Nonexistence of Anomalous Commutators in Spinor Electrodynamics*

C. R. Hagen

Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

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An attempt is made to infer the existence of so-called anomalous terms in the divergence of the axial-vector current from an action-principle approach. However, the presence of the spatial vector ϵ which is used to write the axial-vector current as the limit of a gauge-invariant nonlocal operator is found to preclude the possibility of successfully carrying out such a program. It is demonstrated that this failure is only one of several field-theoretical paradoxes which arise from the use of the ϵ limiting procedure. One is thereby led to the conclusion that the definition of current operators in terms of a limit can give consistent results only in a theory which is finite or one which is made finite by a regularization technique. The effect of such considerations is shown to imply that the Schwinger-Adler result for the axial-vector divergence is to be taken as a relation between matrix elements rather than one between field operators. Of more practical concern is the fact that this serves to resolve a discrepancy which has existed between several calculations of the commutators of $j_5{}^0$ with the electric charge density.

I. INTRODUCTION

SOME years ago it was shown by Schwinger[1] that in spinor electrodynamics with bare fermion mass m_0 the usual equation for the divergence of the axial-vector current

$$\partial_\mu j_5{}^\mu = 2m_0 j_5$$

must be modified by the inclusion of an anomalous contribution of the form $(\alpha_0/4\pi)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ on the right-hand side. Such a term makes its appearance when one takes care to define the current as the limit of a gauge-invariant nonlocal operator bilinear in the fermion fields. This result was recently rediscovered in perturbation theory by Adler[2] who used it to resolve a disturbing puzzle which had arisen in connection with the calculation of neutral pion decay using partial conservation of axial-vector current (PCAC). It was subsequently shown by the author[3] and independently by Jackiw and Johnson[4] that this result can be inferred directly from the equations of motion without recourse to Adler's rather intricate perturbation-theory arguments. This latter calculation requires the careful definition of $j_5{}^\mu(x)$ as

$$j_5{}^\mu(x) = \lim_{x',x''\to x} \tfrac{1}{2}i\psi(x')\beta\gamma_5\gamma^\mu \times\exp\left[ie_0 q\int_{x''}^{x'} dx^\nu A_\nu\right]\psi(x''), \quad (1.1)$$

where the limit is to be understood as implying

$$x' = x+\tfrac{1}{2}\epsilon, \quad x'' = x-\tfrac{1}{2}\epsilon,$$

with ϵ a purely spatial vector. The final result,

$$\partial_\mu j_5{}^\mu = 2m_0 j_5 + (\alpha_0/4\pi)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}, \quad (1.2)$$

however, turns out to be independent of how the $\epsilon=0$ limit is taken, a circumstance which is as remarkable as it is ill understood.

It was observed by Adler that Eq. (1.2) can alternatively be written in terms of a vector

$$\bar{j}_5{}^\mu = j_5{}^\mu - (\alpha_0/2\pi)\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta},$$

which is not gauge-invariant as

$$\partial_\mu \bar{j}_5{}^\mu = 0, \quad (1.3)$$

where for simplicity we take the case $m_0=0$. This then suggests that $\bar{j}_5{}^\mu$ could be the object of more fundamental interest and that its commutation relations might have a simpler form than those of $j_5{}^\mu$. In order to investigate this question and to infer the correct generator of chiral gauge transformations, it is natural to attempt to reformulate the problem in terms of an action-principle formalism[5] and to seek to derive (1.2) entirely by means of variational techniques. Since such an approach customarily yields a surface term[5] which is to be interpreted as the generator of the transformation under consideration, this would provide a check on the direct calculation of commutation relations as well as an elegant framework for the general discussion of anomalous terms.

In Sec. II we demonstrate the application of the action-principle approach to fermion fields coupled to an external electromagnetic field in which the usual current definition is replaced by its nonlocal gauge-invariant form. Although no difficulty is encountered in carrying out this generalization, it is shown that any attempt to handle chiral gauge transformations by such an approach must fail. In Sec. III it is demonstrated that this failure is only one of several circumstances in which the ϵ formalism is found to be unsatisfactory. In particular, this procedure fails to give a covariant vacuum polarization, yields contradictory results in the calculation of the commutation relations of $j_5{}^0$, and—perhaps most alarming of all—implies the breakdown of current conservation in the case of a coupling to an external

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[1] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[2] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[3] C. R. Hagen, Phys. Rev. **177**, 2622 (1969).
[4] C. R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969).
[5] J. Schwinger, Phys. Rev. **11**, 713 (1953).

pseudovector field. After thus demonstrating the inadequacies of the ϵ limiting procedure, a resolution of the problem is presented. It is shown that all the above difficulties can be traced to the divergence of the charge renormalization in the usual perturbation-theory approach to quantum electrodynamics. By adopting a regularization procedure, all of these problems are easily resolved and the shortcomings of the ϵ formalism thereby linked to the time-honored question of whether it is possible to eliminate the divergences in Z_3 by means of an eigenvalue condition on α_0. Finally, it should be noted that in this approach, contrary to others that have recently appeared,[4,6,7] there are no anomalous terms in the commutator of $j_5{}^0$ with the electric charge density.

II. ACTION-PRINCIPLE APPROACH

In order to fully appreciate the difficulties to be encountered in the consideration of the axial-vector current it is useful to show that the point separation device which is used to define the electric current

$$j^\mu(x)=\lim_{x',x''\to x}\tfrac{1}{2}\psi(x')\beta\gamma^\mu q \times\exp\left[ie_0 q\int_{x''}^{x'}dx^\nu A_\nu\right]\psi(x'') \quad (2.1)$$

is not intrinsically irreconcilable to an action-principle formalism. We thus begin with the simplest possible case, namely, spinor electrodynamics in which the electromagnetic field is taken to be unquantized, and proceed to describe the somewhat minor new features introduced relative to the case in which $j^\mu(x)$ is (incorrectly) taken to be $\frac{1}{2}\psi(x)\beta\gamma^\mu q\psi(x)$. As already indicated by the above notation, we take $\psi(x)$ to be Hermitian and describe the internal charge space by means of the antisymmetrical matrix

$$q=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

Since one does not expect the usual kinematical terms of the free Lagrangian to be modified by the interactions, it is convenient to write the Lagrangian in the form

$$\mathfrak{L}=\tfrac{1}{2}i\psi\beta\gamma^\mu\partial_\mu\psi-\tfrac{1}{2}m_0\psi\beta\psi+e_0\int_0^A j^\mu(A)\delta A_\mu. \quad (2.2)$$

Note that the only real assumption made in writing (2.2) is the usual action-principle condition

$$ej^\mu(x)\equiv\frac{\delta}{\delta A_\mu(x)}\int dx'\mathfrak{L}(x'), \quad (2.3)$$

and that the somewhat unfamiliar form of (2.1) is associated with the frequently overlooked dependence of $j^\mu(x)$ on the external field. For the moment it will be convenient to disregard (2.1) and define the current more generally as

$$j^\mu(x)=\lim_{x',x''\to x}\tfrac{1}{2}\psi(x')\beta\gamma^\mu qe(x',x'';A)\psi(x''),$$

where $e(x',x'';A)$ is a real matrix in the charge space whose dependence on its arguments will not be specified except to note that

$$e^T(x',x'';A)=qe(x'',x';A)q.$$

Under a gauge transformation of the first kind, i.e.,

$$\psi\to(1+iq\delta\lambda)\psi, \quad (2.4)$$

one has to first order in $x'-x''\equiv\epsilon$,

$$\delta\mathfrak{L}=-\tfrac{1}{2}\psi\beta\gamma^\mu q\psi\partial_\mu\delta\lambda-\lim_{x',x''\to x}e_0 \times\int\tfrac{1}{2}i\psi(x')\beta\gamma^\nu qe(x',x'';A)\psi(x'')\delta A_\nu\epsilon^\mu\partial_\mu\delta\lambda,$$

and the conservation law

$$\partial_\mu j^{\mu\prime}=0,$$

where

$$j^{\mu\prime}=\tfrac{1}{2}\psi\beta\gamma^\mu q\psi+\tfrac{1}{2}i\lim_{\epsilon\to 0}e_0\int\psi(x+\tfrac{1}{2}\epsilon) \times\beta\gamma^\nu qe(x+\tfrac{1}{2}\epsilon,\,x-\tfrac{1}{2}\epsilon;\,A)\psi(x-\tfrac{1}{2}\epsilon)\delta A_\nu\epsilon^\mu. \quad (2.5)$$

A prime notation has been introduced for the current derived from the transformation (2.4) since there is no *a priori* reason for j^μ and $j^{\mu\prime}$ to coincide. If, however, one seeks to make contact with the case of quantum electrodynamics, the current j^μ must be constrained to be conserved and consequently identified with $j^{\mu\prime}$. This being the case of interest, the prime notation can now be abandoned.

From the symmetry of the second derivative, i.e.,

$$e_0\frac{\delta j^\mu(x)}{\delta A_\nu(x')}=\frac{\delta^2}{\delta A_\mu(x)\delta A_\nu(x')}\int dx\,\mathfrak{L}=e_0\frac{\delta j^\nu(x')}{\delta A_\mu(x)},$$

it immediately follows from (2.5) that

$$\delta j^\mu(x)=\lim_{\epsilon\to 0}\tfrac{1}{2}ie_0(\epsilon^\nu\delta A_\nu)\psi(x+\tfrac{1}{2}\epsilon) \times\beta\gamma^\mu qe(x+\tfrac{1}{2}\epsilon,\,x-\tfrac{1}{2}\epsilon;\,A)\psi(x-\tfrac{1}{2}\epsilon),$$

from which one derives

$$\delta\ln e(x+\tfrac{1}{2}\epsilon,\,x-\tfrac{1}{2}\epsilon;A)=ie_0q\epsilon^\nu\delta A_\nu. \quad (2.6)$$

It is clear that (2.6) integrates to the result

$$e(x',x'';A)=\exp\left(ie_0q\int_{x''}^{x'}dx^\nu A_\nu\right), \quad (2.7)$$

[6] S. L. Adler and D. G. Boulware, Phys. Rev. **184**, 1740 (1969).
[7] R. A. Brandt, Phys. Rev. **180**, 1490 (1969).

thereby providing a derivation of the usual exponential form of $e(x',x'';A)$ as well as demonstrating the compatibility of the action principle with the point-separation technique. The purely spatial nature of ϵ further allows one to infer from (2.4) the form of the generator

$$Q=\int j^0(x)d^3x,$$

where $j^0(x)$ has the structure

$$j^0(x)=\tfrac{1}{2}\psi(x)q\psi(x).$$

It should be emphasized that despite the seemingly universal nature of the above derivation of the form of $e(x',x'';A)$, there exist important cases in which the condition $j^\mu=j^{\mu\prime}$ need not apply. In particular, the elimination of the requirement that j^μ be conserved allows in the case of two dimensions[8] the identification

$$j^\mu=j^{\mu\prime}-(\eta/\pi)A^\mu,$$

where η is an arbitrary real number (and more generally an arbitrary real function of A^μ). From (2.3) one readily sees that this corresponds to the case in which $\mathcal{L}$ has the form

$$\mathcal{L}=\tfrac{1}{2}i\psi\beta\gamma^\mu\partial_\mu\psi-\tfrac{1}{2}m_0\psi\beta\psi+e_0\int_0^A j^{\mu\prime}\delta A_\mu-\frac{e_0\eta}{2\pi}A^\mu A_\mu.$$

Although similar results can be obtained in four dimensions, this is not relevant to the present discussion; we consequently refer the reader elsewhere[9] for a consideration of cases in which Eq. (2.7) fails.

Proceeding now to the case of interest, namely, the γ_5 transformation

$$\psi\to(1+\gamma_5\delta\lambda)\psi,$$

one finds in analogy to (2.5) the current operator

$$\tilde{j}^\mu=\tfrac{1}{2}i\psi\beta\gamma_5\gamma^\mu\psi-\tfrac{1}{2}\lim_{x',x''\to x}e_0\psi(x')\beta\gamma_5\gamma^\nu q\times\exp\left(ie_0q\int_{x''}^{x}dx^\nu A_\nu\right)\psi(x'')A_\nu\epsilon^\mu+O(A^4),\quad(2.8)$$

where we have used charge-conjugation invariance to lump together all terms in which four or more powers of A^μ occur. Since it will be seen that the previously mentioned difficulty appears already in the second power of A, there is clearly no need to further specify such terms at this point. The action principle implies that the current defined by Eq. (2.8) satisfies the equation

$$\partial_\mu\tilde{j}_5{}^\mu=2m_0j_5,$$

where j_5 is given by

$$j_5=\tfrac{1}{2}\psi\beta\gamma_5\psi.$$

A comparison with (1.3) clearly indicates that consistency requires that $\tilde{j}_5{}^\mu$ be identical to Alder's $\bar{j}_5{}^\mu$. It is straightforward to show, however, that the vacuum matrix elements of $\tilde{j}_5{}^\mu$ and $\bar{j}_5{}^\mu$ are not the same. To this end, one notes that to order A^2,

$$\begin{aligned}\langle\tilde{j}_5{}^\mu\rangle=&\lim_{\epsilon\to0}\operatorname{Tr}G(x+\tfrac{1}{2}\epsilon,x-\tfrac{1}{2}\epsilon)\gamma^\mu\gamma_5\\&+\tfrac{1}{2}ie_0\lim_{\epsilon\to0}\operatorname{Tr}G(x+\tfrac{1}{2}\epsilon,x-\tfrac{1}{2}\epsilon)\gamma_5\gamma^\nu q\times\exp(-ie_0q\epsilon A)A_\nu\epsilon^\mu\\=&\tfrac{1}{2}\lim_{\epsilon\to0}\operatorname{Tr}G\gamma^\mu\gamma_5\exp(-ie_0q\epsilon A)\\&+\tfrac{1}{2}ie_0\lim_{\epsilon\to0}\operatorname{Tr}G\gamma_5q\exp(-ie_0q\epsilon A)(\gamma A\epsilon^\mu-\gamma^\mu\epsilon A)\\\equiv&\langle\tilde{j}_5{}^\mu\rangle^{(1)}+\langle\tilde{j}_5{}^\mu\rangle^{(2)},\end{aligned}$$

where $G(x,x')$ is the fermion Green's function in the external field A^μ. The first term is readily seen to coincide with the usual axial-vector current defined by (1.1), i.e.,

$$\langle\tilde{j}_5{}^\mu\rangle^{(1)}=\langle j_5{}^\mu\rangle,$$

so that it is only necessary to demonstrate that

$$\langle\tilde{j}_5{}^\mu\rangle^{(2)}+(e_0{}^2/8\pi^2)\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta}\neq0\qquad(2.9)$$

in order to display the inconsistency. This can be done using calculational techniques identical to those of Ref. 3; consequently, the details need not be presented here. The final result is that the left-hand side of (2.9) is given by

$$\frac{e_0{}^2}{4\pi^2}A_\nu F_{\alpha\beta}\left[\epsilon^{\alpha\beta\gamma\nu}\left(\tfrac{1}{4}\delta_\gamma{}^\mu-\frac{\epsilon^\mu\epsilon_\gamma}{\epsilon^2}\right)-\epsilon^{\alpha\beta\gamma\mu}\left(\tfrac{1}{4}\delta_\gamma{}^\nu-\frac{\epsilon^\nu\epsilon_\gamma}{\epsilon^2}\right)\right];\qquad(2.10)$$

the contradiction is thereby established.

It is important to point out that one must resist the temptation to perform a four-dimensional averaging in order to make (2.10) vanish. Such a step would contradict the canonical-field-theory formulation and would not provide an escape from some of the further difficulties associated with the ϵ limiting procedure which will be noted in Sec. III. Thus it is seen that the point-separation teqhnique for the axial-vector current cannot be consistently incorporated into an action-principle approach, since it has yielded results which are not only noncovariant but which are also in contradiction with the Schwinger-Adler result (1.2). We now proceed to show that this is only one of several paradoxes which are encountered as a consequence of the definition of currents as limits of nonlocal operators.

III. SOME PARADOXES OF FIELD THEORY

One of the earliest problems to arise out of perturbative expansions in the fine-structure constant was the

[8] C. R. Hagen, Nuovo Cimento **51B**, 169 (1967).
[9] C. R. Hagen, Phys. Rev. **178**, 2154 (1969).

troublesome photon-mass term. Despite the fact that gauge invariance formally requires that the current correlation function be conserved, it was found that actual calculations based on a strictly local definition of $j^\mu(x)$ not only failed to satisfy the condition of gauge invariance, but that the unwanted terms were quadratically divergent as well. The fact that these could be cancelled by a bare photon mass served merely to obscure the significance of this result and to dampen efforts to isolate the breakdown of gauge invariance. The resolution of the photon-mass problem nonetheless, was accomplished by the introduction of the explicitly gauge-invariant definition (2.1) of the current operator, and this remains as the greatest single triumph of the ϵ formalism. However, since this entire approach is suspect in view of the results of Sec. II, it is important to examine more carefully the actual degree of success which it has achieved in quantum electrodynamics.

In the case in which A^μ is an external field one can readily show[10] that the vacuum polarization is given by

$$\Pi^{\mu\nu}(q,\epsilon)=i\int\frac{dp}{(2\pi)^4}e^{ip\epsilon}\,\mathrm{Tr}\gamma^\mu\left\{G(p+\tfrac{1}{2}q)\gamma^\nu G(p-\tfrac{1}{2}q)+\left[1+\frac{1}{24}\left(q\frac{\partial}{\partial p}\right)^2\right]\frac{\partial}{\partial p_\nu}G(p)\right\},\quad(3.1)$$

where the notation explicitly indicates the possible dependence of $\Pi^{\mu\nu}$ on the vector ϵ. One notes that it is the second term in the curly brackets in (3.1) which cancels the spurious quadratic divergence, while the last term merely imposes gauge invariance on the remainder. Although Johnson[10] in his derivation of (3.1) sets $\epsilon=0$ at this point, the fact that the integral remains logarithmically divergent makes such a step suspect. In order to investigate the possibility of consistently taking $\epsilon=0$, one can proceed by explicitly performing the calculation of $\Pi^{\mu\nu}(q,\epsilon)$. This may be done with the aid of standard techniques, with the result

$$\Pi^{\mu\nu}(q,\epsilon)=\Pi^{\mu\nu}(q,\epsilon)^{(1)}+\Pi^{\mu\nu}(q,\epsilon)^{(2)},$$

where

$$\Pi^{\mu\nu}(q,\epsilon)^{(1)}=-\frac{1}{2\pi^2}(g^{\mu\nu}q^2-q^\mu q^\nu)\int_0^1du\,u(1-u)\times\int_0^\infty\frac{d\nu}{\nu}\exp\left(-i\nu[m_0{}^2+q^2u(1-u)]+\frac{i\epsilon^2}{4\nu}\right),$$

$$\Pi^{\mu\nu}(q,\epsilon)^{(2)}=(1/12\pi^2)(1/\epsilon^2)[(\epsilon^\mu q^\nu+\epsilon^\nu q^\mu)\times\epsilon q-g^{\mu\nu}(\epsilon q)^2-q^2\epsilon^\mu\epsilon^\nu].$$

In $\Pi^{\mu\nu}(q,\epsilon)^{(1)}$ (the usual expression for the vacuum polarization), the quantity ϵ appears in the combination ϵ^2 and plays only the relatively harmless role of a cutoff for the remaining logarithmic divergence. However, $\Pi^{\mu\nu}(q,\epsilon)^{(2)}$ involves the ambiguous form $\epsilon^\alpha\epsilon^\beta/\epsilon^2$ which cannot give a covariant result unless four-dimensional averaging is performed. Since this would contradict the spirit of canonical field theory, one is again faced with a situation in which the ϵ limiting procedure gives rise to noncovariant results. On the other hand, it should be noted that the ϵ formalism has fully restored the desired element of gauge invariance, since one can readily verify that despite the noncovariance of $\Pi^{\mu\nu}(q,\epsilon)$, we have

$$q_\mu\Pi^{\mu\nu}(q,\epsilon)^{(i)}=q_\nu\Pi^{\mu\nu}(q,\epsilon)^{(i)}=0.$$

In order to recognize the absence of a universal conviction to the effect that the use of ϵ limits must be confined to the case of purely spatial ϵ, we shall now present two instances in which the ϵ approach implies an outright contradiction independent of how one averages. Thus we explicitly recognize the possibility that the reader might prefer to dismiss the two cited failures of the ϵ formalism on the basis of a general impression that somehow four-dimensional averaging must be justifiable. To this end we return to the discussion of the axial-vector current and attempt to calculate the anomalous commutators claimed in Refs. 6 and 7. Consider, in particular, the equal-time commutator

$$[j^0(x),j_5{}^0(x')].$$

Using the canonical commutation relation

$$\{\psi(x),\psi(x')\}=\delta(\mathbf{x}-\mathbf{x}')$$

and the definitions (1.1) and (2.1), one finds by direct calculation that

$$[j^0(x),j_5{}^0(x')]=\tfrac{1}{4}i\lim_{\epsilon\to0}[\psi(x+\tfrac{1}{2}\epsilon),q\gamma_5\psi(x-\tfrac{1}{2}\epsilon)]\times\boldsymbol{\epsilon}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}').$$

Upon taking the vacuum matrix element of this result, one finds as a trivial extension of the axial-vector divergence result[3]

$$\langle[j^0(x),j_5{}^0(x')]\rangle=(ie_0/2\pi^2)(1/\epsilon^2)\times(\boldsymbol{\epsilon}\cdot\mathbf{H})(\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla})\delta(\mathbf{x}-\mathbf{x}'),\quad(3.2)$$

where H_i is the magnetic field.

On the other hand, one can readily contradict (3.2) by a simple derivation of the Jackiw-Johnson[4] and the Adler-Boulware[6] result for the same quantity. This is accomplished by writing (1.2) in the form

$$-i[H,j_5{}^0(x')]+\int d^3x''\frac{\delta_3}{\delta A^\mu(\mathbf{x}'',t)}j_5{}^0(x')\partial_0A^\mu(\mathbf{x}'',t)+\partial_kj_5{}^k-2m_0j_5+(\alpha_0/4\pi)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_\alpha{}^\beta,\quad(3.3)$$

where we have used a subscript notation on δ to indicate

[10] K. Johnson, *1964 Brandeis Lecture Notes* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965), Vol. 2.

a variation on a three-dimensional surface.[11] Upon taking a variational derivative with respect to $A_0(x)$, it is easy to show that (3.3) becomes

$$\langle[j^0(x),j_5^0(x')]\rangle=(ie_0/2\pi^2)\mathbf{H}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}')\,. \quad (3.4)$$

Although the right-hand side of (3.2) can readily be cast into the form $(ie_0/6\pi^2)\mathbf{H}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}')$ by a three-dimensional averaging, it is clear that there remains a discrepancy with (3.4) by a factor of 3. Similarly, Brandt[7] avoids ϵ dependence by four-dimensional averaging and derives a result differing from (3.4) by a factor of 4.

These differences, of course, are readily understood. The independence of (1.2) of the details of the $\epsilon=0$ limit has already been noted and consequently the derivation of (3.4) which proceeds from (1.2) is characterized by the absence of any necessity for averaging. (It is in effect a "one-dimensional average.") Thus, the fact that the direct calculation (3.2) and that of Brandt differ from (3.4) by factors of 3 and 4, respectively, is trivially seen to be a consequence of the dimensionality of the space in which averaging is performed. We thus have encountered a clear contradiction that is summarized by the observation that no averaging procedure can reconcile Eqs. (3.2) and (3.4).

Before terminating this discussion of the shortcomings of the definition (2.1), we give here what might well be the most significant contradiction which it implies. To motivate the calculation to be presented, note that the definition (2.1) and the equation of motion imply that

$$\partial_\mu j^\mu=\lim_{\epsilon\to0}\epsilon^\gamma F_{\mu\nu}\tfrac{1}{2}i\psi(x+\tfrac{1}{2}\epsilon)\beta\gamma^\mu \times\exp(ie_0q\epsilon A)\psi(x-\tfrac{1}{2}\epsilon)\,, \quad (3.5)$$

which points out the fact that once one recognizes the singularity associated with the product of two fermion fields, there is no obvious reason for the right-hand side of (3.5) to vanish and thereby assure the consistency of Maxwell's equations. However, charge-conjugation invariance asserts that in the expansion of the vacuum matrix element of (3.5) only odd powers in the external field A^μ can enter. Since the trace of $G(x+\frac{1}{2}\epsilon,\, x-\frac{1}{2}\epsilon)\gamma^\mu$ to zero order in A^μ is proportional to ϵ^μ, the equation

$$\langle\partial_\mu j^\mu\rangle=\lim_{\epsilon\to0}\epsilon^\nu F_{\mu\nu}\,\mathrm{Tr}\tfrac{1}{2}G(x+\tfrac{1}{2}\epsilon,\, x-\tfrac{1}{2}\epsilon)\gamma^\mu \quad (3.6)$$

must be at least cubic in A^μ by virtue of the antisymmetry of $F^{\mu\nu}$. The expansion of G to second order in A^μ is, of course, the triangle graph which is formally linearly divergent and could consequently give a finite value to (3.6) for ϵ going to zero. However, an explicit calculation shows that here also the result is proportional to ϵ^μ and thus one finds $\langle\partial_\mu j^\mu\rangle=0$.

This sketch of the proof of current conservation illustrates the rather shaky position of the result. In particular, it is easy to imagine the possibility that other couplings could upset this delicate situation, a circumstance we now seek to exploit. Note that if one were to introduce a coupling of ψ to an external pseudovector field B^μ, it is conceivable that the divergence of j^μ could be proportional to $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}G_{\alpha\beta}$, where

$$G_{\alpha\beta}=\partial_\alpha B_\beta-\partial_\beta B_\alpha\,.$$

This structure is Lorentz-invariant, gauge-invariant, and odd under charge conjugation as is the divergence of j^μ.

Such a speculation can be confirmed by considering the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}i\psi\beta\gamma^\mu\partial_\mu\psi-\tfrac{1}{2}m_0\psi\beta\psi+e_0\int_0^A j^\mu(B=0)\delta A_\mu+g_0 \times\int_0^B j_5^\mu(A=0)\delta B_\mu+\int_0^A\int_0^B l^{\mu\nu}\delta A_\mu\delta B_\nu\,,$$

where

$$l^{\mu\nu}=\delta e_0 j^\mu/\delta B_\nu=\delta g_0 j_5^\mu/\delta A_\nu$$

and

$$j^\mu(x)=\tfrac{1}{2}\lim_{x',x''\to x}\psi(x')\beta\gamma^\mu q e(x',x'';A)g(x',x'';B)\psi(x'')\,,$$

$$j_5^\mu(x)=\tfrac{1}{2}i\lim_{x',x''\to x}\psi(x')\beta\gamma_5\gamma^\mu e(x',x'';A)g(x',x'';B)\psi(x'')\,,$$

the function $g(x',x'';B)$ having been introduced to make j^μ and j_5^μ manifestly invariant under γ_5 transformations.[12] The form of this function is found from the techniques of Sec. II to be

$$g(x',x'';B)=\exp\left(g_0\gamma_5\int_{x''}^{x'}dx^\nu B\nu\right).$$

The field equation

$$\{\gamma^\mu[(1/i)\partial_\mu-e_0qA_\mu+ig_0\gamma_5B_\mu]+m_0\}\psi=0$$

now implies that

$$\partial_\mu j^\mu=\tfrac{1}{2}i\lim_{\epsilon\to0}e_0\epsilon^\nu F_{\mu\nu}\psi(x+\tfrac{1}{2}\epsilon)\beta\gamma^\mu \times\exp(ie_0q\epsilon A+g_0\gamma_5\epsilon B)\psi(x-\tfrac{1}{2}\epsilon) -\tfrac{1}{2}\lim_{\epsilon\to0}g_0\epsilon^\nu G_{\mu\nu}\psi(x+\tfrac{1}{2}\epsilon)\beta\gamma_5\gamma^\mu q \times\exp(ie_0q\epsilon A+g_0\gamma_5\epsilon B)\psi(x-\tfrac{1}{2}\epsilon)$$

or

$$\langle\partial_\mu j^\mu\rangle=\lim_{\epsilon\to0}[e_0\epsilon^\nu F_{\mu\nu}\,\mathrm{Tr}\tfrac{1}{2}G(x+\tfrac{1}{2}\epsilon,\, x-\tfrac{1}{2}\epsilon)\gamma^\mu +g_0\epsilon^\nu G_{\mu\nu}\,\mathrm{Tr}\tfrac{1}{2}iG(x+\tfrac{1}{2}\epsilon,\, x-\tfrac{1}{2}\epsilon)\gamma_5\gamma^\mu q]\,.$$

Again the results of Ref. 3 allow a quick evaluation, and one finds that the two terms of (3.7) are identical and

[11] L. S. Brown, Phys. Rev. **150**, 1338 (1966).

[12] For the benefit of the reader who might object to the inclusion of this factor, it should be pointed out that in the case in which B^μ is quantized and massless (in analogy to electrodynamics) this factor must necessarily be included.

of the same sign. The final answer,

$$\langle\partial_\mu j^\mu\rangle=(e_0g_0/8\pi^2)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}G_{\alpha\beta},$$

furthermore, is independent of how the limit is taken, exactly as in the case of (1.2). The result applies equally well to the quantized-field case; consequently, one has a direct contradiction of the Maxwell field equation

$$\partial_\nu F^{\mu\nu}=e_0 j^\mu.$$

This remarkable paradox, it is hoped, has served to remove any lingering doubts that one might have retained concerning the asserted failure of the ϵ limiting procedure. The following section proposes a resolution of the several contradictions discussed in this paper.

IV. AN ESCAPE FROM THE INCONSISTENCIES OF NONLOCAL CURRENT DEFINITIONS

Before spelling out in detail the precise manner in which the field-theoretical paradoxes of Secs. II and III are to be resolved, it is well to ask whether that list comprises an exhaustive enumeration of the defects of the source driven charge field. In point of fact, there is one additional shortcoming associated with such a theory which is crucial to an understanding of the problems which have been discussed here, namely, the divergent character of the vacuum polarization. Since any reasonable criterion for the consistency of the theory described by (3.2) should include the requirement that the vacuum matrix elements of an arbitrary number of current operators exist (i.e., the vacuum-to-vacuum matrix element considered as a functional of A^μ should exist) the fact that (3.1) is logarithmically divergent means that this model is defective *ab initio*. Thus it is not surprising (and even expected) that numerous inconsistencies are encountered under a careful scrutiny of this theory. Since it will be claimed here that this is the fundamental source of all the contradictions noted earlier, it is important to point out that in two and three dimensions (where the vacuum polarization is finite) none of the paradoxes discussed in this paper are found to occur. Thus this is necessarily a problem in spaces of four (or more) dimensions.

Once one makes this observation concerning the possible relationship between vacuum polarization and the inconsistencies of field theory, the resolution of the problem is fairly immediate. In particular, there are standard tools (e.g., regularization) which can be used to make the theory (2.2) finite. While admitting to a general distaste for regularization techniques and their attendant problems (i.e., indefinite metric), this particular device is certainly the simplest and quite probably the most pedagogically effective approach to the resolution of the problem.

We proceed by taking the Lagrangian (2.2) (hereafter called $\mathfrak{L}_{m_0}$) and adding to it the term

$$\mathfrak{L}_M=\tfrac{1}{2}i\Psi\beta\gamma^\mu\partial_\mu\Psi-\tfrac{1}{2}M\Psi\beta\Psi+ie_0\int_0^A J^\mu(A)\delta A_\mu,$$

where $J^\mu(A)$ is given by (2.1) with ψ and e_0 replaced by Ψ and ie_0, respectively. The inclusion of the complete Lagrangian associated with the regulator field is made explicit here so as to facilitate its incorporation into our action-principle formalism. The imaginary unit in the coupling term clearly implies an indefinite metric and serves to specify precisely the regularization technique to be employed.

Despite the appearance of the indefinite metric, one has the usual circumstance that for sufficiently large M it can have no observable effects at any finite energy and the regulator consequently serves only to impose formal conditions of regularity on the theory. In the case of an external field, the cutoff must always be retained inasmuch as the theory (2.2) cannot be well defined. For quantized A^μ, however, an eigenvalue condition[13] on α_0 could allow for the elimination of all cutoff effects from the theory, in which case both the observable and the unobservable parameters of the theory would be finite and well defined.

It is clear from an inspection of (3.1) that the Lagrangian $\mathfrak{L}_{m_0}+\mathfrak{L}_M$ immediately resolves the problem of the vacuum polarization. The ultraviolet divergence is clearly eliminated by the regulator prescription of subtracting from $\Pi^{\mu\nu}$ the same expression with m_0 replaced by M. The vector ϵ can consequently be set equal to zero and a covariant vacuum polarization obtained. On the other hand, the derivation of (1.2) is considerably affected inasmuch as all mass-independent terms must disappear. One thus derives in place of (1.2) the result

$$\partial_\mu(j_5{}^\mu+J_5{}^\mu)=2m_0j_5+2MJ_5,\qquad(4.1)$$

where an obvious notation has been introduced for the axial-vector current and pseudoscalar density of the regulator field. This is in fact the correct version of the Schwinger-Adler result, since it is easy to verify that the vacuum matrix element of (4.1) yields the vacuum matrix element of (1.2). In particular, one can readily infer from Eqs. (16) to (18) of Ref. 2 that for $M\to\infty$ the matrix element of $J_5{}^\mu$ vanishes while Eqs. (18) to (22) imply that in the same limit

$$2M\langle J_5\rangle=(\alpha_0/4\pi)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}.$$

This also serves to verify the contention made earlier that the Schwinger-Adler result (1.2) is to be taken as an equation between matrix elements rather than between field operators. On the other hand, it should be noted that this conclusion does not significantly affect Adler's resolution of the partially conserved axial-

[13] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954); M. Baker and K. Johnson, *ibid.* **183**, 1292 (1969).

vector current (PCAC) paradox in neutral pion decay, inasmuch as the question of whether (1.2) or (4.1) is the correct relation is largely immaterial to that argument. Furthermore, Adler's arguments against the Bell-Jackiw[14] regularization procedure remain valid, but, of course, do not apply to the entirely different regularization technique employed here.

The breakdown of charge conservation noted earlier can similarly be resolved by regularization after inclusion of the coupling to the external pseudovector field B^μ. The mass-independent term $(e_0 g_0/8\pi^2)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}G_{\alpha\beta}$ clearly drops out, giving

$$\partial_\mu(j^\mu+J^\mu)=0\,,$$

and of course the presence of the J^μ current will remain undetectable at all finite energies.

Finally, we note that the action-principle approach presented in Sec. II now goes through with no difficulty. It yields the result (4.1) together with the generator of chiral gauge transformations

$$Q_5=\int d^3x(j_5{}^0+J_5{}^0)\,.$$

The commutator which replaces (3.3) is clearly

$$-i[H,j_5{}^0(x')+J_5{}^0(x')]+\int d^3x''\frac{\delta_3}{\delta A^\mu(\mathbf{x}'',t)}\times[j_5{}^0(x')+J_5{}^0(x')]\partial_0 A^\mu(\mathbf{x}'',t)+\partial_k(j_5{}^k+J_5{}^k)=2m_0 j_5+2MJ_5\,,$$

which yields, upon taking a variational derivative with respect to $A^0(x)$,

$$[j^0(x)+iJ^0(x),j_5{}^0(x')+J_5{}^0(x')]=0\,.$$

It is entirely straightforward to verify this result by direct calculation; one concludes that there are no anomalous commutators in this formulation, in marked contrast to results obtained in the approaches of Jackiw and Johnson,[4] Adler and Boulware,[6] and of Brandt.[7] The reason for this difference, of course, is the insistence here (in contrast to a view recently expressed by Jackiw and Preparata[15] and Adler and Tung[16]) that perturbation theory must be performed in such a way as to be consistent with unrenormalized field theory. Regularization is the device used to accomplish that end, without which one is led to the inconsistencies previously noted.

In summary, then, the observation that regularization is an essential ingredient of the external source theory has served to eliminate all the paradoxes which have been discussed in this paper. The fact that such a device is required in the fully quantized theory[17] is more widely recognized and here also no contradictions are encountered if one consistently takes the regulator terms into account in all calculations. A significant conclusion which follows from this result is that although the ϵ limiting device can be expected to be useful when questions of gauge invariance arise, it should be noted that situations in which it also plays the role of a cutoff, or where it gives finite anomalous terms, must be suspect in a four-dimensional space-time. The so-called anomalous term has instead been seen to arise from the use of a regulator which in the case of the axial-vector current has the remarkable property of yielding the finite extra term of Eq. (1.2) in the limit of large M. This also serves to explain the previously mysterious independence of (1.2) of the details of the $\epsilon=0$ limit in the original derivations from the field equations. Viewing the anomalous term as a consequence of the regulator formalism, this circumstance emerges as a simple corollary of the fact that the ϵ-independent term $2MJ_5$ happens to yield finite elements which precisely coincide with the Schwinger-Adler anomalous term $(\alpha_0/4\pi)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$.

[14] J. S. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969).

[15] R. Jackiw and G. Preparata, Phys. Rev. Letters **22**, 975 (1969).

[16] S. L. Adler and W. K. Tung, Phys. Rev. Letters **22**, 978 (1969).

[17] In this case, of course, it is necessary to introduce a photon regulator field in addition to the fermion regulator of mass M.