l-Plane and Khuri-Plane Singularities in the Veneziano Model*

G. C. JOSHI AND A. W. MARTIN†

Department of Physics, Rutgers, The State University, New Brunswick, New Jersey 08903

(Received 9 June 1969)

The singularity structure of the Veneziano model in the angular momentum and Khuri planes is discussed. The continued partial-wave amplitude is shown to have an infinite number of Regge poles, spaced by integers, together with an infinite number of fixed poles at wrong-signature nonsense points in the l plane. The residues of both the moving and fixed poles are calculated, and consequences of their structure are discussed. In the Khuri plane, the Veneziano model is shown to be characterized by an infinite number of moving poles, but there are no fixed poles. The residues of the moving poles are presented.

I. INTRODUCTION

An elegant model for relativistic scattering amplitudes has recently been proposed by Veneziano.[1] Crossing symmetry is exactly satisfied by the model, and Regge asymptotic behavior is ensured by the requirement that the trajectories rise linearly.[2] A number of physical consequences of the model have been explored by several authors[3] with a good degree of success. Also, though not unique, the Veneziano amplitude provides a solution of the finite-energy sum rules.[4]

Because of this phenomenological success, the question of the singularity structure in the complex angular momentum plane, implied by the Veneziano model is of considerable interest. In this paper we study the partial-wave projection of the model as a function in the angular momentum plane. The partial-wave amplitude is holomorphic to the right of a certain line in the complex l plane; a situation familiar in the study of the Frossart-Gribov representation.[5] We then analytically continue the amplitude to the left of this line, encountering families of Regge poles and an infinite set of fixed poles at wrong-signature nonsense points.

We also examine the singularity structure of the Khuri amplitude,[6] defined as the Mellin transform of the absorptive parts of the total scattering amplitude. Our method of analytic continuation of the Khuri amplitude in the complex index plane is essentially the same as that for the partial-wave amplitude. We show that the poles in the index plane have residues given by quite simple expressions. Furthermore, no fixed poles in this plane are found.

For simplicity we consider the scattering of identical spinless particles, so that the total amplitude is symmetric in t and u. The mathematical techniques employed, however, are directly applicable to unequal-mass processes with no t-u symmetry. The only requirements are that $\alpha(t)$ and $\alpha(u)$ be strictly linear functions with identical slopes, and that the intercepts of the trajectories be less than unity.

While it is not mathematically necessary that these slopes and intercepts be strictly real, we shall treat them as such in what follows. This is, of course, the narrow-width resonance approximation. Similarly, there is no restriction on the function $\alpha(s)$ in our method, but to simplify the equations we will take it to be a linear function as well.

The Veneziano amplitude we consider is

$$F(s,t,u)=V(s,t)+V(s,u)+V(t,u) \tag{1}$$

with

$$V(x,y)=\frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} \tag{2}$$

and

$$\alpha(x)=a+bx. \tag{3}$$

We neglect the over-all multiplicative constant in the amplitude. We wish to point out that the results to follow are not limited to the form (2) but can be applied to (2) with arbitrary positive integers in the arguments of the gamma functions. This follows from the freedom of redefining the intercepts of the trajectories and the fact that one gamma function, because of the equal slopes, is always simply a function of s.

In Sec. II we discuss the analyticity properties of the partial-wave amplitudes derived from the first two terms in (1). We show that this contribution to the full partial-wave amplitude is characterized by an infinite number of simple poles in the l plane which are spaced by integers and whose positions are determined by $\alpha(s)$. We discuss various properties of the residues of these moving poles, including a necessary relation to the Veneziano supplementary condition,[1] and show that the continued partial-wave amplitude has the correct threshold behavior in s.

In Sec. III we consider the partial-wave amplitude following from the third term in (1). This contribution is holomorphic in the l plane except for simple fixed

* Research supported in part by the National Science Foundation.

† Present address: Department of Physics, University of Massachusetts–Boston, 100 Arlington Street, Boston, Mass. 02116.

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] S. Mandelstan, in *Proceedings of the International Conference on Particles and Fields, Rochester, 1967* (Wiley-Interscience, Inc., New York, 1967), p. 605; Phys. Rev. **166**, 1539 (1968).

[3] See, e.g., C. Lovelace, Phys. Letters **28B**, 264 (1968); C. J. Goebel, M. L. Blackmon, and K. C. Wali, Phys. Rev. **182**, 1487 (1969).

[4] See, e.g., M. A. Virasoro, Phys. Rev. **177**, 2309 (1969).

[5] S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963); H. Banerjee and G. C. Joshi, Phys. Rev. **137**, B1576 (1965).

[6] N. N. Khuri, Phys. Rev. **132**, 914 (1963).

poles at wrong-signature nonsense points. The residues of these poles are such that if the partial-wave amplitude is unitarized through the N/D formalism, then the kernel of the integral equation for N will be nondegenerate and the well-known Gribov-Pomeranchuk phenomenon will result.[7]

In Sec. IV we derive the Khuri amplitude for the Veneziano model. The first two terms in (1) give rise to an infinite number of moving poles in the index plane, spaced by integers. The residues of these Khuri poles are much simpler in structure than are those for the Regge poles. The contribution to the Khuri amplitude of the third Veneziano term is found to be analytic throughout the index plane. Finally, Sec. V contains our conclusions, and in the Appendix we prove that the residues of fixed poles at right-signature nonsense points in the l plane vanish identically.

II. REGGE POLES

Consider the partial-wave projection

$$a(l,s)=\frac{1}{2}\int_{-1}^{1} dz_s P_l(z_s)[V(s,t)+V(s,u)], \quad (4)$$

where

$$z_s=1+t/2q^2.$$

We use the representation[8]

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\sum_{n=0}^{\infty}\frac{\Gamma(n+1-y)}{n!\Gamma(1-y)(n+x)}, \quad \text{Re}y>0$$

to obtain

$$V(s,t)=-\frac{1}{\Gamma[\alpha(s)]}\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{n![n+1-\alpha(t)]}, \quad \text{Re}\alpha(s)<0$$

with an identical expression for $V(s,u)$.

With the linear expression (3) for $\alpha(t)$ and $\alpha(u)$, the partial-wave projection (4) can be directly carried out to yield Legendre functions of the second kind. Because of the $t-u$ symmetry in our specialized case, the odd-signatured partial-wave amplitude $a^-(l,s)$ vanishes identically, and the even-signatured amplitude is given by

$$a^+(l,s)=-\frac{1}{bq^2}\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{n!\Gamma(\alpha(s))}Q_l\left(1+\frac{n+1-a}{2bq^2}\right). \quad (5)$$

This is simply the Froissart-Gribov formula with the absorptive parts in the t and u channels [for the first two terms in (1)] represented by infinite sums of δ functions. We note that the sum in (5) now converges for $\text{Re}\alpha(s)<l$, because of the asymptotic behavior of the Legendre function, and permits the continuation in $\alpha(s)$ up to this value.

In the conventional way we define the auxiliary amplitude

$$A(l,s)=(4q^2)^{-l}a^+(l,s),$$

and, by Carlson's theorem, continue (5) into the complex l plane in the region $\text{Re}l>\text{Re}\alpha(s)$, where it is holomorphic. For the continuation of $A(l,s)$ to the left of this line, (5) is not a convenient representation, so we employ the Laplace transform[9]

$$Q_l(z)=\int_0^{\infty} dy\, e^{-zy}(\pi/2y)^{1/2}I_{l+1/2}(y), \qquad \text{Re}z>1, \quad \text{Re}l>-1.$$

Inserting this representation into (5), we are permitted to interchange the order of summation and integration for $\text{Re}l>\text{Re}\alpha(s)$, and we encounter the sum

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{n!}e^{-\theta yn}=\Gamma(\alpha(s)+1)(1-e^{-\theta y})^{-\alpha(s)-1},$$

where

$$\theta=(2bq^2)^{-1}$$

and $\text{Re}\theta>0$ is required for the convergence of the sum.

After this step, the auxiliary partial-wave amplitude becomes

$$A(l,s)=M(l,s)\int_0^{\infty} dy\, e^{-\phi y}(1-e^{-\theta y})^{-\alpha(s)-1} \times(\pi/2y)^{1/2}I_{l+1/2}(y), \quad (6)$$

where

$$\phi=1+(1-a)/2bq^2 \quad (7)$$

and

$$M(l,s)=-4\alpha(s)/b(4q^2)^{l+1}. \quad (8)$$

We note that (6) still does not permit the analytic continuation of $A(l,s)$ to the left of the line $\text{Re}l=\text{Re}\alpha(s)$ because of the divergence of the integral at the lower limit. To overcome this limitation we consider the Taylor expansion with the remainder of the following function:

$$f(y)=\left(\frac{1-e^{-\theta y}}{y}\right)^{-\alpha(s)-1}=\sum_{k=0}^{M}f_k y^k/k!+R_M(y), \quad (9)$$

where f_k is the kth derivative of $f(y)$ evaluated at $y=0$.

It is necessary to keep the sum in (9) finite because the full Taylor expansion has a finite radius of convergence, and we wish to carry out the integration in (6) over the positive y axis. However, since the singularities in (6) come from the lower limit and

$$R_M(y)\to y^{M+1} \quad \text{as} \quad y\to 0$$

[7] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters **2**, 239 (1962); C. E. Jones and V. L. Teplitz, Phys. Rev. **159**, 1271 (1967); S. Mandelstam and L. L. Wang, *ibid.* **160**, 1490 (1967); G. C. Joshi and R. Ramachandran, *ibid.* **166**, 1832 (1967).

[8] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 8, Eq. (2).

[9] Bateman Manuscript Project, *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 195, Eq. (5).

by construction, the remainder term in (9) will contribute no l-plane singularities for $\mathrm{Re}l > \mathrm{Re}\alpha(s)-(M+1)$. In what follows, we will ignore the contribution of this remainder term since M can be chosen arbitrarily large, but finite. This procedure makes the analysis of such questions as the asymptotic behavior in s very difficult but it does not affect the determination of the pole positions and the values of their residues.

Because the evaluation of the residues requires the f_k coefficients, we note some of their properties. It is not difficult to show that (using $\theta^{-1}=2bq^2$)

$$f_0=(2bq^2)^{\alpha(s)+1},$$

$$f_k=(2bq^2)^{\alpha(s)+1-k}\sum_{m=0}^{k-1}(-1)^m \times C_m{}^k\Gamma(\alpha(s)+1+k-m)/\Gamma(\alpha(s)+1), \quad (10)$$

where the $C_m{}^k$ are positive, purely numerical coefficients but are rather complicated in structure. We list here the simpler ones:

$$C_0{}^k=2^{-k},\quad C_1{}^k=\tfrac{1}{3}k(k-1)2^{1-k},\quad C_{k-1}{}^k=(k+1)^{-1}.$$

Using (9) in (6), we obtain

$$A(l,s)=M(l,s)\sum_{k=0}^{M}f_k/k!\int_0^\infty dy \times e^{-\phi y}y^{k-\alpha(s)-1}(\pi/2y)^{1/2}I_{l+1/2}(y).$$

This integral is a well-known Laplace transform[10] and yields associated Legendre functions,

$$\int_0^\infty dy\, e^{-\phi y}y^\mu(\pi/2cy)^{1/2}I_{l+1/2}(cy) =c^{-1}(\phi^2-c^2)^{-\mu/2}e^{-i\pi\mu}Q_l{}^\mu(\phi/c) =\frac{\pi^{1/2}c^l\Gamma(l+\mu+1)}{2^{l+1}\phi^{l+\mu+1}\Gamma(l+\frac{3}{2})} \times F(\tfrac{1}{2}l+\tfrac{1}{2}\mu+\tfrac{1}{2},\tfrac{1}{2}l+\tfrac{1}{2}\mu+1;l+\tfrac{3}{2};c^2/\phi^2), \quad (11)$$

with the conditions for the convergence of the integral

$$\mathrm{Re}(l+\mu)>-1,\quad \mathrm{Re}\phi>|\mathrm{Re}c|.$$

This result allows us to write the partial-wave amplitude in the form

$$A(l,s)=M(l,s)\sum_{k=0}^{M}\frac{f_k\pi^{1/2}\Gamma(l+k-\alpha(s))}{k!2^{l+1}\phi^{l+k-\alpha(s)}} \times F(\tfrac{1}{2}l+\tfrac{1}{2}k-\tfrac{1}{2}\alpha(s),\tfrac{1}{2}l+\tfrac{1}{2}k+\tfrac{1}{2} -\tfrac{1}{2}\alpha(s);l+\tfrac{3}{2};\phi^{-2})/\Gamma(l+\tfrac{3}{2}), \quad (12)$$

where ϕ and $M(l,s)$ are given by (7) and (8)' respectively.

Equation (12) can now be continued into the complex l plane, and because $F(a,b;c;z)/\Gamma(c)$ is an entire function of a, b, and c for fixed $|z|<1$, it follows that the only singularities of $A(l,s)$ are those of $\Gamma(l+k-\alpha(s))$. In this way we find that the first two terms in the Veneziano model [Eq. (1)] lead to partial-wave amplitudes that are holomorphic in the l plane except for simple poles at $l=\alpha(s)-m$, with $m=0, 1, 2, \ldots$.

The residues of these moving poles can be expressed as complicated, but finite, sums. The residue of the pole at $l=\alpha(s)-m$ is

$$\beta_m(s)=M(\alpha(s)-m,s)\sum_{k=0}^{m}\frac{(-1)^{k+m}\pi^{1/2}f_k\phi^{m-k}}{k!(m-k)!2^{\alpha(s)-m+1}} \times F(\tfrac{1}{2}k-\tfrac{1}{2}m,\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2};\alpha(s) -m+\tfrac{3}{2};\phi^{-2})/\Gamma(\alpha(s)-m+\tfrac{3}{2}). \quad (13)$$

The simplest case is the residue of the first pole ($m=0$), for which we obtain

$$\beta_0(s)=-\pi^{1/2}\alpha(s)(\tfrac{1}{4}b)^{\alpha(s)}/\Gamma(\alpha(s)+\tfrac{3}{2}). \quad (14)$$

It is interesting to note that this residue vanishes for $\alpha(s)=0$ and for $\alpha(s)=-\frac{3}{2}-n$, with $n=0, 1, 2, \ldots$. Also, the residue vanishes as $\mathrm{Re}\alpha(s)\to+\infty$.[11,12]

Consider the residue of the first daughter pole at $l=\alpha(s)-1$. From (13) we find

$$\beta_1(s)=-\frac{\pi^{1/2}\alpha(s)(\frac{1}{4}b)^{\alpha(s)-1}}{2\Gamma(\alpha(s)+\frac{1}{2})}[\alpha(s)+2a-4bq^2-1], \quad (15)$$

which vanishes for $\alpha(s)=0$ and $\alpha(s)=-\frac{1}{2}-n$, with $n=0, 1, 2, \ldots$. We also note that because, by (3),

$$\alpha(t)+\alpha(u)=2a-4bq^2$$

for our equal-mass process, the condition for $\beta_1(s)$ to vanish identically is

$$\alpha(s)+\alpha(t)+\alpha(u)=1,$$

which is the Veneziano supplementary condition for this process.

The structure is a necessary consequence of the fact that the supplementary condition eliminates alternative trajectories in the total Veneziano amplitude. Because of the complicated structure of (10) and (13), we have not been able to prove that all of the odd-m residues vanish identically if the supplementary condition is fulfilled, but we believe it to be true.

In conclusion, we note that $A(l,s)$ [Eq. (12)], using the q^2 dependence of $M(l,s)$, ϕ, and the coefficients f_k, approaches a constant value as $q^2\to 0$, which is the correct threshold behavior for the auxiliary partial-wave amplitude.

---

[10] G. E. Roberts and H. Kaufman, *Tables of Laplace Transforms* (W. B. Saunders Co., Philadelphia, 1966), p. 72 (4); Ref. 9, p. 196 (9). Note that in the latter reference, Barnes's definition of the associated Legendre functions of the second kind is used rather than the usual definition employed elsewhere in Ref. 9.

[11] H. Goldberg, Phys. Rev. Letters **19**, 1303 (1967).

[12] C. J. Goebel, Phys. Rev. Letters **21**, 383 (1968).

III. FIXED POLES

In this section we discuss the partial-wave projection of the third term in the Veneziano amplitude (1). For the present, we restrict l to positive-integer values and write

$$b_l(s)=\frac{1}{2}\int_{-1}^{1} dz_s P_l(z_s)V(t,u) \tag{16}$$

with

$$V(t,u)=\Gamma(1-\alpha(t))\Gamma(1-\alpha(u))/\Gamma(\alpha(s)+\lambda), \tag{17}$$

where

$$\lambda=1-3a-4m^2b.$$

In writing (17), we have used the assumed linearity of $\alpha(t)$ and $\alpha(u)$ to express the argument of the last gamma function solely in terms of s and have made the further, and independent, simplification of using the linear form (3) for $\alpha(s)$ as well. This latter simplification, clearly, has no effect on the mathematics of the partial-wave projection.

To evaluate the integral in (16) we use the following representation for the gamma function[13]:

$$\Gamma(z)=\int_{-\infty}^{\infty} dx\, e^{zx-e^x}, \quad \mathrm{Re}z>0$$

so that (16) may be written

$$b_l(s)=[2\Gamma(\alpha(s)+\lambda)]^{-1}\int_{-1}^{1} dz_s P_l(z_s) \times\iint_{-\infty}^{\infty} dxdy\, \exp[2bq^2(y-x)z_s] \times\exp[(1-a+2bq^2)(x+y)-e^x-e^y]. \tag{18}$$

The orders of integration in (18) may be interchanged and the identity[14]

$$\frac{1}{2}\int_{-1}^{1} dz_s P_l(z_s)e^{cz_s}=(\pi/2c)^{1/2}I_{l+1/2}(c)$$

may be employed to carry out the z_s integration.

We now make the change of variables

$$x=t-\tfrac{1}{2}r, \quad y=t+\tfrac{1}{2}r, \quad \iint_{-\infty}^{\infty} dxdy \to \iint_{-\infty}^{\infty} drdt$$

to write (18) in the form

$$b_l(s)=\Gamma^{-1}(\alpha(s)+\lambda)\int_{-\infty}^{\infty} dr(\pi/4bq^2r)^{1/2}I_{l+1/2}(2bq^2r) \times\int_{-\infty}^{\infty} dt\, \exp[2(1-a+2bq^2)t-e^t(e^{r/2}+e^{-r/2})]. \tag{19}$$

With the final change of variable $t=\ln x$, the t integral in (19) becomes a simple Laplace transform[15] which converges provided $\mathrm{Re}(1-a+2bq^2)>0$. The result of these manipulations is

$$b_l(s)=[\alpha(s)+\lambda]\int_{-\infty}^{\infty} dr(e^{r/2}+e^{-r/2})^{-\alpha(s)-\lambda-1} \times(\pi/4bq^2r)^{1/2}I_{l+1/2}(2bq^2r). \tag{20}$$

This expression is still not suitable for analytic continuation into the complex l plane because the modified Bessel function in (20) is cut along the negative axis for complex l. However, for integer l we have the symmetry property

$$(-1/z)^{1/2}I_{l+1/2}(-z)=(-1)^l(1/z)^{1/2}I_{l+1/2}(z),$$

and the function has no cuts in the z plane. Because the rest of the integrand in (20) is symmetric in r in our equal-mass process, we carry out the manipulation for integer l

$$\int_{-\infty}^{\infty} dr \to \int_0^{\infty} dr+\int_{-\infty}^{0} dr \to [1+(-1)^l]\int_0^{\infty} dr$$

and note the obvious fact that the odd-signatured partial-wave amplitude vanishes identically.

Defining the even-signatured amplitude in the usual way, we obtain the integral representation

$$b^+(l,s)=2[\alpha(s)+\lambda]\int_0^{\infty} dr\, e^{-[1+\alpha(s)+\lambda]r/2} \times(1+e^{-r})^{-1-\alpha(s)-\lambda}(\pi/4bq^2r)^{1/2}I_{l+1/2}(2bq^2r). \tag{21}$$

It is amusing to note that (21) is almost identical to (6) except for the sign change in

$$(1-e^{-\theta y})\to(1+e^{-r}).$$

But this sign change is crucial to the pole structure in the complex l plane, for it changes the Regge poles of the previous section into fixed poles for the amplitude under consideration.

Before demonstrating this, we consider the Froissart-Gribov representation that follows directly from (21) Inserting the binomial expansion

$$(1+e^{-r})^{-1-\alpha(s)-\lambda}=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\alpha(s)+\lambda+1)}{n!\Gamma(\alpha(s)+\lambda+1)}e^{-nr}$$

into (21) and using the appropriate Laplace transform,[16]

[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press Inc., New York, 1965), p. 934, (10).

[14] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C. 20025, 1965), p. 445, (10.2.36).

[15] Reference 9, p. 137, Eq. (1).

[16] Reference 9, p. 195, Eq. (5), with a simple change of variable.

we find

$$b^+(l,s)=bq^{-2}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\alpha(s)+\lambda+1)}{n!\Gamma(\alpha(s)+\lambda)}\times Q_l\left(\frac{\alpha(s)+\lambda+1+2n}{4bq^2}\right),\quad(22)$$

which is very similar to (5) except for the factor $(-1)^n$. Again, this factor signifies the distinction between Regge poles and fixed poles.

To develop the fixed-pole structure we first define the auxiliary partial-wave amplitude by

$$B(l,s)=(4q^2)^{-l}b^+(l,s),$$

and, by Carlson's theorem, continue (22) into the complex l plane in the region $\mathrm{Re}l>-1$, where it is holomorphic. We now employ the Taylor expansion with remainder, as before,

$$g(r)=(1+e^{-r})^{-1-\alpha(s)-\lambda}=\sum_{k=0}^{M}g_kr^k/k!+R_M(r),\quad(23)$$

where g_k is the kth derivative of $g(r)$ evaluated at $r=0$. Since, by construction,

$$R_M(r)\to r^{M+1}\quad\text{as}\quad r\to 0,$$

the remainder term will contribute no l-plane singularities for $\mathrm{Re}l>-(M+2)$.

The residues of the fixed poles depend upon the coefficients g_k, so we note some of their properties. It is not difficult to verify that

$$g_0=2^{-1-\alpha(s)-\lambda},$$

$$g_k=\sum_{m=0}^{k-1}(-1)^mD_m{}^k2^{-1-\alpha(s)-\lambda-k+m}\frac{\Gamma(1+\alpha(s)+\lambda+k-m)}{\Gamma(1+\alpha(s)+\lambda)},$$

where $D_m{}^k$ are positive, purely numerical coefficients. We list here the simpler ones:

$$D_0{}^k=D_{k-1}{}^k=1,\quad D_1{}^k=\tfrac{1}{2}k(k-1),\quad D_{k-2}{}^k=2^{k-1}-1.$$

In general, these coefficients satisfy the recursion relation

$$D_{m+1}{}^{k+1}=(k-m)D_m{}^k+D_{m+1}{}^k.$$

The next step is to insert (23) into (21) and to obtain for the auxiliary partial-wave amplitude

$$B(l,s)=2[\alpha(s)+\lambda](4q^2)^{-l}\sum_{k=0}^{M}g_k/k!\int_0^\infty dr\,e^{-(1+\alpha(s)+\lambda)r/2}\times r^k(\pi/4bq^2r)^{1/2}I_{l+1/2}(2bq^2r)+R_M(l,s),\quad(24)$$

where $R_M(l,s)$ represents the integral over the remainder term in (23) and is holomorphic for $\mathrm{Re}l>-(M+2)$. Since we are primarily interested here in the location of the fixed poles and the values of their residues, and since M may be chosen arbitrarily large (but finite), we will henceforth ignore the remainder term.

The Laplace transform in (24) converges for $\mathrm{Re}(l+k)>-1$ and $\mathrm{Re}(1-a)>0$, and it is evaluated according to (11). The resulting amplitude may be written in the form

$$B(l,s)=N(l,s)\sum_{k=0}^{M}\frac{g_k\pi^{1/2}\Gamma(l+k+1)}{k!2^{l+1}(2bq^2)^k\phi^{l+k+1}}\times F(\tfrac{1}{2}l+\tfrac{1}{2}k+\tfrac{1}{2},\tfrac{1}{2}l+\tfrac{1}{2}k+1;l+\tfrac{3}{2};\phi^{-2})/\Gamma(l+\tfrac{3}{2}),$$

where

$$N(l,s)=4[\alpha(s)+\lambda]/b(4q^2)^{l+1}$$

and ϕ is again given by (7). This amplitude coming from the third term in the Veneziano model is seen to be very similar in structure to (12). It exhibits the correct threshold behavior, and for the same reasons as discussed in Sec. II, its singularities in the complex l plane are those of $\Gamma(l+k+1)$. It follows that $B(l,s)$ has an infinite number of simple, fixed poles at $l=-m$, $m=1,2,3,\ldots$.

However, as we will show, only the wrong-signature fixed poles (m odd) survive, for the residues of the right-signature fixed poles vanish identically. The residue of the simple pole at $l=-m$ is given by the finite sum

$$\gamma_m=-N(-m,s)\sum_{k=0}^{m-1}\frac{(-1)^{k+m}g_k\pi^{1/2}2^{m-1}}{k!(m-1-k)!(2bq^2)^k\phi^{k+1-m}}\times F(\tfrac{1}{2}k+\tfrac{1}{2}-\tfrac{1}{2}m,\tfrac{1}{2}k+1-\tfrac{1}{2}m;\tfrac{3}{2}-m;\phi^{-2})/\Gamma(\tfrac{3}{2}-m),\quad(25)$$

and, for example, the residue at $l=-1$ is

$$\gamma_1=b^{-1}[\alpha(s)+\lambda]2^{1-\alpha(s)-\lambda}.$$

This structure has the following consequence. If the partial-wave amplitude were unitarized by means of the N/D method, the fact that the residue of the fixed pole at $l=-1$ is proportional to $2^{-\alpha(s)}$ would lead to a nondegenerate kernel in the integral equation for N. In the absence of moving branch points, such a nondegenerate kernel results in an essential singularity at $l=-1$ in the partial-wave amplitude.[7]

It can be verified by explicit calculation from (25) that $\gamma_2=\gamma_4=0$, but this procedure becomes very tedious for the residues of the remaining right-signature fixed poles. To prove that these residues vanish in general, we turn to the Froissart-Gribov representation (22). Using the fact that the residue of $Q_l(z)$ at $l=-m$, $m=1,2,\ldots$, is $P_{m-1}(z)$ we obtain from (22) the alternative formula for the residues of the fixed poles

$$\gamma_m=4b^{-1}(4q^2)^{m-1}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\alpha(s)+\lambda+1)}{n!\Gamma(\alpha(s)+\lambda)}\times P_{m-1}\left(\frac{2n+\alpha(s)+\lambda+1}{4bq^2}\right).\quad(26)$$

It should be remarked that this sum is being evaluated upon its circle of convergence, but it does converge due to the factor $(-1)^n$. In the Appendix it is proved that $\gamma_m=0$ for m even, and as a consequence there are only wrong-signature fixed poles present in $B(l,s)$.

IV. KHURI POLES

In the section we examine the Khuri-plane[6] singularities of the Veneziano model. As before, we will discuss separately the contributions of the first two terms and the third term in (1). It is convenient to use the Khuri amplitudes

$$c^{\pm}(\nu,s)=c_1(\nu,s)\pm c_2(\nu,s)\,,$$

where c_1 and c_2 are defined in Ref. 6 as the Mellin transforms of the t- and u-channel absorptive parts, respectively. For our t-u-symmetric case the amplitude $c^-(\nu,s)$ vanishes identically.

For the contribution of the first two terms of the Veneziano formula to $c^+(\nu,s)$, we use the infinite-sum-of-δ-functions representation for the absorptive parts that can be read off directly from (5). The Mellin transform of this representation yields

$$c^+(\nu,s)=-2b^\nu\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{n!\Gamma(\alpha(s))}(n+1-a)^{-\nu-1}\,, \quad (27)$$

and the sum is convergent for $\mathrm{Re}\nu>\mathrm{Re}\alpha(s)$. To evaluate this sum we use the identity[17]

$$(n+1-a)^{-\nu-1}=\int_0^\infty dt\, t^\nu e^{-(n+1-a)t}/\Gamma(\nu+1)\,. \quad (28)$$

Inserting (28) into (27), we can interchange the order of summation and integration for $\mathrm{Re}\nu>\mathrm{Re}\alpha(s)$, and the resulting sum is simply a binomial expansion.

These steps yield

$$c^+(\nu,s)=-2\alpha(s)b^\nu\int_0^\infty dt$$
$$\times t^\nu e^{-(1-a)t}(1-e^{-t})^{-\alpha(s)-1}/\Gamma(\nu+1)\,, \quad (29)$$

but this form is still not suitable for analytic continuation in the variable ν because of the divergence of the integral at the lower limit for $\mathrm{Re}\nu\leqslant\mathrm{Re}\alpha(s)$. As in the Regge-pole case of Sec. II, we perform the analytic continuation by considering the Taylor expansion

$$f(t)=\left(\frac{1-e^{-t}}{t}\right)^{-\alpha(s)-1}=\sum_{k=0}^{M}f_k\frac{t^k}{k!}+R_M(t)\,, \quad (30)$$

where f_k is the kth derivative of $f(t)$ evaluated at $t=0$. These coefficients f_k are given by (10) with $2bq^2$ replaced by unity.

[17] Reference 8, p. 1, Eq. (5).

We insert (30) into (29) and obtain (again ignoring the remainder term)

$$c^+(\nu,s)=-\frac{2\alpha(s)b^\nu}{\Gamma(\nu+1)}\sum_{k=0}^{M}\frac{f_k}{k!}\int_0^\infty dt\, t^{k+\nu-\alpha(s)-1}e^{-(1-a)t}.$$

Evaluating the simple Laplace transform gives the compact result

$$c^+(\nu,s)=-\frac{2\alpha(s)b^\nu}{\Gamma(\nu+1)}\sum_{k=0}^{M}\frac{f_k\Gamma(k+\nu-\alpha(s))}{k!(1-a)^{k+\nu-\alpha(s)}}\,, \quad (31)$$

which can now be continued into the complex ν plane. We see that the Khuri amplitude is analytic except for an infinite number of simple poles coming from the factor $\Gamma(k+\nu-\alpha(s))$. These poles occur at $\nu=\alpha(s)-m$, with $m=0, 1, 2,\ldots$, and their residues, from (31), are given by

$$\beta_m=-\frac{2\alpha(s)b^{\alpha(s)-m}}{\Gamma(\alpha(s)+1-m)}\sum_{k=0}^{m}\frac{(-1)^{k+m}(1-a)^{m-k}f_k}{k!(m-k)!}. \quad (32)$$

In particular, the residue of the leading pole is

$$\beta_0=-2b^{\alpha(s)}/\Gamma(\alpha(s))\,,$$

which could have been deduced from (14) and the results of the Appendix of Ref. 6. The residue of the first daughter pole is

$$\beta_1=-\alpha(s)b^{\alpha(s)-1}[\alpha(s)+2a-1]/\Gamma(\alpha(s)).$$

As a general property we note from (32) that the residues vanish at $\alpha(s)=m-1, m-2, \ldots, 0, -1, -2, \ldots$, and at additional values of $\alpha(s)$ determined by the finite sum in (32). This sum is always a polynomial in $\alpha(s)$ of order m, as can be seen from (10).

We now discuss the Khuri-plane singularities coming from the third term in (1). The δ-function representation for the absorptive parts can be read off from (22), and the Mellin transform of this representation gives

$$c^+(\nu,s)=2b^\nu\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+1+\alpha(s)+\lambda)}{n!\Gamma(\alpha(s)+\lambda)}(n+1-a)^{-\nu-1}.$$

This sum converges for all finite values of ν, because of the $(-1)^n$, indicating that $c^+(\nu,s)$ is an entire function of ν. But it is worth while to verify this in a different manner. Using (28) as before, interchanging the orders of integration and summation, and evaluating the binomial expansion, we obtain

$$c^+(\nu,s)=2b^\nu[\alpha(s)+\lambda]\int_0^\infty dt$$
$$\times t^\nu e^{-(1-a)t}(1+e^{-t})^{-\alpha(s)-\lambda-1}/\Gamma(\nu+1)\,. \quad (33)$$

The integral in (33) diverges at the lower limit for $\mathrm{Re}\nu\leqslant-1$, but it is clear by inspection that the resulting singularities are like those of the gamma function, namely, simple poles at $\nu=-1, 2, \ldots$. To prove this in detail, one simply inserts the Taylor expansion (23) into (33) and evaluates the resulting Laplace transform. The outcome is

$$c^{+}(\nu,s)=\frac{2[\alpha(s)+\lambda]b^{\nu}}{\Gamma(\nu+1)}\sum_{k=0}^{M}\frac{g_k\Gamma(\nu+k+1)}{k!(1-a)^{\nu+k+1}}+R_M(\nu,s)\,,$$

which can be analytically continued throughout the ν plane to the right of the line $\mathrm{Re}\nu=-M-2$, with M arbitrarily large but finite. This verifies the fact that the Khuri amplitude coming from the third Veneziano term in an entire function in the index plane.

V. CONCLUSIONS

We have studied the singularity structure of the Veneziano model in both the angular momentum and Khuri planes. For simplicity we chose to work with an amplitude symmetric in the Mandelstam variables t and u. However, the mathematical procedures used in the partial-wave projection are immediately applicable to an arbitrary Veneziano amplitude for equal- or unequal-mass scattering processes. The only mathematical restriction is that the Regge trajectories $\alpha(t)$ and $\alpha(u)$ be strictly linear in their arguments. There is no restriction on the trajectory $\alpha(s)$.

It was convenient to analyze separately the contributions of the first two terms and the third term in the Venziano representation because their singularity structures are quite different. The first two terms give rise to an infinite number of Regge poles in the l plane, spaced by integers. The residues of these poles are complicated in structure, but a number of general properties can be inferred. The residues vanish at $\alpha(s)=0$ and at negative half-integer values of $\alpha(s)$. They also tend to zero as $\alpha(s)\to+\infty$.

A further property was noted for the first daughter residue: It vanishes identically if the Veneziano supplementary condition is fulfilled. This is a necessary consequence of the general properties of the supplementary condition in the total Veneziano amplitude. While we were not able to prove that all odd-daughter residues share this property, it is very likely true.

In this connection we note that our results for the residues can be immediately applied to the residues of the resonance poles in the physical s-channel partial-wave amplitudes. The condition that these residues be positive leads to constraints upon the trajectory parameters. This analysis will be presented elsewhere. Finally, we noted that the continued partial-wave amplitude exhibits the correct threshold behavior in the energy variable.

The third term in the Veneziano model gives rise to an infinite number of fixed poles in the l plane. At first sight it appears that these fixed poles occur for both right- and wrong-signature nonsense values of the angular momentum. We were able to prove, however, that the residues of the right-signature fixed poles vanish identically, as is the case with the Mandelstam representation. The residues of the wrong-signature fixed poles were calculated and the following consequence noted. The residues are such that if one proceeds to unitarize the continued partial-wave amplitude by means of the N/D method, the resulting integral equation for N will contain a nondegenerate kernel. In the absence of moving branch points (which are absent in the Veneziano model with strictly linear trajectories), this leads to an essential singularity at $l=-1$.

It is a very interesting fact that the partial-wave amplitudes coming from the Veneziano model have many properties in common with those from the Mandelstam representation. That is, there seems to be a rough one-to-one correspondence between the three terms in the Veneziano amplitude and the three double-spectral-function terms. The properties of the fixed poles noted above are one example.

In this regard we would like to comment that the analysis presented in this paper for the first two terms of the Veneziano model can be rather simply extended to the case of complex, nonlinear trajectories $\alpha(t)$ and $\alpha(u)$. Our preliminary analysis indicates the presence of Regge branch points arising from the nonlinear terms. By analogy to the Mandelstam representation case we expect these branch cuts to correspond to the well-known AFS cuts.[18] The third Veneziano term is more difficult to handle when the trajectories are nonlinear, but we expect Regge branch cuts to arise from this term as well.[19] This analysis will be presented elsewhere.

Once the l-plane structure of the Veneziano model is known, it is straightforward to deduce the singularity structure in the Khuri plane. We showed that the first two Veneziano terms lead to an infinite number of moving poles, spaced by integers in the Khuri plane. The residues of these poles were calculated and their properties discussed. Finally, we demonstrated that the third Veneziano term leads to an entire function in the Khuri plane.

We are hopeful that the singularity structure of the Veneziano model in the angular momentum and Khuri planes presented here will be of use in the major problem of imposing unitarity upon the Veneziano amplitude. Another question of interest, that can be attacked with the methods employed here, is the asymptotic behavior of the partial-wave amplitude as s tends to infinity in all complex directions. While the analysis is straightforward for $\mathrm{Re}s\to+\infty$, it appears to be more complicated for $\mathrm{Re}s\to-\infty$.

Note added in manuscript. After this work was completed, we learned of recent work of Alessandrini and

[18] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962) (referred to here as AFS).

[19] S. Mandelstam, Nuovo Cimento **30**, 1113 (1963); **30**, 1127 (1963); **30**, 1148 (1963); V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. **139**, B184 (1965); J. C. Polkinghorne, J. Math. Phys. **6**, 1960 (1965).

Amati,[20] who approach the problem in a manner quite similar to ours. The present paper contains a considerable extension and a more rigorous treatment of the subject matter. We also learned of work of Fivel and Mitter,[21] who have considered similar problems, although their approach is quite different. The expressions for Regge residues as calculated by Fivel and Mitter agree with ours.

ACKNOWLEDGMENTS

We wish to thank Professor R. W. Childers and Professor A. Pagnamenta for several helpful suggestions.

APPENDIX

In this Appendix we prove that the residues of the fixed poles at right-signature nonsense values of the angular momentum vanish identically. From (26), we observe that these residues are proportional to

$$\gamma_{2m+2} \propto \sum_{n=0}^{\infty} (-1)^n \Gamma(n+\beta) P_{2m+1}[(n+\tfrac{1}{2}\beta)/2bq^2]/n!\,, \tag{A1}$$

where $\beta=\alpha(s)+\lambda+1$ and $m=0, 1, 2, \ldots$. Since the Legendre polynomials in (A1) consist of odd powers of their argument, it is sufficient to prove that

$$S_m(\beta)=\sum_{n=0}^{\infty} (-1)^n \Gamma(n+\beta)(n+\tfrac{1}{2}\beta)^{2m+1}/n!=0$$

for $m=0, 1, 2, \ldots$.

To do this, we first make use of the identity

$$\sum_{n=0}^{\infty} (-1)^n \Gamma(n+\beta+k)/n! = \Gamma(\beta+k)/2^{\beta+k}$$

to prove that $S_0(\beta)=0$. This follows from the steps

$$S_0(\beta)=\sum_{n=0}^{\infty} (-1)^n \Gamma(n+\beta+1)/n! -\tfrac{1}{2}\beta \sum_{n=0}^{\infty} (-1)^n \Gamma(n+\beta)/n!$$
$$=\Gamma(\beta+1)/2^{\beta+1}-\tfrac{1}{2}\beta\Gamma(\beta)/2^{\beta}=0\,.$$

We next establish the recursion relation

$$S_{m+1}(\beta)=\tfrac{1}{4}\beta^2 S_m(\beta)-S_m(\beta+2)\,, \tag{A2}$$

which follows from

$$S_{m+1}(\beta)=\sum_{n=0}^{\infty} (-1)^n \Gamma(n+\beta)(n+\tfrac{1}{2}\beta)^{2m+1} \times (n^2+\beta n+\tfrac{1}{4}\beta^2)/n!$$
$$=\tfrac{1}{4}\beta^2 S_m(\beta)+\sum_{n=1}^{\infty} (-1)^n \Gamma(n+\beta+1) \times (n+\tfrac{1}{2}\beta)^{2m+1}/(n-1)!$$
$$=\tfrac{1}{4}\beta^2 S_m(\beta)-\sum_{n=0}^{\infty} (-1)^n \Gamma(n+\beta+2) \times (n+\tfrac{1}{2}\beta+1)^{2m+1}/n!$$
$$=\tfrac{1}{4}\beta^2 S_m(\beta)-S_m(\beta+2)\,.$$

Since, by (A2), the vanishing of $S_m(\beta)$ implies the vanishing of $S_{m+1}(\beta)$, and since $S_0(\beta)$ vanishes, the proof is completed by induction. We conclude from (A1) that the residues of the right-signature fixed poles arising from the third term in the Veneziano amplitude vanish identically, and that only the wrong-signature fixed poles persist.

[20] V. Alessandrini and D. Amati, Phys. Letters **29B**, 193 (1969).

[21] D. I. Fivel and P. K. Mitter, Phys. Rev. **183**, 1240 (1969)