. 7(a). The fixed-pole part of (4.4) is represented by a "contact" term with no form factors in q_i^2, similar to the fixed-pole piece of the current-algebra solution. We also see that our Pomeranchon (like currents) should be thought of as a no-quark object.[23] The "hadronic" Pomeranchon contribution can be represented as in Fig. 7(b)—without ELI terms, but with form factors $F(q_1^2)F(q_2^2)$ and exotic resonances (four-quark states) in crossed channels.[23] The close analogy between the Pomeranchon solution and the current-algebra solution is apparent both in our construction and in the diagrammatic representations.

[23] S. Mandelstam, Phys. Rev. Letters (to be published). Mandelstam has shown how to introduce the Pomeranchon as a no-quark trajectory along with exotic resonances on lower trajectories in the model of Ref. 10.

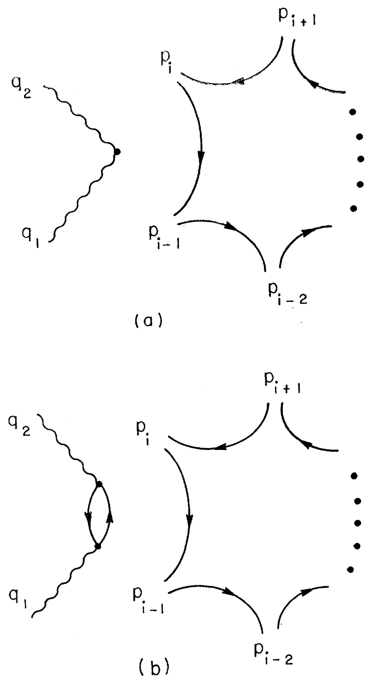

FIG. 7. Pomeranchuk as no-quark state: (a) with no form factors and $J=0$ singularity; (b) with form factors and exotic resonances in the crossed channels but with no ELI poles.

V. CONCLUSION

First we would like to mention several ways of modifying our basic amplitudes [(2.11), (3.3), (3.19), and (4.4)] by adding additional terms. Since the tensor

$$q_2^\mu q_1^\nu - q_1 \cdot q_2 g^{\mu\nu}$$

is divergenceless, it can be multiplied by a suitable invariant amplitude and added to $M^{\mu\nu}$ without affecting the divergences (i.e., the time-time and time-space commutators). In fact, our basic amplitudes have zero space-space commutators, and these terms can be used to make them nonzero.[12] Any term of the form $q_1^2 \mathcal{P}(q_1)^\mu{}_{\mu'} T^{\mu'\nu'} \mathcal{P}_{\nu'}{}^\nu(q_2) q_2^2$, or a similar term for single-current amplitudes, is clearly acceptable and does not change the divergences ($T^{\mu\nu}$ should be chosen so as not to affect the ELI poles or leading-order factorization).

The condition (2.10) on the trajectories is crucial to our construction and has been assumed throughout this paper. In our simple model it holds, but in more general models where it may not hold one apparently must resort to brute-force methods of satisfying the divergence conditions. For example, for single-current amplitudes, one could add to V^μ invariant amplitudes parametrized by beta functions multiplied by tensors $[(q\cdot p_j)p_i^\mu - (q\cdot p_i)p_j^\mu]$—note that such terms do not contribute at $q_\mu \to 0$.

There are also terms which affect the divergences. Terms proportional to q^2 and $q_1^2 q_2^2$ can be added to V^μ and $M^{\mu\nu}$, respectively, without violating the requirements for physical photons. It is more difficult to modify the divergences in other ways, but, for example, terms proportional to B_{N+1} can be added to, say, $M_{2(n)}$, where q_1+q_2, p_1, ..., p_N are the external momenta in B_{N+1}. This adds a term proportional to $q_1\cdot q_2$ to the q_1 divergence. This is completely consistent with the theorem of I, since it does not introduce poles in variables overlapping the t channel into the divergence.

More importantly, we should introduce form factors with arbitrary numbers of vector-meson poles consistent with the vector mesons on the lower trajectories in the factorized hadron solution. As we see from Ref. 1, the problem of arbitrary form factors becomes quite involved due to the necessity of avoiding ancestor trajectories. However, the essential point that $F(q_1^2)F(q_2^2) - F(t)$ can be expanded into two terms, one proportional to $q_1\cdot q_2$ and another proportional to $q_1^2 q_2^2$ [see (3.15)], still holds, as one can demonstrate by a Taylor-series expansion. We feel that this will permit general form factors to be introduced in much the same manner as in Ref. 1. But this construction procedure should be developed simultaneously with the implementation of the correct correspondence to vector-meson processes [condition (iv)] and quadratic factorization (vb) on lower trajectories.

Work is proceeding on the successive introduction of higher-mass vector-meson poles and factorization on lower trajectories. Clearly, at some stage our brute-force methods must be replaced by a more elegant technique to obtain a fully factorized solution in the N-point beta-function model.

The axial-vector currents should also be studied in this model. In the special case of one axial-vector current and three pions ($N=3$), PCAC leads to the well-known condition on the trajectories[24] $\alpha_\rho(m_\pi^2) = \alpha_\pi(m_\pi^2) + \frac{1}{2}$. We are investigating the problem of introducing axial-vector currents with pion-pole-dominated divergences into the N-hadron amplitude.

Beyond the scope of the N-point beta-function model with the Chan-Paton isospin factor are the problems of baryon trajectories, exotic resonances, and the Pomeranchon. Mandelstam has discussed these problems for the hadronic amplitude from the point of view of a relativistic quark model.[23] Here the form of the hadronic solution is far less clear, and the attempt to introduce currents may help to develop this more realistic zero-width model. Clearly, one must replace the $SU(6)$ symmetry of the present Mandelstam model[10,23] by a chiral-symmetry scheme that allows the pion mass to be zero with a finite ρ-meson mass, if there is any hope of introducing both reasonable vector and axial-vector currents.

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[24] C. Lovelace, Phys. Letters **28B**, 204 (1968); M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969); H. J. Schnitzer, *ibid.* **22**, 1154 (1969); R. Arnowitt *et al.*, *ibid.* **22**, 1158 (1969).

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APPENDIX A

We first give several alternative expressions for $B_i^\mu(q)$ and prove the identity $q_\mu B_i^\mu \equiv 0$.

From (2.7) we have

$$B_i^\mu(q) = q^\mu B_{N+1} + 2 \sum_m{}_R\, p_m^\mu B_{N+1}(\alpha_{V,l<m}-1), \quad \text{(A1)}$$

corresponding to the ordering of momenta $p_1, \ldots, p_{i-1}, q, p_i, \ldots, p_N$ and the choice of variables similar to Fig. 2. The subscript R of the summation indicates a sum over momenta to the right of q excluding the momentum immediately to its left, p_{i-1}. We may obtain directly a sum L over momenta to the left of q by following the steps leading to (2.7) but with the anticyclic permutation of the momenta (which leaves B_{N+2} unchanged):

$$B_i^\mu(q) = -q^\mu B_{N+1} - 2 \sum_m{}_L\, p_m^\mu B_{N+1}(\alpha_{m<l,V}-1). \quad \text{(A2)}$$

Comparing (A1) and (A2) and using momentum conservation yields the identity

$$B_{N+1} = B_{N+1}(\alpha_{V,l<m}-1) + B_{N+1}(\alpha_{m<l,V}-1) \quad \text{(A3)}$$

for all m. This can also be derived directly from the integral representation (2.1) and the trivial relation

$$1 = u_1u_2\cdots u_{m-1} + (1-u_1u_2\cdots u_{m-1}). \quad \text{(A4)}$$

A further expression can be obtained by converting the integral representation (2.6) into one corresponding to the multi-Regge diagram with the momenta cyclically permuted one position to the right. This procedure yields (for $i=1$)

$$B_1^\mu(q) = \int_0^1 du_1\cdots du_{N-2}\Big[-(q^\mu+2p_N^\mu)(1-u_1) + (q^\mu+2p_1^\mu)u_1 + 2p_2^\mu \frac{(1-u_1)u_1u_2}{1-u_1u_2} + \cdots + 2p_{N-2}^\mu \frac{(1-u_1)u_1\cdots u_{N-2}}{1-u_1\cdots u_{N-2}}\Big] I_{N+1}(u_1,\ldots,u_{N-2}). \quad \text{(A5)}$$

This expression is actually most easily obtained by beginning anew from B_{N+2} cyclically permuted one position to the right from Fig. 1. The expression (A5) has the advantage of exhibiting explicitly both soft-photon terms.

The result, $q_\mu B_1^\mu = 0$, may now be easily shown by using the identity

$$0 = \int_0^1 du_1\cdots du_{N-2} \frac{d}{du_1} \times [u_1^{-\alpha_{NV}}(1-u_1)^{-\alpha_{V1}}(1-u_1u_2)^{-\Delta_{V2}}\cdots \times (1-u_1\cdots u_{N-2})^{-\Delta_{V,N-2}}] I_N(u_2,\ldots,u_{N-2}),$$

which is trivially true when the u_1 integral is defined and is true by analytic continuation elsewhere. Carrying out the differentiation yields

$$0 = \int_0^1 du_1\cdots du_{N-2}\Big[-\alpha_{NV}(1-u_1) + \alpha_{V1}u_1 + \Delta_{V2}\frac{(1-u_1)u_1u_2}{1-u_1u_2} + \cdots + \Delta_{V,N-2}\frac{(1-u_1)u_1\cdots u_{N-2}}{1-u_1\cdots u_{N-2}}\Big] \times I_{N+1}(u_1,\ldots,u_{N-2}). \quad \text{(A6)}$$

From (2.3), (2.4), (2.5), and (2.10) we obtain

$$\alpha_{NV} = (q+p_N)^2 + a_{NV} = 2q\cdot(\tfrac{1}{2}q+p_N) + m_N^2 + a_{NN} = 2q\cdot(\tfrac{1}{2}q+p_N), \quad \text{(A7a)}$$

$$\alpha_{Vk} - \alpha_{1k} = 2q\cdot(\tfrac{1}{2}q+p_1+\cdots+p_k), \quad \text{(A7b)}$$

and

$$\Delta_{Vk} = 2q\cdot p_k. \quad \text{(A7c)}$$

Notice that (2.10) was crucial in deriving (A7a) and (A7b). Substituting (A7) in (A6) and comparing with (A5) immediately yields $q_\mu B_i^\mu \equiv 0$.

We now discuss identities useful for two-current amplitudes. Corresponding to the choice of variables of Fig. 8, we consider

$$0 = \int_0^1 du_0\cdots du_{N-2}\frac{d}{du_0} \times [u_0^{-\alpha_{N,V1}}(1-u_0)^{-\alpha}(1-u_0u_1)^{-\Delta_{V1,1}}\cdots \times (1-u_0\cdots u_{N-2})^{-\Delta_{V1,N-2}}] I_{N+1}(u_1,\ldots,u_{N-2}), \quad \text{(A8)}$$

where $\Delta_{V_11} = (\alpha_{V_11} - \alpha_{V_21}) - \alpha$, but for typographical convenience we have written $V1$ for V_1. Differentiating and using equations like (A7) then gives

$$0 = \int_0^1 du_0\cdots du_{N-2}\Big[-2q_1\cdot(\tfrac{1}{2}q_1+p_N)(1-u_0) + \alpha\Big(1-\frac{1-u_0}{1-u_0u_1}\Big) + 2q_1\cdot(\tfrac{1}{2}q_1+q_2+p_1) \times \frac{(1-u_0)u_0u_1}{1-u_0u_1} + \cdots + 2q_1\cdot p_{N-2}\frac{(1-u_0)u_0\cdots u_{N-2}}{1-u_0\cdots u_{N-2}}\Big] \times I_{N+2}(u_0,\ldots,u_{N-2}).$$

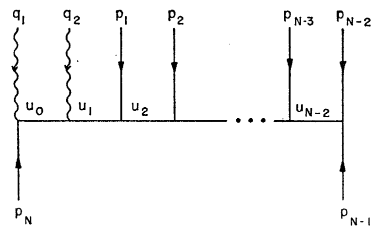

FIG. 8. Choice of variables for Eq. (A8).

Comparing with (A5), we obtain

$$\begin{aligned}(\tfrac{1}{2}q_1^2+q_1\cdot q_2)B_{N+2}(\alpha-1)&\\ +\sum_m{}_R\; q_1\cdot p_m B_{N+2}(\alpha-1;\,\alpha_{V_1,l<m}-1)&\\ =\tfrac{1}{2}\alpha[B_{N+2}(\alpha-1)-B_{N+2}]&\\ =-\tfrac{1}{2}\alpha B_{N+2}(\alpha_{1<l,V_1}-1)\,.&\qquad (A9)\end{aligned}$$

Several useful formulas may be obtained from (A9). For example, choosing

$$\alpha=\alpha_{V_1V_2}-1=a_{V_1V_2}+(q_1+q_2)^2-1=t-m_V^2$$

gives

$$\begin{aligned}(\tfrac{1}{2}q_1^2+q_1\cdot q_2)B(\alpha_{V_1V_2}-2)&\\ +\sum_m q_1\cdot p_m B_{N+2}(\alpha_{V_1V_2}-2;\,\alpha_{V_1,l<m}-1)&\\ =\tfrac{1}{2}(t-m_V^2)[B_{N+2}(\alpha_{V_1V_2}-2)&\\ -B_{N+2}(\alpha_{V_1V_2}-1)]\,.&\qquad (A10)\end{aligned}$$

Rearranging of terms and using $t=2q_1\cdot(\frac{1}{2}q_1+q_2)+q_2^2$ gives

$$\begin{aligned}(\tfrac{1}{2}q_1^2+q_1\cdot q_2)B_{N+2}(\alpha_{V_1V_2}-1)&\\ +\sum_m q_1\cdot p_m B_{N+2}(\alpha_{V_1V_2}-2;\,\alpha_{V_1,l<m}-1)&\\ =\tfrac{1}{2}(q_2^2-m_V^2)[B_{N+2}(\alpha_{V_1V_2}-2)&\\ -B_{N+2}(\alpha_{V_1V_2}-1)]\,.&\qquad (A11)\end{aligned}$$

Note that for $q_2^2=m_V^2$ we recover the identity $q_uB^\mu=0$. Finally, the current-algebra identity

$$\begin{aligned}(\tfrac{1}{2}q_1^2+q_1\cdot q_2)B_{N+2}(-1)&\\ +\sum_m q_1\cdot p_m B_{N+2}(-1;\,\alpha_{V_1,l<m}-1)=\tfrac{1}{2}B_{N+1}&\qquad (A12)\end{aligned}$$

is obtained by taking the limit $\alpha\to 0$. This identity is particularly interesting because it relates B_{N+2} to B_{N+1}. If one returns to (A8), one sees that the right-hand side of (A12) can be viewed as a surface term at $u_0=1$ occurring for $\alpha=0$, which is the first nonsense point on the left-hand side.

APPENDIX B

We demonstrate here that, for $N=3$ and $q^2=0$, the single-current amplitude given here is the same as the photoproduction amplitude of Ref. 1. For simplicity we neglect internal symmetries and use the simplified formalism of I [Eqs. (3.4)–(3.7)] for physical photons.

For $N=3$ there is just one independent hadron ordering. We may choose

$$Q_i=\tfrac{1}{3}(e_i-e_{i-1})+C\,,$$

and hence

$$\begin{aligned}V^\mu(q)=\tfrac{2}{3}(e_1-e_3)B_1{}^\mu(q)+\tfrac{2}{3}(e_2-e_1)B_2{}^\mu(q)&\\ +\tfrac{2}{3}(e_3-e_2)B_3{}^\mu(q)+CB^{\mu(\Sigma)}(q)\,.&\qquad (B1)\end{aligned}$$

We take $s=(q+p_1)^2$, $t=(q+p_3)^2$, and $u=(q+p_2)^2$ and compute the s-channel physical helicity-1 amplitude $H_1{}^s$. Using $\epsilon(1)\cdot q=\epsilon(1)\cdot p_1=0$, we readily find for the kinematic-singularity-free amplitude

$$\begin{aligned}A\equiv\phi^{-1/2}H_1{}^s=\phi^{-1/2}\epsilon_\mu(1)V^\mu(q)&\\ =-[1/2^{1/2}\lambda^{1/2}(s,m_1^2,q^2)]\{\tfrac{2}{3}(e_1-e_3)B(1-\alpha_s,\,-\alpha_t)&\\ +\tfrac{2}{3}(e_2-e_1)B(-\alpha_u,\,1-\alpha_s)&\\ -\tfrac{2}{3}(e_3-e_2)B(-\alpha_t,\,-\alpha_u)+C[B(1-\alpha_s,\,-\alpha_t)&\\ +B(-\alpha_u,\,1-\alpha_s)-B(-\alpha_t,\,-\alpha_u)]\}\,,&\qquad (B2)\end{aligned}$$

where ϕ is the usual Kibble function. For $q^2=0$,

$$\begin{aligned}\lambda^{1/2}&\equiv[s^2-2(m_1^2+q^2)s+(m_1^2-q^2)^2]^{1/2}\\ &=s-m_1^2=\alpha_s=-(\alpha_t+\alpha_u)\,,\end{aligned}$$

and we find

$$\begin{aligned}A=2^{-1/2}[\tfrac{2}{3}(e_1-e_3)\tilde{B}(-\alpha_s,\,-\alpha_t)&\\ +\tfrac{2}{3}(e_2-e_1)\tilde{B}(-\alpha_u,\,-\alpha_s)+\tfrac{2}{3}(e_3-e_2)\tilde{B}(-\alpha_t,\,-\alpha_u)&\\ +CS(-\alpha_s,\,-\alpha_t,\,-\alpha_u)]\,,&\qquad (B3)\end{aligned}$$

where

$$\tilde{B}(-\alpha_x,\,-\alpha_y)=\Gamma(-\alpha_x)\Gamma(-\alpha_y)/\Gamma(1-\alpha_x-\alpha_y)$$

and

$$\begin{aligned}S(-\alpha_x,\,-\alpha_y,\,-\alpha_z)&\\ =\tilde{B}(-\alpha_x,\,-\alpha_y)+\tilde{B}(-\alpha_y,\,-\alpha_z)+\tilde{B}(-\alpha_z,\,-\alpha_x)\,.&\end{aligned}$$

Equation (B3) is equivalent to the result of Ref. 1 written in terms of the charges.

Similarly, for the two-current case we calculate the helicity amplitudes for $N=2$ and $q_1^2=q_2^2=0$ and compare our results with the Compton scattering amplitudes of Ref. 1. The resultant parametrizations for the nonadjacent-currents term (contributing to $I=0$ and 2 in the t channel) are

$$\begin{aligned}H_{1-1}{}^t&=-(\phi/t)2e^2\tilde{B}[-\alpha_\pi(s),\,-\alpha_\pi(u)]\,,\\ H_{11}{}^t&=(\phi/t)2e^2\tilde{B}[-\alpha_\pi(s),\,-\alpha_\pi(u)]\\ &\quad-2e^2B[1-\alpha_\pi(s),\,1-\alpha_\pi(u)]\,,\qquad (B4)\end{aligned}$$

which agree with Ref. 1 only for the double-helicity-flip amplitude. In Ref. 1, the nonflip amplitude was given by $-2tm_\pi{}^2e^2\tilde{B}[-\alpha_\pi(s),\,-\alpha_\pi(u)]$, which resulted in an $M=1$ pion. Here an $M=0$ pion is obtained, simply because in the N-point beta function, all leading trajectories are parity singlets and hence $M=0$ trajectories. All other aspects of Ref. 1, including the current-algebra amplitude, are equivalent to the appropriate special cases of our general solution.