Algebraic Structure of Current Algebra, Superconvergence Sum Rules, and the Infinite-Momentum Method*

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We present here a derivation of Weinberg's formulas for the algebraic structure of current algebra and superconvergence sum rules in the case of massive pions, using the infinite-momentum method. We show that Weinberg's formulas are closely related to the fact that there is no $T=2$ part in some double commutation relations between axial charges $Q_5{}^a$, $Q_5{}^b$ and the generators of inhomogeneous Lorentz transformations.

I. INTRODUCTION

DURING the past few years, a large number of current-algebra sum rules and superconvergence relations have been derived using various theoretical ideas that include current algebra, partial conservation of axial-vector current (PCAC), dispersion relations, Regge-pole theory, etc. A powerful assumption that has been used to extract useful relations is the saturation assumption, which asserts that a few low-lying single-particle states dominate the dispersion integral. Many algebraic relations between coupling constants and masses have been obtained in this way.[1] In particular, Gilman and Harari[2] saturated all π-ρ scattering sum rules at $t=0$ by π, ω, and A_1, and obtained many results that are in good agreement with experiment. They also considered the problem of saturation for $t\neq 0$ sum rules.[3] In the hope of providing a general algebraic formalism for $t=0$ scattering that might serve as a basis for the applications of chiral dynamics in the future, Weinberg derived an algebraic relation involving masses and axial-vector coupling matrices.[4] His relation is more general than that of Gilman and Harari. Specifically, Weinberg concluded that the axial-vector coupling matrices X and the isospin matrices T form a representation of chiral $SU(2)\times SU(2)$, and that the mass matrix m^2 behaves as the sum of a chiral scalar and a chiral four-vector with respect to the commutation relations with X and T. Furthermore, he was able to express one of the superconvergence relations for pion scattering as a statement about the matrices X, and concluded that the matrix mJ_y is also the sum of a chiral scalar and the fourth component of a chiral four-vector.[5]

Historically, the statement about the commutation relations among the X's and T's was first derived by Dashen and Gell-Mann using the $|\mathbf{p}|\to\infty$ method.[6] However, it was stated in Ref. 4 that there are difficulties in trying to derive the algebraic relation by the $|\mathbf{p}|\to\infty$ method.

It is the purpose of this paper to derive Weinberg's results using the $|\mathbf{p}|\to\infty$ method when the pion is massive. Besides merely providing an alternative derivation, the present approach has the following advantage: We can generalize the method to derive similar algebraic relations for the local current algebra originally proposed by Dashen and Gell-Mann.[7] In Sec. II we start with the derivation of Weinberg's first algebraic relation $[X^a,[X^b,m^2]]_{T=2}=0$ using the $|\mathbf{p}|\to\infty$ method. In Sec. III we consider the behavior of the axial charge $Q_5{}^a$ under Lorentz transformation. In that section we derive Weinberg's second algebraic relation

$$[X^a,[X^b,mJ_y]]_{T=2}=0$$

after introducing the concept of the little group of the Lorentz transformations. In Sec. IV we shall discuss some of the possible generalizations.

II. DERIVATION OF ALGEBRAIC RELATION $[X^a,[X^b,m^2]]_{T=2}=0$

We consider in this section the case in which the pion is massive. We assume that the time derivative of the axial charge $Q_5{}^b(t)$ satisfies

$$[D^b(t),Q_5{}^a(t)]=\delta^{ab}S(t)\,, \tag{1}$$

where $D^b(t)=-i[Q_5{}^b(t),H]=dQ_5{}^b/dt$,[8] and $S(t)$ is an isoscalar. Equation (1) is true in the σ model[9] as well as in the free-quark model.[10]

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[1] Here we only mention some of them: V. Mathur and L. Pandit, Phys. Rev. Letters **19**, 523 (1965); R. Gatto, L. Maiani, and G. Preparata, *ibid.* **16**, 377 (1966); H. Harari, *ibid.* **16**, 964 (1966); **17**, 56 (1966); I. S. Gerstein and B. W. Lee, *ibid.* **16**, 1060 (1966); see also Ref. 2.

[2] F. J. Gilman and H. Harari, Phys. Rev. Letters **18**, 1150 (1967); see also P. H. Frampton and J. C. Taylor, Nuovo Cimento **49**, 152 (1967).

[3] F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968).

[4] S. Weinberg, Phys. Rev. **177**, 2605 (1969). The mass formula derived by Weinberg is a generalization of that of Gilman and Harari. See F. J. Gilman and H. Harari, Phys. Rev. Letters **19**, 723 (1967).

[5] S. Weinberg, Phys. Rev. Letters **22**, 1023 (1969).

[6] R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966*, edited by A. Perlnutter, G. Sudarshan, and B. Kurşunoğlu (W. H. Freeman and Co., San Francisco, 1966).

[7] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966).

[8] The operator D has been introduced before. See, e.g., S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40A**, 1171 (1965); also, F. J. Gilman and H. Harari, Phys. Rev. Letters **19**, 723 (1967).

[9] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[10] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

We can rewrite Eq. (1) as

$$[Q_5{}^a,[Q_5{}^b,H]]_{T=2}=0\,, \tag{2}$$

where the label $T=2$ on any matrix B^{ab} means

$$B^{ab}{}_{T=2}=B^{ab}+B^{ba}-\tfrac{2}{3}\delta^{ab}B^{cc}. \tag{3}$$

In order to get physical results from Eq. (2), we follow Fubini and Furlan[11] in taking the matrix element between single-particle states $|\alpha\mathbf{p}_\alpha\lambda\rangle$ and $|\beta\mathbf{p}_\beta\lambda\rangle$. Here $\mathbf{p}_\alpha$ and $\mathbf{p}_\beta$ are in the z direction, and λ is the helicity. The single-particle states are normalized in such a way that

$$\langle\alpha\mathbf{p}_\alpha\lambda|\alpha\mathbf{p}_\alpha'\lambda'\rangle=\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}_\alpha-\mathbf{p}_\alpha')\,. \tag{4}$$

Now we make the following important assumption: *We can neglect the contributions from the pair states, as well as from the continuum to the $T=2$ part of the matrix element of Eq.* (2), *in the limit of* $|\mathbf{p}|\to\infty$. This is related to the fact that the integrand in the continuum contribution behaves as $\nu^{\alpha_T(0)-1}$ as $\nu\to\infty$,[12] with $\alpha_{T=2}(0)<0$,[13] and the contribution of the pair states behaves as $|\mathbf{p}|^{-2}$ as $|\mathbf{p}|\to\infty$.[12] Under this assumption, we conclude in the spirit of single-particle saturation that

$$[X^a(\lambda),[X^b(\lambda),E]]_{T=2}=0\,, \tag{5}$$

where $X^a(\lambda)$ is defined by

$$[X^a(\lambda)]_{\alpha\gamma}\delta^3(\mathbf{p}-\mathbf{p}')\equiv\lim_{|\mathbf{p}|\to\infty}\langle\alpha\mathbf{p}\lambda|Q_5{}^a|\gamma\mathbf{p}'\lambda\rangle\,, \tag{6}$$

and the energy matrix E is related to the mass matrix m^2 by

$$E=(m^2+|\mathbf{p}|^2)^{1/2}=|\mathbf{p}|+m^2/2|\mathbf{p}|+O(|\mathbf{p}|^{-2})\,. \tag{7}$$

In fact, Eq. (5) should be read as

$$[X^a(\lambda),[X^b(\lambda),|\mathbf{p}|+m^2/2|\mathbf{p}|+O(|\mathbf{p}|^{-2})]]_{T=2}+O(|\mathbf{p}|^{-2})=0.$$

From this we extract the relation

$$[X^a(\lambda),[X^b(\lambda),m^2]]_{T=2}=0\,. \tag{8}$$

This is precisely the result obtained previously by Weinberg. We have succeeded in deriving Weinberg's first algebraic relation in the case of a massive pion using the $|\mathbf{p}|\to\infty$ method without any complication.

III. DERIVATION OF ALGEBRAIC RELATION $[X^a,[X^b,mJ_y]]_{T=2}=0$

We have seen in Sec. II that we derive Weinberg's relation $[X^a,[X^b,m^2]]_{T=2}=0$ by studying the behavior of $Q_5{}^b$ under time translation, i.e., studying D^b. We here ask ourselves the similar question, What is the behavior of $Q_5{}^b$ under the pure Lorentz transformation?

We recall that the axial charge $Q_5{}^b$ is defined as

$$Q_5{}^b\equiv\int d^3x\,A^{b0}(0,\mathbf{x})=\int d^4x\,\delta(t)A^{b0}(0,\mathbf{x})$$

$$=\int d^4x\,\partial_\nu\theta(-n\cdot x)A^{b\nu}\,, \tag{9}$$

where $n=(\mathbf{1,0})$ is the timelike unit vector, and θ is the usual θ function:

$$\theta(a)=1\quad\text{if } a>0$$
$$=0\quad\text{if } a<0.$$

It follows that the charge of $Q_5{}^b$ under the pure Lorentz transformation is given by

$$\delta Q_5{}^b=UQ_5{}^bU^{-1}-Q_5{}^b$$

$$=\int d^4x\,\partial_\nu\theta(-n'\cdot x)A^{b\nu}-\int d^4x\,\partial_\nu\theta(-n\cdot x)A^{b\nu}$$

$$=\int d^4x\,\partial_\nu[\theta(-n'\cdot x)-\theta(-n\cdot x)]A^{b\nu}$$

$$=-\int d^4x[\theta(-n'\cdot x)-\theta(-n\cdot x)]\partial^\nu A^b{}_\nu\,, \tag{10}$$

where $\theta(-n'\cdot x)-\theta(-n\cdot x)\neq 0$ only in the region bounded by the hypersurface $n\cdot x=0$ and $n'\cdot x=0$. One obtains from this

$$[Q_5{}^a(0),\delta Q_5{}^b]$$
$$=-\int d^4x\{\theta(-n'\cdot x)-\theta(-n\cdot x)\}[Q_5{}^a(0),\partial^\nu A^b{}_\nu(0,\mathbf{x})]\,,$$

when we consider the infinitesimal transformation, $\partial^\nu A^b{}_\nu\to\partial^\nu A^b{}_\nu(0,\mathbf{x})$. We assume as before that

$$[Q_5{}^a,\partial^\nu A^b{}_\nu]_{T=2}=0\ \text{at equal times,} \tag{11}$$

which is true in the σ model and in the free-quark model.

Under this assumption we have $[Q_5{}^a,\delta Q_5{}^b]_{T=2}=0$ for an infinitesimal pure Lorentz transformation, or

$$[Q_5{}^a[Q_5{}^b,\mathbf{N}]]_{T=2}=0\,,\quad N^i=M^{0i} \tag{12}$$

where N is the generator of the pure Lorentz transformation.

In fact, here we are interested only in a special subgroup of the homogeneous Lorentz transformation, namely, the little group[14] that leaves the momentum $p^\mu=(E_\gamma\equiv(m_\gamma{}^2+p^2)^{1/2},\,0,\,0,\,p)$ invariant. The unitary operator corresponding to the element in this little

[11] S. Fubini and G. Furlan, Physics **1**, 229 (1965).
[12] See, e.g., S. Adler and R. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968), Chap. 4.
[13] R. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1968). See Ref. 3.

[14] For a general discussion of the problem of the little group and the definition of spin, we refer to the article by A. S. Wightman, in *Dispersion Relations and Elementary Particles* (Wiley–Interscience, Inc., New York, 1960), pp. 161–226.

group can be expressed as $e^{in_\kappa W^\kappa}$, where

$$W^\kappa = \tfrac{1}{2}\epsilon^{\kappa\lambda\rho\sigma} P_\lambda M_{\rho\sigma}.$$

The following properties of W^λ are easily established:

$$\begin{aligned} &W^\lambda P_\lambda = 0, \quad [W^\lambda, P^\mu] = 0, \\ &[W^\lambda, W^\mu] = -i\epsilon^{\lambda\mu\rho\sigma} P_\rho W_\sigma, \\ &[W^\lambda, M^{\mu\nu}] = g^{\lambda\mu} W^\nu - g^{\lambda\nu} W^\mu, \end{aligned} \tag{13}$$

where $M^{\mu\nu} = -M^{\nu\mu}$ are generators of the homogeneous Lorentz group.

Explicitly,

$$\begin{aligned} W^0 &= -\mathbf{p}\cdot\mathbf{J}, \\ \mathbf{W} &= -p^0\mathbf{J} + \mathbf{p}\times\mathbf{N}, \quad N^i = M^{0i} \end{aligned} \tag{14}$$

and

$$[W^1, M^{03}] = [W^2, M^{03}] = 0.$$

Using Eq. (12) and the fact that $Q_5{}^b$ is invariant under translation and rotation, one proves easily that

$$[Q_5{}^a, [Q_5{}^b, W^1]]_{T=2} = 0, \tag{15a}$$

$$[Q_5{}^a, [Q_5{}^b, W^2]]_{T=2} = 0. \tag{15b}$$

It is well known that the little group provides a covariant description of the spin[14]; in particular,

$$\begin{aligned} &\hat{n}_\perp\cdot\mathbf{W}\,|[m_\gamma, s]0\sigma\rangle = -m_\gamma \hat{n}_\perp\cdot(\mathbf{J})_{\sigma\sigma'}\,|[m_\gamma, s]0\sigma'\rangle, \\ &\hat{n}_\perp\cdot\mathbf{W}\,|[m_\gamma, s], \mathbf{p}\sigma\rangle = -m_\gamma \hat{n}_\perp\cdot(\mathbf{J})_{\sigma\sigma'}\,|[m_\gamma, s]\mathbf{p}\sigma'\rangle \\ &\qquad (\mathbf{p} \text{ in the } z \text{ direction}). \end{aligned} \tag{16}$$

Here $\hat{n}_\perp$ is any unit vector in the 1-2 plane, and $\mathbf{J}$ acts only on the helicity indices σ. The second equation is derived from the first by using the fact that W^1 and W^2 are invariant under the boost in the z direction.

To extract the physical content of Eqs. (15a) and (15b), we take their matrix elements between two single-particle states $|\alpha\mathbf{p}_\alpha\lambda_\alpha\rangle$ and $|\beta\mathbf{p}_\beta\lambda_\beta\rangle$, and let $|\mathbf{p}|\to\infty$. In this limit the little group is the same for all particles.

The following is our essential assumption: *We assume that we can neglect the contributions from pair states and the continuum to the $T=2$ part of the matrix element of $[Q_5{}^a, [Q_5{}^b, \hat{n}_\perp\cdot\mathbf{W}]]$ in the limit of $|\mathbf{p}|\to\infty$.*

The single-particle contributions can be readily evaluated in this limit. We end up with the result

$$[X^a, [X^b, m\mathbf{J}\cdot\hat{n}_\perp]]_{T=2} = 0. \tag{17}$$

Here $\mathbf{J}$ is the usual angular momentum matrix defined to act on the helicity indices only. Equation (17) was first derived by Weinberg using the superconvergence relation.

IV. DISCUSSION

So far, we have succeeded in deriving Weinberg's results in the case of a massive pion using the $|\mathbf{p}|\to\infty$ method of Fubini and Furlan. We have also pointed out the connection between Weinberg's results and the absence of a $T=2$ part in some double commutators between the axial charges $Q_5{}^a$, $Q_5{}^b$ and the generators of inhomogeneous Lorentz transformations. In this section we want to mention some of the possible generalizations.

(1) Our present method can be easily generalized to the case of local current algebra proposed originally by Gell-Mann and Dashen. One relation that can be obtained reads

$$[X^a(\mathbf{q}_\perp), [X^b(\mathbf{q}'_\perp), m^2 + \mathbf{p}_\perp{}^2]]_{T=2} = 0.$$

This can be viewed as a generalization of Weinberg's relation.

(2) The little group has been proved to be an important tool in the Sec. II. We may go one step further than Sec. II by considering a finite transformation. In fact, if the pion is massless, we can prove that $[Q_5{}^b, \exp(i\hat{n}_\perp\cdot\mathbf{W}\varphi)] = 0$. Although the $|\mathbf{p}|\to\infty$ method is no longer appropriate for the case of the massless pion, we can use the dispersion approach of Fubini[15] to obtain useful results from $[Q_5{}^b, \exp(i\hat{n}_\perp\cdot\mathbf{W}\varphi)] = 0$. The result we can obtain is $[X^a[X^b, \exp(-i\hat{n}_\perp\cdot\mathbf{J}m\varphi)]]_{T=2} = 0$. We will not attempt to derive this or discuss its possible consequences here. They will be left for a future publication.

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[15] S. Fubini, Nuovo Cimento 43A, 475 (1966).