Simple Model for Low-Energy Pion-Pion Scattering*

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A simple model for low-energy pion-pion scattering is given, which expresses the S-wave phase shifts in terms of the ρ-meson parameters. A vital assumption is that the isospin amplitudes are as free as possible from singularities—that is, that the threshold branch points are weak, and can be treated by successive approximation. The present version of the model obeys elastic unitarity above threshold. If a parameter is chosen in a range such that the amplitudes are crossing-symmetric below threshold to a high degree of accuracy, and such that the $\pi^0\pi^0$ S-wave amplitude obeys a set of constraints proven from axiomatic field theory, then the numerical predictions agree reasonably with experiment. A method of improving the model using partial-wave dispersion relations is indicated. Comparison with other, more complicated, models suggests that these often contain many superfluous features which obscure the essential origins of their predictions. It is concluded, for example, that there is no real difference between duality and interference models, as far as prediction of S-wave π-π scattering lengths are concerned.

1. INTRODUCTION

THIS paper describes a simple model of low-energy pion-pion scattering. The model is based on what, we believe, may be the smallest possible set of reasonable assumptions. In addition to satisfying elastic unitarity and to having a high degree of crossing symmetry, the model has the property that the $\pi^0\pi^0$ scattering amplitude satisfies a set of rigorous constraints below threshold. Its lowest features, i.e., scattering lengths and phase shifts below $\sim$500 MeV, are found to be quite similar to those obtained from other, more complicated, models. We are thus led to conclude that ours is, perhaps, a minimal model for low-energy pion-pion scattering.

Our central assumption is that the scattering amplitudes involved, considered as functions of their energy-momentum variables, are as smooth and well behaved as possible. In particular, we suppose that the normal threshold singularities are weak, and can be treated as small perturbations by a method of successive approximations. We consider here only the first approximation.

This hypothesis of maximal smoothness stems from the unproven but nevertheless very appealing idea that all physically relevant quantities are as free as is possible (consistent with general principles) from singularities and pathological variations. Our procedure, following this postulate, is as follows.

First, we write down a crossing-symmetric polynomial expansion of the $\pi\pi$ isospin amplitudes in the unphysical triangle below each channel threshold, assuming the absence of isospin $I=2$ resonances. The series contains terms up to quadratic in the usual channel invariants (s,t,u), this being the lowest-order expansion which permits satisfaction of a certain set of general constraints on the S-wave $\pi^0\pi^0$ amplitude.[1]

Second, three of the four parameters that appear in the amplitudes are fixed by (a) inserting a zero below threshold, as required by the Adler consistency condition[2] (which applies off the mass shell), and (b) requiring smooth matching of the P-wave $I=1$ amplitude at threshold to a simple two-parameter effective-range extrapolation from the ρ meson, whose mass and width are assumed to be given.

The smoothness postulate is involved in both (a) and (b), and (a) requires a rather long extrapolation off the mass shell.

The sign of the final parameter is determined by the constraints on the $\pi^0\pi^0$ S-wave amplitude.

The third step is the insertion of unitarity singularities into the partial-wave amplitudes. As a first approximation only the right-hand cut is included, the effects of the crossed thresholds—i.e., the left-hand cut—being neglected. The prescription used to make the amplitudes unitary is chosen to make as little change as possible in the partial waves below threshold. As a result, the $\pi^0\pi^0$ S wave continues to obey its constraints, and crossing symmetry is preserved to a very good approximation, provided that the free parameter is of a certain (small) size.

At this point we are able to predict S-wave scattering lengths (which, satisfactorily, turn out to be small[3]) and low-energy phase shifts in reasonable agreement with experiment. However, there is no sign of an $I=0$ S-wave resonance σ, although perhaps this state would appear as a result of including unitarity singularities in a better approximation. Possible ways to do this while maintaining crossing symmetry are mentioned. There is no reason to expect that better approximations to analyticity will significantly alter the scattering-length results.

One of the conclusions to be drawn from this very simple model is that prediction of the threshold $\pi\pi$

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[1] A. Martin, Nuovo Cimento **47**, 265 (1967); A. K. Common, *ibid.* **53A**, 946 (1968).

[2] S. L. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).

[3] See, e. g., S. Weinberg, Rapporteur talk, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968); Phys. Rev. **166**, 1568 (1968), Ref. 15.

parameters is not an exacting test of any particular dynamical philosophy. Any other reasonable model of $\pi\pi$ scattering, which may possibly be far more complicated, should display approximately the same general properties and so give very similar scattering-length predictions. For example, in this context it is impossible to distinguish between "duality" and "interference" models.[4]

The paper is organized as follows. Section 2 deals with the construction of the $\pi\pi$ amplitudes and with their unitarization, and explains further the assumptions which they embody. Section 3 gives numerical predictions of the model, and Sec. 4 suggests an extension using dispersion relations, and compares this approach to some other theoretical models.

2. AMPLITUDES

Our relatively uncontroversial assumptions are that the $\pi\pi$ interaction obeys Bose statistics and conserves isospin, and that the amplitudes are crossing-symmetric and Lorentz-invariant. Also, we include at the outset several important rigorous consequences[1] of weak analyticity and unitarity assumptions, although both properties themselves are approximated in accordance with our postulate of maximal smoothness.

Our starting expressions for the s-channel isospin amplitudes $A^I(s,t)$, in the absence of $I=2$ resonances, are[5]

$$A^0(s,t)=\tfrac{3}{2}[F(s,t)+F(s,u)]-\tfrac{1}{2}F(t,u)\,, \qquad (1)$$

$$A^1(s,t)=F(s,t)-F(s,u)\,, \qquad (2)$$

$$A^2(s,t)=F(t,u)\,, \qquad (3)$$

where $F(x,y)=F(y,x)$, and the condition $s+t+u=4\mu^2$ (μ is the pion mass) includes the mass-shell constraint.

Equations (1)–(3) have been derived by Shapiro and Yellin[5] and by Yahil,[5] an they express explicitly Bose statistics, crossing symmetry, and Lorentz invariance. Experimentally the absence of doubly charged dipion resonances is rather well established,[6] so that we may regard the basis for (1)–(3) as fairly secure.

The smoothness hypothesis leads us to write

$$F(s,t)=A+B(s+t)+Cst+D(s^2+t^2) \qquad (4)$$

in the region $0\leqslant s,\ t,\ u\lesssim 4\mu^2$, where A, B, C, and D are real constants.

The Adler condition,[2] which follows either from partial conservation of axial-vector current (PCAC)[7] or from conspiracy theory,[8] applies to all pionic amplitudes, and here means

$$F(\mu^2,\mu^2)=0\,, \qquad (5)$$

off the mass shell, when the value of the third invariant is μ^2.

We assume with our smoothness hypothesis that (5) applies unchanged on the mass shell (that is, where the third invariant equals $2\mu^2$), so that we have

$$A+2\mu^2B=-\mu^4(C+2D)\,. \qquad (6)$$

This is, of course, a strong assumption, although it is lent support by the work of Khuri.[9] We regard the basis for (5) itself as now fairly well established (whatever the phenomenological ambiguities of conspiracy theory[10]) in view of the impressive successes of PCAC in the Goldberger-Treiman relation[11] and in soft-pion calculations[12] of the Adler-Weisberger[13] type.

We define partial-wave amplitudes $A_l^I(s)$ by

$$A^I(s,t)=2\sum_l\,(2l+1)A_l^I(s)P_l(1+2t/(s-4\mu^2))\,, \qquad (7)$$

where the summation over l is restricted such that $l+I$ is a non-negative even integer.

The lowest partial waves in each isospin channel are then from (1)–(4) and (6)

$$A_0{}^0(s)=\tfrac{1}{4}[5A+4B(s+2\mu^2)-\tfrac{1}{6}C(19s-4\mu^2)(s-4\mu^2)+\tfrac{2}{3}D(11s^2-16\mu^2s+32\mu^4)]\,, \qquad (8)$$

$$A_1{}^1(s)=\tfrac{1}{6}(s-4\mu^2)[B+Cs+D(4\mu^2-s)]\,, \qquad (9)$$

$$A_0{}^2(s)=\tfrac{1}{2}[A-B(s-4\mu^2)+\tfrac{1}{6}(C+4D)(s-4\mu^2)^2]\,. \qquad (10)$$

Above threshold, a unitary partial-wave amplitude can be parametrized by

$$[A_l{}^I(s)]^{-1}=\rho(s)[\cot\delta_l{}^I(s)-i]\,, \qquad (11)$$

where $\rho(s)=2q/\sqrt{s}$ and $4q^2=s-4\mu^2$. The phase shift $\delta_l{}^I(s)$ is real in the elastic region $4\mu^2\leqslant s\leqslant 16\mu^2$. However, it is known from experiments on peripheral pion production[14] that $\pi\pi$ scattering is probably totally elastic below about 1 GeV in the center-of-mass system ($s\lesssim 50\mu^2$), and we shall henceforth assume this to be exactly true.

Dispersion-relation calculations[15] and phenomenological analysis[16] show that a good representation of the P-wave $I=1$ amplitude for $50\mu^2\gtrsim s\geqslant 4\mu^2$ is

$$q^2[A_1{}^1(s)]^{-1}=(1/a_1)(1-q^2/k^2)-i\rho(s)\,, \qquad (12)$$

[4] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

[5] J. Shapiro and J. Yellin (unpublished) [quoted by J. Shapiro, Phys. Rev. **179**, 1345 (1969), Ref. 7]; A. Yahil, *ibid.* **185**, 1786 (1969). Yahil's assumption is that resonances in, e.g., the s channel are associated with explicit s dependence of the appropriate amplitude.

[6] A. H. Rosenfeld, in Proceedings of the Philadelphia Conference on Meson Spectroscopy (unpublished).

[7] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960); J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 757 (1960); M. Gell-Mann and M. Lévy, *ibid.* **16**, 705 (1960); J. Bernstein, M. Gell-Mann, and L. Michel, *ibid.* **16**, 560 (1960).

[8] S. Mandelstam, Phys. Rev. **168**, 1884 (1968).

[9] N. N. Khuri, Phys. Rev. **153**, 1477 (1967).

[10] See, for a review, G. E. Hite, Rev. Mod. Phys. (to be published).

[11] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).

[12] See, for a review, S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968).

[13] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); W. I. Weisberger, *ibid.* **14**, 1047 (1965).

[14] P. B. Johnson *et al.*, Phys. Rev. **176**, 1651 (1968).

[15] M. G. Olsson, Phys. Rev. **162**, 1338 (1967).

[16] J. Pišút and M. Roos, Nucl. Phys. **B6**, 325 (1968).

where $4k^2=m^2-4\mu^2$ and $a_1=\frac{1}{8}m^2\Gamma/k^5$. The parameters m and Γ are the ρ-meson mass and width, which we assume to be given. Equation (12) seems to be in error by less than 10%.

Expanding (12) for small $q^2>0$, we find real coefficients for the first two terms. Equating these to the coefficients of the powers of q^2 in (9), we deduce

$$B+4\mu^2D=\tfrac{3}{2}a_1(1-\mu^2/k^2) \tag{13}$$

and

$$C-D=\tfrac{3}{4}a_1/k^2. \tag{14}$$

From (8) and (10), the S-wave scattering lengths in $I=0$ and $I=2$ are

$$a_0=(5/4)A+6\mu^2(B+4\mu^2D) \tag{15}$$

and

$$a_2=\tfrac{1}{2}A. \tag{16}$$

So from (13) we obtain the sum rule

$$2a_0-5a_2=18a_1\mu^2(1-\mu^2/k^2). \tag{17}$$

Weinberg's original soft-pion sum rule[17] reads $2a_0-5a_2=18a_1\mu^2$, and was derived with the further approximation (in our notation)

$$C=D=O(\mu^2/m^2)=0. \tag{18}$$

The introduction of quadratic energy dependence in (5) has led to "hard"-pion corrections of order μ^2/k^2, which is about 15%. This result [Eq. (17)] agrees numerically very well with the rather securely based dispersion-sum-rule approach of Olsson.[18]

The quadratic terms in (4) also mean that a set of constraints[1] on the $\pi^0\pi^0$ S-wave amplitude

$$f=\tfrac{1}{3}(A_0{}^0+2A_0{}^2) \tag{19}$$

can be satisfied.[19] These constraints read

$$df/ds<0,\quad 0\leqslant s\leqslant 1.05\mu^2 \tag{20}$$

$$>0,\quad 1.7\mu^2\leqslant s\leqslant 4\mu^2 \tag{21}$$

$$d^2f/ds^2>0,\quad 0\leqslant s\leqslant 1.7\mu^2 \tag{22}$$

$$f(4\mu^2)>-4, \tag{23}$$

$$f(0)<f(4\mu^2), \tag{24}$$

$$f(3.136)\leqslant f(0), \tag{25}$$

and

$$\mu^2f(0)\geqslant\tfrac{1}{2}\int_{2\mu^2}^{4\mu^2}f(s)ds. \tag{26}$$

Also we have

$$\int_0^{4\mu^2}[A_0{}^0(s)-4A_0{}^2(s)]ds\leqslant 6\mu^2A_0{}^2(0). \tag{27}$$

All these inequalities follow from crossing symmetry, plus positivity of absorptive parts in single-variable dispersion relations for $\pi\pi$ scattering. The positivity condition is a result of unitarity, and the necessary dispersion relations can be established from axiomatic field theory.

If (18) holds, (20)–(27) cannot be obeyed, because f is simply a constant. However, with nonzero C and D coefficients, a straightforward calculation shows that *all eight conditions are simultaneously satisfied if and only if* $C<4D$.

We therefore introduce the convenient parameter

$$X=\tfrac{4}{3}k^2(4D-C)/a_1, \tag{28}$$

and from (6), (13), and (14) find

$$A=-\mu^2a_1[3-\tfrac{1}{4}(13+5X)\mu^2/k^2], \tag{29}$$

$$B=a_1[\tfrac{3}{2}-(2+X)\mu^2/k^2], \tag{30}$$

$$C=\tfrac{1}{4}(X+2)a_1/k^2, \tag{31}$$

and

$$D=\tfrac{1}{4}(X+\tfrac{1}{2})a_1/k^2. \tag{32}$$

X must be positive to satisfy (20)–(27), and it would be rather surprising, given the scale of (29)–(32), if its numerical value were to exceed 2 or 3. Note that the choice $X=1$ leads to

$$F(s,t)=(B+4\mu^2D)(s+t-2\mu^2)+(C-D)(s+t-2\mu^2)^2, \tag{33}$$

which from (13) and (14) is a two-parameter expansion of F around the Adler zero. Possibly this is close to an optimum truncation of the power series for F.

In passing, we point out that if we make the soft-pion approximation (18), and scale the amplitude in terms of the charged-pion decay amplitude $F_\pi(\approx 95$ MeV) using PCAC and the postulated[20] axial-charge commutation relations, we get

$$B=-\tfrac{1}{2}F_\pi{}^{-2}. \tag{34}$$

This is consistent with (13) if the KSRF relation[21] and ρ dominance of the pion electromagnetic form factor are both valid.[22]

Having arrived at parametric forms for the $\pi\pi$ amplitudes below and (with our smoothness hypothesis) up to threshold, the next step is to unitarize them. The simplest procedure would be to equate the inverse of the real expressions (8)–(10) to the quantity $\rho(s)\cot\delta_l{}^I(s)$—that is, to reinterpret these partial-wave amplitudes as simple K-matrix elements.

[17] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[18] M. G. Olsson, University of Wisconsin Report No. Coo-228, 1969 (unpublished).

[19] A one-parameter model of low-energy π-π scattering which satisfies these constraints has been constructed by G. Auberson, O. Piguet, and G. Wanders [Phys. Letters **28B**, 41 (1968)], with rather similar results.

[20] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).

[21] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

[22] This remark has been made also by L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968). Their good prediction of Γ, given m, results from the accuracy of the KSRF relation, and of Eq. (12).

Unfortunately the singularity at $s=0$ in $\rho(s)$ makes the resulting unitary expression for $f(s)$ [given by (19) in terms of the unitarized versions of $A_0{}^0$ and $A_0{}^2$] violate rather badly the constraints (20)–(27). Also, crossing symmetry in $0\leq s,\ t,\ u\leq 4\mu^2$ is completely destroyed.

The remedy is to use the generalized effective-range procedure first derived by Chew and Mandelstam.[23] We introduce

$$H(s)=\frac{1}{\pi}\rho(s)\ln\left(\frac{\rho(s)+1}{\rho(s)-1}\right) \tag{35}$$

$$=\frac{1}{\pi}\rho(s)\ln\left(\frac{1+\rho(s)}{1-\rho(s)}\right)-i\rho(s),\quad s\geq 4\mu^2 \tag{36}$$

which is a function with a branch point only at $s=4\mu^2$, and which is numerically less than unity in $0\leq s\leq 4\mu^2$.

Then we identify[24]

$$\rho(s)\cot\delta_l{}^I(s)=[A_l{}^I(s)]^{-1}+\mathrm{Re}H(s)\,,\quad s\geq 4\mu^2 \tag{37}$$

so that the unitarized partial-wave amplitude is

$$A_l{}^{I(u)}(s)=A_l{}^I(s)[1+H(s)A_l{}^I(s)]^{-1}. \tag{38}$$

The unitarized amplitudes match the nonunitary ones at threshold and at the Adler zero, and they differ throughout $0\leq s\leq 4\mu^2$ by rather little, provided X is not too large, so that for reasonable m, Γ we have $|H(s)A_l{}^I(s)|\ll 1$.[21]

For example, with $X=1$, $m=765$ MeV, and $\Gamma=120$ MeV, it is found that $A_0{}^{0(u)}$ and $A_0{}^0$ differ by at most 5% in $0\leq s\leq 4\mu^2$. The values of $A_0{}^{2(u)}$ and $A_0{}^2$ differ by a smaller fraction, and the real parts of $A_1{}^{1(u)}$ and $A_1{}^1$ agree to within 3% in $0\leq s\leq 10\mu^2$.

Consequently (as we have verified by calculation) the condition $X>0$ remains to an excellent approximation the criterion for satisfaction of the constraints (20)–(27). Also, crossing symmetry below threshold is preserved to high accuracy, and our normalization to the ρ meson is affected to a negligible extent.

3. NUMERICAL RESULTS

Figures 1 and 2 show, respectively, the $I=0$ and $I=2$ S-wave phase shifts as functions of the $\pi\pi$ center-of-mass energy, in four situations of interest, which are as follows.

Case 1 is the soft-pion approximation defined by (18), and is essentially the same as the result of Brown and Goble.[22] It is included for the sake of comparison. Cases 2–4 have nonzero C and D coefficients, and show the effects of increasing X from zero to unity. The normalization is fixed by $m=765$ MeV, $\Gamma=120$ MeV, so that the scattering lengths are related by [from (17)][25]

$$2a_0-5a_2=0.46. \tag{39}$$

[23] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[24] This is the procedure used by Brown and Goble (Ref. 22).

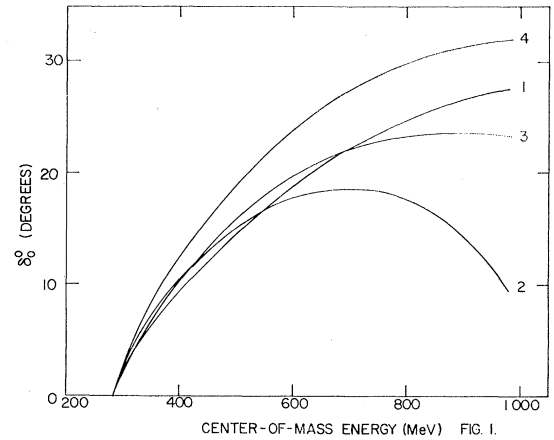

FIG. 1. The $I=0$ S-wave phase shift $\delta_0{}^0$ as a function of dipion c.m. energy, in the four cases described in the text.

Table I summarizes some relevant parameters of the four solutions, including values of the Chew-Mandelstam[23] coupling constant

$$\lambda=-\tfrac{1}{2}A^2(\tfrac{4}{3}\mu^2,\tfrac{4}{3}\mu^2) \tag{40}$$

and

$$\Delta_K=|\delta_0{}^0(s=m_K{}^2)-\delta_0{}^2(s=m_K{}^2)|\,, \tag{41}$$

where m_K is the kaon mass.

In each case $\delta_0{}^0$ and $\delta_0{}^2$ follow the trends indicated by experimental analyses[13] [although there is now some phenomenological[26] and theoretical[27] evidence for a broad $I=0$ S-wave resonance (σ) near 750 MeV, which we do not reproduce]. Both a_0 and a_2 are uniformly small, so that the initial supposition of weak threshold singularities is at least consistent. Note that all the

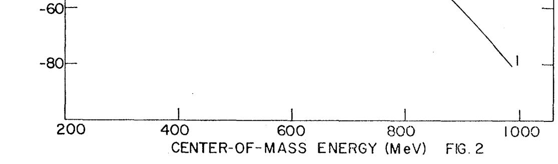

FIG. 2. The $I=2$ S-wave phase shift $\delta_0{}^2$. The notation is the same as in Fig. 1.

[25] Note also that $a_0/a_2=-\frac{7}{2}+O(X\mu^2/k^2)$. Our assumption of no exotic resonances corresponds in this instance to the current-algebraic supposition that the σ term (Weinberg, Ref. 17) is isoscalar, so that we closely approximate the usual result $a_0/a_2=-\frac{7}{2}$.

[26] For an up-to-date list of references see, e.g., Ref. 14.

[27] V. Barger and D. Cline, Phys. Rev. **182**, 1849 (1969); R. C. Johnson, Phys. Rev. Letters **22**, 1143 (1969).

TABLE I. Values of the parameter X, the S-wave scattering lengths a_0 and a_2, the Chew-Mandelstam coupling constant λ, and the difference Δ_K of the S-wave phase shifts at the kaon mass, for the four solutions described in the text.

Solution	X	a_0	a_2	λ	Δ_K(deg)
1	$\cdots$	0.157	−0.045	−0.015	22.7
2	0	0.135	−0.038	−0.013	23.3
3	$\frac{1}{3}$	0.137	−0.037	−0.013	23.6
4	1	0.142	−0.035	−0.013	23.4

values of λ are well within the bounds set by Shaw,[28] and that Δ_K is of the order suggested by most phenomenological investigations of $\pi\pi$ final-state interactions in weak decays.[29]

There is D-wave scattering in $I=0$ and in $I=2$, and for the scattering lengths $a_D{}^I$ we deduce the expressions

$$a_D{}^0=a_1(5X+4)/30k^2, \tag{42}$$

$$a_D{}^2=a_1(X-1)/15k^2. \tag{43}$$

We have then in $I=0$ the lower bound

$$a_D{}^0 \gtrsim 2.1\times 10^{-3}\mu^{-4},$$

which is about seven times as large as expected from the tail of a purely elastic f^0 (mass 1260 MeV, width 145 MeV).[30] The combination $2a_D{}^2+a_D{}^0$ is positive, of course, if $X>0$, as the rigorous results[1] require. Note that only for $X<1$ is the $I=2$ D-wave phase shift negative, as duality suggests.[31]

Plainly, below 500 MeV there is little difference between the four sets of phase shifts. At higher energies, in the δ-meson region (650–850 MeV) there is still not much to choose between cases 1, 3, and 4 for the $I=0$ phase shifts, but the $I=2$ phases show considerable differences. Case 1 has $\delta_0{}^2$ decreasing rapidly, while cases 2 and 3 have $\delta_0{}^2$ roughly constant, and in case 4 $\delta_0{}^2$ is increasing. Experimentally,[14] $\delta_0{}^2$ in this region seems to be constant at about $-10°$ to $-20°$, so that either case 2 or 3 is acceptable on these grounds. Case 3 is preferred theoretically.

As X is increased beyond unity, $|a_2|$ at first decreases, a_0 (and $\delta_0{}^0$) increases monotonically, and $\delta_0{}^2$ changes sign closer and closer to threshold. Eventually, at $X\approx 13$, a_2 becomes positive. Further increases in X decrease $[A_l{}^I(s)]^{-1}$, and so from (37) we have in both channels $\rho(s)\cot\delta_l{}^I(s)\rightarrow \mathrm{Re}H(s)$ (and so a_0, $a_2\rightarrow\infty$, $\Delta_K\rightarrow 0$) as $X\rightarrow\infty$. Neither $\delta_0{}^0$ nor $\delta_0{}^2$ can reach 90° (except at threshold in the limit $X\rightarrow\infty$).

Large values of X, however, lead to unacceptably large values of a_0, a_2, $\delta_0{}^2$, and λ, and imply severe violations of crossing symmetry below threshold in the unitarized amplitudes. They are therefore to be rejected.

Finally, changes in m and Γ of order 20 MeV (as permitted by current phenomenology[30]) do not alter significantly the over-all features of the results.

We conclude that the present experimental situation favors $0\lesssim X\lesssim\frac{1}{2}$, a satisfactory range of values from a theoretical point of view.

4. DISCUSSION AND CONCLUSIONS

In its present form, the model has a number of attractive features, and it gives reasonable numerical predictions. Its improvement involves taking into account the left-hand cuts of the partial-wave amplitudes, while maintaining crossing symmetry.

One natural way to do this is through partial-wave dispersion relations, written either for the amplitude itself[23] or for its inverse.[32] Then the model is a basis for the construction of trial functions, or else is a starting point for an iterative procedure, and in either case serves to fix subtraction constants below threshold. Whichever form of dispersion relations is chosen, the results should be the same,[33] and they should lead to the redundancy of the parameter X, because the output amplitudes will be analytic, crossing-symmetric, and unitary.[34]

The work of Tryon[35] appears to be in some ways equivalent to this sort of extension of the model, and shows that an acceptable solution of the partial-wave dispersion relations exists which displays a broad σ meson together with small values of a_0 and a_2. There is therefore no reason to suppose that more elaborate calculations will necessarily lead to substantially different scattering-length results[36] (although this point is currently the subject of detailed calculations).

The numerical values of a_0 and a_2 given in Table I are rather similar to those predicted by several specifically dynamical models, namely, a_0 is small and positive, and a_2 is smaller and negative. Therefore, the question arises as to the correspondence between the various different starting points.

The models of Schwarz,[37] of Moffat,[38] and of Johnson and Collins[36] are interference models, whereas the model of Lovelace[39] is based on the diametrically opposed[40]

[28] G. Shaw, Phys. Letters 28B, 44 (1968).

[29] B. Gobbi *et al.*, Phys. Rev. Letters 22, 685 (1969); R. P. Ely *et al.*, Phys. Rev. 180, 1319 (1969), and references therein.

[30] Particle Data Group, Rev. Mod. Phys. 41, 1 (1969).

[31] M. G. Olsson and G. Y. Kaiser, University of Wisconsin Report No. Coo-222, 1969 (unpublished).

[32] J. W. Moffat, Phys. Rev. 121, 926 (1961).

[33] This is true provided that there are no significant complex zeros of the partial-wave amplitude.

[34] In the present model, X could be determined by, for instance, maximizing crossing symmetry in some region below threshold—however, this was felt to be an artificial refinement.

[35] E. P. Tryon, Phys. Rev. Letters 20, 769 (1968); in Proceedings of the Argonne-Purdue Conference on $\pi\pi$ and πK Interactions, 1969 (unpublished).

[36] The model of R. C. Johnson and P. D. B. Collins [Phys. Rev. 185, 2020 (1969)], while starting from a very different viewpoint, leads to similar conclusions, although in this case $\delta_0{}^0$ does not exceed 60°, as it does in some of the results of Tryon (Ref. 35).

[37] J. H. Schwarz, Phys. Rev. 175, 1852 (1968).

[38] J. W. Moffat, Trieste Report No. IC/69/42 (unpublished).

[39] C. Lovelace, Phys. Letters 28B, 265 (1968); in Proceedings of the Argonne-Purdue Conference on $\pi\pi$ and πK Interactions, 1969 (unpublished).

[40] The distinction between "interference" and "duality" models has been shown to be rather less clear-cut than hitherto supposed by work of D. Lichtenberg, R. G. Newton, and E. Predazzi, Phys. Rev. Letters 22, 1215 (1969).

dynamical philosophy of duality. The approach of Schwarz is fairly simple, while the other three models are rather complex. However, the properties common to all are (i) crossing symmetry, (ii) the lack of $I=2$ resonances, (iii) the presence of zeros below threshold,[41] and (iv) the ρ meson with more-or-less correct parameters. The treatment of unitarity varies widely.

It seems therefore that as far as a_0 and a_2 are concerned, the essential ingredients in these models are not the particular dynamical philosophies (interference or duality) nor the explicit satisfaction of unitarity, but rather the presence of the four features listed above which are the essential characteristics of the present model. We expect them to be common to all successful descriptions of low-energy $\pi\pi$ scattering.

[41] The zeros are not in the same place in each model—also they arise from apparently rather different (but presumably deeply connected) sources.

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Behavior of Baryon-Baryon and Baryon-Antibaryon Total Cross Sections at High Energy*

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The energy-dependent parts of baryon-baryon and baryon-antibaryon total cross sections at high energies are obtained from (i) factorization of the residues of the leading meson trajectories, and (ii) the connection between the absence of resonances in "exotic" channels and the flatness of meson-meson and meson-baryon total cross sections in these channels. (Such a connection leads to exchange degeneracies, all of which are consistent with experiment.) The following is proven: (a) Nucleon-nucleon (and certain other $B=2$) total cross sections are flat whether or not trajectory couplings obey $SU(3)$; (b) the remaining $B=2$ cross sections are flat if tensor and vector exchanges couple via $SU(3)$, with equal F/D ratios at the baryon vertex; (c) non-Pomeranchuk contributions to baryon-antibaryon total cross sections persist in the **10**, $\overline{\mathbf{10}}$, and **27** channels despite the relaxation of assumptions regarding $B=2$ systems. [Results similar to (a) and (b) have been *assumed* to hold in previous approaches.]

I. INTRODUCTION

In the Regge-pole description of high-energy nucleon-nucleon interactions, the large real parts of the NN forward amplitude[1] and the rapid decrease of $\sigma_T(N\bar{N})$ with increasing energy[2] indicate that sizable non-Pomeranchuk contributions are present above 8 GeV/c. The relative flatness of $\sigma_T(NN)$ in this energy range indicates, however, that the contributions of $+$ and $-$ signature non-Pomeranchuk trajectories to the imaginary part of the forward spin-averaged NN amplitude almost cancel. This cancellation has been termed exchange degeneracy; it entails a particular "mixture of mesons" appropriate for describing the NN interaction in the combination of helicity states contributing to σ_T.[3]

The usual explanation of the near degeneracy of $+$ and $-$ signature contributions to $\sigma_T(BB)$ assumes the comparative weakness of $B=2$ s-channel forces as compared with presumably much stronger $B=0$ u-channel forces. From this assumption about the forces,

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