Veneziano Model and the Adler-Weisberger Sum Rules for K-π Scattering

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It is shown that the contributions to the Adler-Weisberger sum rule (AWSR) in the simplest Veneziano model for K-π scattering are too large by a factor of 2. A modification of the simplest Veneziano amplitude is proposed which satisfies (i) the Adler self-consistency condition, (ii) the AWSR, and (iii) the $K^*(890)$ having the experimentally observed width of 50 MeV. If this modified Veneziano amplitude is required to satisfy the Adler-Weisberger-type sum rule for zero-mass K mesons scattering from physical pions, the ratio F_K/F_π is predicted to be 1.26, in good agreement with experiment.

I. INTRODUCTION

RECENTLY, Veneziano[1] has suggested an ansatz for scattering amplitudes that has stimulated a great deal of activity. The Veneziano amplitude (VA) embodies in a remarkably simple way a number of desirable features of scattering amplitudes: (i) It is crossing symmetric; (ii) it has an infinite number of poles in the desired channels; (iii) it displays the expected Regge behavior in all channels; and (iv) it provides a solution to the finite-energy sum rules of Dolen, Horn, and Schmid[2] and is a realization of the strong duality (Regge pole = resonances) proposed by Schmid.[3] The drawbacks of the VA are also well known and are hardly negligible; the most important of these are the presence in the VA of poles in the physical region and the related difficulty of its failure to satisfy unitarity.

An unexpected feature of the VA for π-π scattering was pointed out by Lovelace.[4] He showed that by imposing on the (necessarily linear, exchange-degenerate) ρ-f trajectory $\alpha_\rho(s)$ the condition $\alpha_\rho(s=m_\pi^2)=\frac{1}{2}$, the VA for π-π scattering automatically satisfies the Adler self-consistency condition (ASC).[5] Since the measured ρ trajectory does satisfy $\alpha_\rho(m_\pi^2)=\frac{1}{2}$ to a good degree of accuracy,[6] this is another very attractive feature of the VA. Ademollo, Veneziano, and Weinberg[7] subsequently showed that the ASC for $\pi+A\rightarrow B+C$ in the Veneziano model leads to a surprising quantization of Regge intercepts and hadron masses. They obtain a number of relations between hadron masses that are in reasonably good agreement with the observed hadron spectrum.

Lovelace found that in addition to satisfying the ASC, his simple π-π model agreed with the predictions of chiral symmetry in a number of its other features (e.g., the position and width of the ϵ meson and the π-π scattering lengths). Stimulated by the admittedly striking successes of this model, he conjectured a bold generalization: "Chiral symmetry for soft mesons + absence of exotic resonances = Veneziano formula with no secondary terms."

It is the purpose of this paper to examine the validity of this conjecture in K-π scattering using the simplest VA for this process which was given, for example, by Kawarabayshi, Kitakado, and Yabuki.[8] In Sec. II we will see that if the single free parameter in this VA is fitted to the observed width of the $K^*(890)$, the Adler-Weisberger sum rule (AWSR) is violated by a factor of 2.

In Sec. III an elementary modification of the VA is proposed containing three free parameters. These constants are fitted by requiring this modified Veneziano amplitude (MVA) to satisfy (i) the ASC, (ii) the AWSR, and (iii) the $K^*(890)$ having the observed width. Then, by imposing on this MVA the AWSR obtained for scattering of zero-mass K mesons on physical pions, the ratio F_K/F_π is predicted to be 1.26, in good agreement with its experimental value. In addition, this MVA predicts the width of the $K_N(1420)$ to be about 50 MeV, which is quite close to its observed value.

II. SIMPLEST VENEZIANO AMPLITUDE AND AWSR

We begin by postulating the following simple form for the crossing-even and -odd K-π scattering amplitudes[8,9]:

$$T^{(\pm)}(s,t,u)=-\lambda\left[\frac{\Gamma(1-\alpha_\rho(t))\Gamma(1-\alpha_{K^*}(s))}{\Gamma(1-\alpha_\rho(t)-\alpha_{K^*}(s))}\pm(s\rightarrow u)\right].\quad(1)$$

The exchange-degenerate ρ-f and K^*-K_N trajectories (α_ρ and α_{K^*}, respectively) are assumed to be linear, with universal slopes $a\simeq0.89$ BeV^{-2}. The ASC in $\pi\pi$ scattering requires $\alpha_\rho(m_\pi^2)=\frac{1}{2}$ and the ASC imposed on Eq. (1) requires $\alpha_{K^*}(m_K^2)=\frac{1}{2}$. Thus, the trajectories

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1 G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

2 R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968); see also K. Igi, *ibid.* **130**, 920 (1963); L. A. P. Balázs and J. M. Cornwall, *ibid.* **160**, 1313 (1967).

3 C. Schmid, Phys. Rev. Letters **20**, 689 (1968).

4 C. Lovelace, Phys. Letters **28B**, 264 (1968).

5 S. Adler, Phys. Rev. **137**, B1022 (1965).

6 Lovelace (Ref. 4) gives recent fits to $\alpha_\rho(t)$ together with references to the relevant experiments.

7 M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969).

8 K. Kawarabayshi, S. Kitakado, and H. Yabuki, Phys. Letters **28B**, 432 (1969).

9 S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); W. I. Weisberger, *ibid.* **14**, 1047 (1965).

are taken to be

$$\alpha_\rho(t)=0.89t+0.48\,,\qquad \alpha_{K^*}(s)=0.89s+0.28\,. \tag{2}$$

These values for the trajectories are in resonable agreement with the observed π-π and K-π resonances, and we will accept their validity henceforth.

To determine λ, the only unknown in Eq. (1), we look at the $K^*(890)$ pole in $T^{(\pm)}$. Defining the K^*-K-π vertex in the conventional fashion,[10,11] we find that

$$\lambda=\tfrac{2}{3}g_{K^*}{}^2\,, \tag{3}$$

where the width for $K^*\to K\pi$ is

$$\Gamma(K^*\to K\pi)=\frac{g_{K^*}{}^2}{6\pi}\frac{|\mathbf{q}|_{\text{c.m.}}{}^3}{m_{K^*}{}^2}\,, \tag{4}$$

$|\mathbf{q}|_{\text{c.m.}}$ being the center-of-mass three-momentum of the final state in the K^* rest frame.

Now, the AWSR for zero-mass pions to scatter from physical kaons ($\pi_q+K_p\to\pi_q+K_p$) is

$$\frac{1}{\pi}\int_0^\infty\frac{d\nu}{\nu^2}\operatorname{Im}[T^{(\pi^-K^+)}(\nu)-T^{(\pi^+K^+)}(\nu)]=\frac{1}{F_\pi{}^2}\,. \tag{5}$$

Here $\nu=p\cdot q$, $T^{(\pi^\pm K^+)}(\nu)$ is the forward ($t=0$) scattering amplitude for zero-mass ($q^2=0$) $\pi^\pm$ to scatter from the target K^+, and F_π is the pion decay constant which, when evaluated using the charged-pion lifetime, is equal to approximately 95 MeV.[12]

According to Eq. (1), the π^+K^+ scattering amplitude has no absorptive part in the s channel (absence of $I=\frac{3}{2}$ resonances) and the integrand in Eq. (5) is easily evaluated to be

$$\operatorname{Im}T^{(\pi^-K^\pm)}(\nu)=-2\pi\lambda\Gamma(1-b)\sum_{K=0}^\infty\frac{(-)^N}{N!}\frac{\delta(1-\alpha_{K^*}(s)+N)}{\Gamma(1-b-\alpha_{K^*}(s))}\,, \tag{6}$$

where $\alpha_\rho(0)=b$. For zero-mass pions, $s=m_K{}^2+2\nu$ and the use of $\alpha_{K^*}(m_K{}^2)=\frac{1}{2}$ enables us to locate the Nth pole in $T^{(\pi^-K^+)}$ at $2a\nu_N=N+\frac{1}{2}$. Then insertion of Eq. (6) in Eq. (5) leads to the sum rule

$$-4a\lambda\Gamma(1-b)\sum_{N=0}^\infty\frac{(-)^N}{N!}\frac{1}{(N+\frac{1}{2})^2}\frac{1}{\Gamma(-b-N)}=\frac{1}{F_\pi{}^2}\,. \tag{7}$$

This sum can be explicitly evaluated and is remarkably simple if we approximate $b=\frac{1}{2}$.[13] Then Eq. (7) reduces to

$$-4a\lambda\Gamma(\tfrac{1}{2})\times[-\Gamma(\tfrac{1}{2})]=1/F_\pi{}^2\,,$$

i.e.,

$$4\pi aF_\pi{}^2\lambda=1\,. \tag{8}$$

If λ is fitted by combining Eqs. (3) and (4) and using[14] $\Gamma_{K^*}=50$ MeV ($g_{K^*}{}^2/4\pi=2.5$), then we find that the left-hand side of Eq. (8) is equal to 2 instead of 1. Therefore, in the simplest Veneziano model of K-π scattering, if the $K^*(890)$ width is fixed at its experimental value, the AWSR is violated by a factor of 2. Upon investigation of the sum involved in Eq. (7), it is easy to see the physical origin of this overestimate; the odd daughter of the $K^*(890)$ (the κ meson) has a very large width ($\sim$450 MeV, $\Gamma_\kappa/\Gamma_{K^*}\simeq 9$) and the $\kappa+K^*$ contribution to the AWSR is already too large by a factor of 1.3.

There are, of course, many ways out of this dilemma. One can always appeal to the failure of the VA to satisfy unitarity; however, if the 100% correction needed to fix the AWSR were to come from unitarity corrections, we would be quite reluctant to say that the VA is a good approximation to the real amplitude. Another possibility, which we will pursue in Sec. III, is to add satellite terms to the original VA in such a way as to still satisfy the ASC and also to reduce the contribution to the AWSR of the κ meson. In any case, our results indicates that for K-π scattering, we must modify Lovelace's conjecture which equates the simplest Veneziano model and chiral symmetry.

III. MODIFIED VENEZIANO AMPLITUDE

In order to ameliorate the difficulties encountered when using the simplest VA for K-π scattering, we propose a MVA:

$$T^{(\pm)}(s,t,u)=-\lambda_1\frac{\Gamma(1-\alpha_\rho(t))\Gamma(1-\alpha_{K^*}(s))}{\Gamma(1-\alpha_\rho(t)-\alpha_{K^*}(s))}-\lambda_2\frac{\Gamma(2-\alpha_\rho(t))\Gamma(1-\alpha_{K^*}(s))}{\Gamma(2-\alpha_\rho(t)-\alpha_{K^*}(s))}-\lambda_3\frac{\Gamma(1-\alpha_\rho(t))\Gamma(2-\alpha_{K^*}(s))}{\Gamma(2-\alpha_\rho(t)-\alpha_{K^*}(s))}\pm(s\to u)\,. \tag{9}$$

This proposed MVA has three parameters which we will determine by requiring that it (i) satisfy the ASC, (ii) have a pole corresponding to the $K^*(890)$ with width $\Gamma_{K^*}=50$ MeV, and (iii) satisfy the AWSR for $m_\pi=0$.

Since $\alpha_{K^*}(m_K{}^2)=\frac{1}{2}=\alpha_\rho(m_\pi{}^2)$, the first term of Eq. (9) satisfies the ASC. If we choose $\lambda_2+\lambda_3=0$, it is an easy matter to verify that the entire amplitude will then also satisfy the ASC.

If we then look at the $K^*(890)$ pole in $T^{(\pi^-K^+)}$ [as in Eq. (3)], we find that the λ's are related to g_{K^*} by

$$\lambda_1+\lambda_2=\tfrac{2}{3}g_{K^*}{}^2\,. \tag{10}$$

[10] Our conventions are as follows: The S matrix is related to T by $S=[-i(2\pi)^4\delta^4(p_f-p_i)/(2\pi)^6(16q_0p_0q_0'p_0')^{1/2}]T$, (pq) and $(p'q')$ being initial and final momenta; the metric is $g_{00}=-g_{kk}=1$.

[11] For example, the $K^*K\pi$ vertex is $g_{K^*}\eta^\mu(K-\pi)_\mu\times$C.G., where η is the K^* polarization, K and π are the momenta of the pseudoscalars, the K^* momentum is $K^*=K+\pi$, and C. G. is short for the $SU(2)$ Clebsch-Gordan coefficient.

[12] The definition of F_π is $\langle 0|A_\mu{}^a(0)|\pi^bK\rangle=i\delta^{ab}F_\pi K_\mu$, where $A_\mu{}^a$ is the usual axial-vector current assumed to satisfy $[A_0{}^a(x),\times A_0{}^b(0)]\delta(x_0)=i\epsilon_{abc}V_0{}^c(0)\delta^4(x)$.

[13] The use of $b=0.48$ changes things in no essential way; Eqs. (8) and (13) are somewhat more complicated, but the numerical results are virtually identical.

[14] Particle Data Group, Rev. Mod. Phys. **41**, 109 (1969).

To determine the remaining parameter, we turn to the evaluation of the AWSR [Eq. (5)]. Denoting

$$L=\frac{1}{\pi}\int\frac{d\nu}{\nu^2}[\mathrm{Im}T^{(\pi^-K^+)}-\mathrm{Im}T^{(\pi^+K^+)}], \quad (11)$$

the use of the $T^{(-)}$ of Eq. (9) plus the constraint $\lambda_3=-\lambda_2$ permits us to write

$$L=2\lambda_1 b\frac{2a}{\Gamma(1+b)}\sum_{N=0}^{\infty}\frac{1}{N!}\frac{\Gamma(1+b+N)}{(N+\frac{1}{2})^2} -2\lambda_2(1-b)\frac{2a}{\Gamma(b)}\sum_{N=0}^{\infty}\frac{1}{N!}\frac{\Gamma(b+N)}{(N+\frac{1}{2})^2} -2\lambda_2 b\frac{2a}{\Gamma(1+b)}\sum_{N=0}^{\infty}\frac{1}{N!}\frac{\Gamma(1+b+N)}{(N+\frac{3}{2})^2}. \quad (12)$$

The sums involved in Eq. (12) can easily be evaluated and are quite simple if $b=\frac{1}{2}$.[13] Writing

$$L=\frac{2a\lambda_1}{\Gamma(\frac{3}{2})}S_1-\frac{2a\lambda_2}{\Gamma(\frac{1}{2})}S_2-\frac{2a\lambda_2}{\Gamma(\frac{3}{2})}S_3,$$

we have

$$S_1=\sum_{N=0}^{\infty}\frac{1}{N!}\frac{\Gamma(\frac{3}{2}+N)}{(N+\frac{1}{2})^2}=\pi\Gamma(\tfrac{1}{2}), \quad (13a)$$

$$S_2=\sum_{N=0}^{\infty}\frac{1}{N!}\frac{\Gamma(\frac{1}{2}+N)}{(N+\frac{1}{2})^2}=-\pi\Gamma(\tfrac{1}{2})[\psi(\tfrac{1}{2})-\psi(1)], \quad (13b)$$

$$S_3=\sum_{N=0}^{\infty}\frac{1}{N!}\frac{\Gamma(\frac{3}{2}+N)}{(N+\frac{3}{2})^2}=\pi\Gamma(\tfrac{3}{2})[\psi(\tfrac{3}{2})-\psi(1)]. \quad (13c)$$

Here, $\psi(z)$ is the logarithmic derivative of the gamma function:

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z).$$

Using Eq. (13), we find the very compact form for L to be

$$L=4\pi a(\lambda_1-\lambda_2). \quad (14)$$

The requirement that $T^{(-)}$ satisfy the AWSR means that

$$4\pi a(\lambda_1-\lambda_2)=1/F_\pi^2. \quad (15)$$

We have now determined the three parameters of Eq. (9). If we look at the $\kappa(890)$ pole in $T^{(\pm)}$, we find that the width $\Gamma_\kappa=95$ MeV, considerably smaller than the corresponding width in the simplest model discussed above. This result was to be expected on the basis of the comments made at the end of Sec. II.

Up to this point we have been just fitting parameters. It is now necessary to ask whether our fully determined MVA has further properties which are in reasonable agreement with experiment. The remainder of this section will be devoted to an investigation of three consequences of our MVA: (i) the width of the $J^P=2^+$ $K_N(1420)$, (ii) the K-π scattering lengths, and (iii) the AWSR obtained by using partial conservation of axial-vector current (PCAC) for the kaons instead of for the pions.

Extraction of partial widths of resonances from Veneziano amplitudes is now a familiar pastime. The $K_N(2^+)K\pi$ vertex is defined by

$$\langle K_N|J_\pi|K\rangle=(g_{K_N}/m_{K_N})\eta^{\mu\nu}(K_N)K_\mu\pi_\nu\times\text{C.G.}, \quad (16)$$

where m_{K_N} is the mass of the K_N, $\eta^{\mu\nu}$ is its polarization vector, K and π denote the momenta of the K and π, and C.G. is the isospin Clebsch-Gordan coefficient. In terms of g_{K_N}, the partial width for $K_N\to K\pi$ is

$$\Gamma(K_N\to K\pi)=\frac{g_{K_N}^2}{4\pi}\frac{1}{15}\frac{|\mathbf{q}|_{\text{c.m.}}^5}{m_{K_N}^4}. \quad (17)$$

Extraction of the K_N pole from $T^{(\pm)}$ and use of Eq. (10) enables us to evaluate g_{K_N} to be

$$g_{K_N}^2/m_{K_N}^2=8ag_{K^*}^2, \quad (18)$$

implying a width $\Gamma(K_N\to K\pi)=51$ MeV, in good agreement with experiment.[14]

The K-π scattering lengths are defined in terms of $T^{(\pm)}$ by[15]

$$a^{(\pm)}=\frac{1}{8\pi}\frac{1}{m_\pi+m_K}T^{(\pm)}[s=(m_\pi+m_K)^2,\, t=0,\, u=(m_\pi-m_K)^2]. \quad (19)$$

One finds that

$$a^{(-)}=0.073m_\pi^{-1}\quad(\text{CA}: 0.059m_\pi^{-1}),$$
$$a^{(+)}=0.016m_\pi^{-1}\quad(\text{CA}: 0). \quad (20)$$

The current-algebra (CA) estimates for $a^{(\pm)}$ are given in the parentheses above. Thus, the K-π scattering lengths are reasonable, which is hardly surprising since we have arranged $T^{(-)}$ to satisfy the AWSR.

As a final test of our MVA, we propose to postulate that it satisfy the Adler-Weisberger-type sum rule obtained for scattering massless kaons from a given target. This formula is

$$\frac{1}{\pi}\int_0^\infty\frac{d\nu}{\nu^2}\,\mathrm{Im}[T^{(\pi^+K^-)}(\nu)-T^{(\pi^+K^+)}(\nu)]_{m_K=0}=\frac{1}{F_K^2}. \quad (21)$$

The bracketed quantities are the forward K-π scattering amplitudes taken for massless kaons. The kaon decay constant F_K is defined in analogy with F_π and experimentally (using a value of the Cabibbo angle $\tan^2\theta=0.053$):

$$(F_K/F_\pi)_{\text{expt}}\simeq1.1. \quad (22)$$

[15] S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

Using the MVA of Eq. (9) and defining $\alpha_{K^*}(0)=d$, one finds that Eq. (21) can be written as

$$\frac{1}{F_K^2}=2\lambda_1 b\frac{2a}{\Gamma(1+b)}\sum_{N=0}^{\infty}\frac{\Gamma(1+b+N)}{N!(1+N-d)^2}-2\lambda_2 b\frac{2a}{\Gamma(b)}\sum_{N=0}^{\infty}\frac{\Gamma(b+N)}{N!(1+N-d)^2}-2\lambda_2 b\frac{2a}{\Gamma(1+b)}\sum_{N=0}^{\infty}\frac{\Gamma(1+b+N)}{N!(2+N-d)^2}. \quad (23)$$

Again approximating $b=\frac{1}{2}$, the sums necessary are easily evaluated to be

$$\Sigma_1=\sum_{N=0}^{\infty}\frac{\Gamma(\frac{3}{2}+N)}{N!(1+N-d)^2}=\pi\frac{\Gamma(1-d)}{\Gamma(\frac{1}{2}-d)}\times[\psi(1-d)-\psi(\tfrac{1}{2}-d)],$$

$$\Sigma_2=\sum_{N=0}^{\infty}\frac{\Gamma(\frac{1}{2}+N)}{N!(1+N-d)^2}=-\pi\frac{\Gamma(1-d)}{\Gamma(\frac{3}{2}-d)}\times[\psi(1-d)-\psi(\tfrac{3}{2}-d)], \quad (24)$$

$$\Sigma_3=\sum_{N=0}^{\infty}\frac{\Gamma(\frac{3}{2}+N)}{N!(2+N-d)^2}=\pi\frac{\Gamma(2-d)}{\Gamma(\frac{3}{2}-d)}\times[\psi(2-d)-\psi(\tfrac{3}{2}-d)].$$

The use of Eqs. (24) in Eq. (23) yields, after some algebra,

$$\frac{1}{F_K^2}=2\pi a(\lambda_1-\lambda_2)\frac{\Gamma(1-d)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}-d)}\times\{(1-2d)[\psi(1-d)-\psi(\tfrac{3}{2}-d)]+2\}. \quad (25)$$

The AWSR for $m_\pi=0$ gave us Eq. (15), whose use in Eq. (25) gives us the $(F_\pi/F_K)^2$ ratio

$$(F_\pi/F_K)^2=\frac{\Gamma(1-d)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}-d)}\{1+(\tfrac{1}{2}-d)\times[\psi(1-d)-\psi(\tfrac{1}{2}-d)]\}. \quad (26)$$

The use of $d=0.28$ from Eq. (2) determines this interesting ratio to be

$$F_K/F_\pi=1.26, \quad (27)$$

which is in quite reasonable agreement with the experimental value of Eq. (22).

IV. CONCLUSIONS

What can we say we have learned from this study? In the first place, we have seen that the simplest Veneziano formula for K-π scattering violates the AWSR by a factor of 2, demonstrating a significant deviation for this VA from the predictions of chiral symmetry. Unless we are willing to believe that unitarity corrections are of the order of 100%, it seems clear that we cannot always expect the simplest VA to provide an accurate account of a given scattering amplitude.[16]

The prescription we have proposed in Sec. III for modifying the simple Veneziano model (when necessary) is the most straightforward thing to do within the spirit of the model. If the predictions of the MVA are to be taken seriously (and the reasonable value for F_K/F_π is certainly an encouraging sign), we deduce that it may often be necessary to add a few satellite terms to the original VA. For processes involving pions, the ASC may then be fulfilled by a combination of the gamma-function pole, *à la* Ademollo, Veneziano, and Weinberg, and cancellation of combinations of terms as was necessary in our MVA. Only when the amplitudes satisfy such basic requirements as the ASC and the AWSR should they be taken as plausible approximations to the scattering amplitudes in the real world.

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[16] For π-π scattering, a treatment analogous to that of Sec. II shows that the AWSR is overestimated by about 30% for a ρ width of 110 MeV. It does not seem implausible that proper incorporation of unitarity might restore this discrepancy; the single Veneziano term for $\pi\pi$ scattering may well be a good approximation as the results of Lovelace (Ref. 4) indicate.