Kinematic Singularities of the Ball-Chew-Pignotti Multiparticle Amplitudes*

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For multiparticle reactions involving massive particles of any spin, the amplitudes introduced by Bali, Chew, and Pignotti (BCP) are considered as functions of the scalar products between four-momenta. A method previously used by Trueman for 2-to-2 particle reactions and by this author for multiparticle helicity amplitudes is used to classify and explicitly extract the kinematic singularities of the BCP amplitudes. This method concentrates on the Lorentz-group parameters that define the state vectors in terms of which the amplitudes are constructed. The basic assumption is that the kinematic singularities of the amplitudes are due solely to the singular behavior of these group parameters on certain surfaces, given by the vanishing of particular Gram determinants, in the space of the invariant variables. The kinematic singularities take a form which seems suitable for analyzing kinematic constraints in a factorizable multiperipheral model.

I. INTRODUCTION

In a previous paper[1] we investigated the kinematic singularities of helicity amplitudes for multiparticle reactions between massive particles of any spin. The procedure employed was based on a method used by Trueman for 2-to-2 particle processes.[2] Trueman observed that the state vectors used in forming helicity amplitudes become ill defined on certain surfaces in the space of the invariant variables. This is so because the Lorentz-group parameters—viz., for helicity states the hyperbolic and polar angles of the particles's three-momenta—become singular when expressed in terms of the scalar variables as soon as particular Gram determinants formed from the four-momenta vanish. Under the assumption that this is the only source of kinematic singularities in the helicity amplitudes, Trueman was able to explicitly extract these singularities by giving the expansion of the amplitudes near each singularity surface. The method was then generalized to multiparticle helicity amplitudes by this author.

From the point of view of applications, the multiparticle helicity amplitudes do not seem very useful. Instead, Bali, Chew, and Pignotti (BCP)[3,4] introduced another set of amplitudes which are most convenient in formulating multiperipheral—in particular, multi-Regge—models; we shall call them the BCP amplitudes. In introducing their amplitudes, BCP applied group-theory techniques developed by Toller and his collaborators.[5–8] The BCP amplitudes have been further elaborated, in particular, by Chew and DeTar (CD).[9] Multiparticle amplitudes which are essentially the BCP amplitudes have been investigated by Toller[7,10] and by Koba.[11]

In this paper we consider the BCP amplitudes as functions of the invariant variables formed from the four-momenta of the reacting particles, and we investigate their kinematic singularities in these variables. As for helicity amplitudes, the basic assumption is that these singularities occur whenever one of the Lorentz-group parameters, considered as a function of the invariant variables, is singular. Although the details deviate from the procedure for the helicity amplitudes, the technique used here is by and large the same. It permits us to explicitly extract the kinematic singularities by giving the expansion of the amplitudes near each singularity surface.

We do not in this paper treat 2-to-2 particle reactions; we only consider processes with at least three particles in the final state. The BCP amplitudes in the former case reduce to helicity amplitudes in a crossed (t) channel,[6,11] and their singularities are already known.[2]

We begin in Sec. II by reviewing the definition of the BCP amplitudes[3,4] in the CD version,[9] noting that in order to obtain unambiguous amplitudes we must be more specific in places where CD leave a choice open. In this procedure, we give the explicit expressions for the Lorentz-group parameters in terms of the invariant variables. The kinematic singularities of the amplitudes are then treated in detail in Sec. III under the assumption that they arise because of the singular behavior of the group parameters. The results are summarized in Sec. IV in a way most suitable for application. A few concluding remarks on our approach appear in Sec. V. An appendix reviews our notation for determinants.

No attempt is made in this paper to actually apply our results in an analysis of concrete multiperipheral models.

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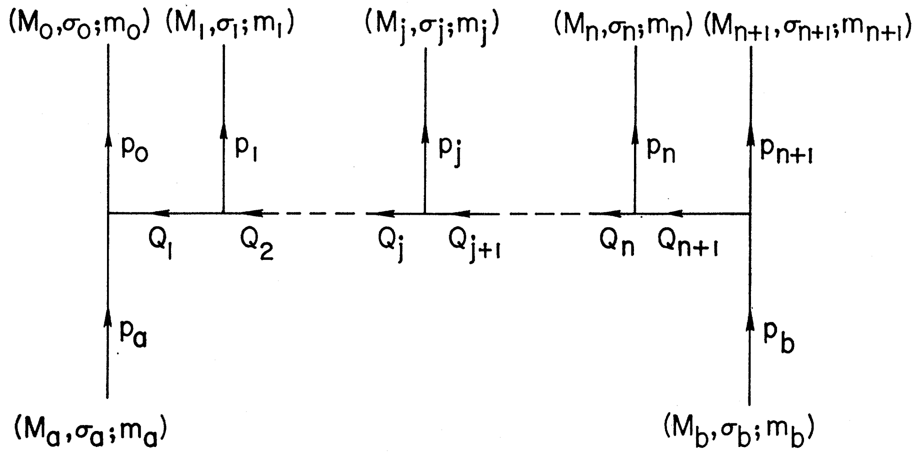

FIG. 1. Multiperipheral chain for the reaction (2.1) with the notation for momenta, masses, spins, and magnetic quantum numbers.

II. BCP AMPLITUDES

Consider the multiparticle reaction

$$a+b\to 0+1+\cdots+n+(n+1)\,,\quad n\geq 1 \qquad (2.1)$$

in which particle j, for $j=a, b, 0, 1, \ldots, n+1$, has mass $M_j\neq 0$, four-momentum $p_j=(E_j,\mathbf{p}_j)$, spin σ_j, and magnetic quantum number m_j; the precise meaning of m_j is given later.

The particles are ordered in some definite, although arbitrary, way to yield the "multiperipheral chain" of Fig. 1, and we introduce the four-momentum transfers

$$Q_j=-p_a+\sum_{j=0}^{j-1} p_j\,,\quad \text{for } j=1, 2, \ldots, n+1\,, \qquad (2.2)$$

and the momentum transfers squared,

$$t_j=Q_j^2\,. \qquad (2.3)$$

We consider physical values of the four-vectors until Sec. II D; in particular, each Q_j is supposed to be a spacelike vector, so that with the metric $(+,-,-,-)$ one has $t_j<0$.

The multiperipheral choice of variables should in the context of this paper be regarded as purely a bookkeeping device, not implying any assumption of dynamical character, although of course our investigations ultimately aim at multiperipheral models.

A. Choice of Lorentz Systems

Following BCP,[3] and, more particularly, CD[9] (see also Ref. 7), we now introduce a series of Lorentz reference systems to be used in defining the particle states when constructing the BCP amplitudes. As already mentioned, in this procedure we have to be more specific than CD concerning the choice of space axes; we comment in Sec. V on this question.

First, define the rest system b_0 for particle 0 to have a three-dimensional coordinate frame with its z axis along $\mathbf{Q}_1=-\mathbf{p}_a$ and its y axis along $\mathbf{Q}_1\times\mathbf{p}_1$ (see Fig. 2).

A rest system b_a for particle a is obtained next by a boost $B_z(\alpha_0)$ along the z axis from the system b_0. Here, and throughout this paper, a Lorentz transformation always means an *active* transformation; in the present context, for example, the boost $B_z(\alpha_0)$ transforms the four-momentum for particle 0 from rest into its value in the system b_a. As a consequence, one has

$$\cosh\alpha_0=(p_ap_0)/(M_aM_0)\,, \qquad (2.4a)$$

$$\sinh\alpha_0=\Delta_2(p_a,p_0)^{1/2}/(M_aM_0)\,; \qquad (2.4b)$$

the notation for Gram determinants follows Ref. 1, reviewed in the Appendix. Moreover, the z axis in b_a is along $\mathbf{p}_0=\mathbf{Q}_1$ and the y axis is along $\mathbf{Q}_1\times\mathbf{p}_1$, as exhibited in Fig. 2.

Note that in the CD terminology our choice of b_a implies that $r_a=1$, so that their system $(0,r)$ coincides with b_a.

Next, one defines a Lorentz system $(1,l)$ in which Q_1 has only a (positive) z component, the z axis being parallel to $\mathbf{p}_a$, while the y axis is still along $\mathbf{Q}_1\times\mathbf{p}_1$. It follows that the system b_a is obtained by a z boost $B_z(q_0)$ from $(1,l)$, and that

$$\cosh q_0=\Delta_2(p_a,Q_1)^{1/2}/[M_a(-t_1)^{1/2}]\,, \qquad (2.5a)$$

$$\sinh q_0=(p_aQ_1)/[M_a(-t_1)^{1/2}]\,. \qquad (2.5b)$$

For further use we also note the expressions

$$\cosh(\alpha_0-q_0)=\Delta_2(p_0,Q_1)^{1/2}/[M_0(-t_1)^{1/2}]\,, \qquad (2.6a)$$

$$\sinh(\alpha_0-q_0)=(-p_0Q_1)/[M_0(-t_1)^{1/2}]\,. \qquad (2.6b)$$

In analogy to the definition of b_0, one defines for each final-state particle j for $j=1, 2, \ldots, n+1$, a special rest system b_j by requiring the z axis to lie along $\mathbf{Q}_j$, and the y axis to be parallel to $\mathbf{p}_{j-1}\times\mathbf{Q}_j$ (Fig. 3). By a z boost $B_z(\alpha_j)$, one then obtains a Lorentz system (j,r) in which Q_j has no energy component, the z axis is along $\mathbf{Q}_j$, and the y axis along $\mathbf{p}_{j-1}\times\mathbf{Q}_j$. It follows that in this system

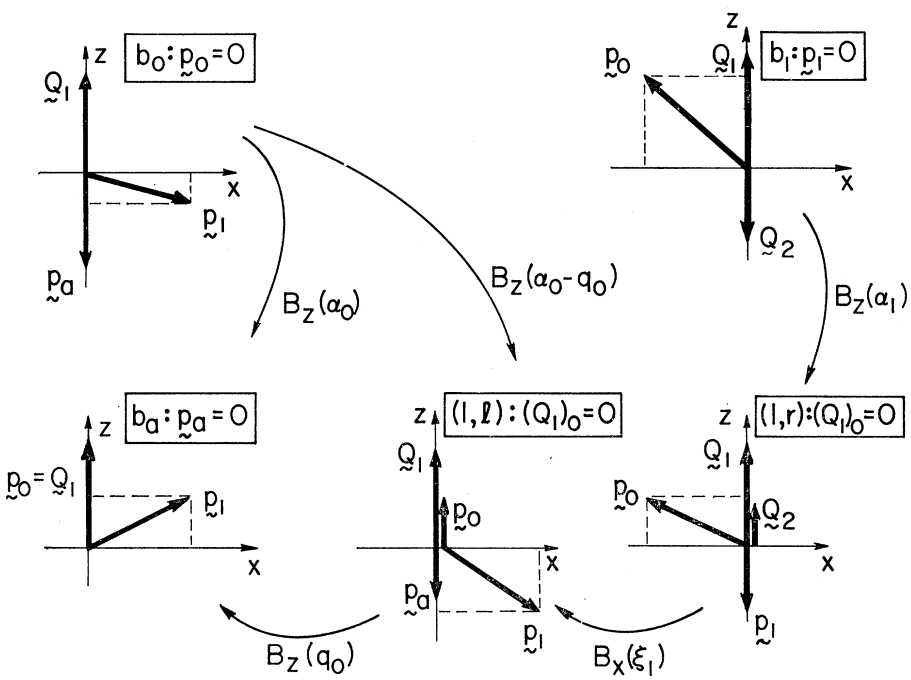

FIG. 2. Lorentz systems associated with the leftmost particles of the multiperipheral chain. Only the xz planes are shown; the three vectors drawn have no y components.

$\mathbf{p}_j$ is parallel to the z axis, as exhibited in Fig. 3, and that

$$\cosh\alpha_j=\Delta_2(p_j,Q_j)^{1/2}/[M_j(-t_j)^{1/2}], \tag{2.7a}$$

$$\sinh\alpha_j=(-p_jQ_j)/[M_j(-t_j)^{1/2}],\quad \text{for}\quad j=1,2,\ldots,n,n+1. \tag{2.7b}$$

Next, one defines another Lorentz system (j,l), for $j=2,\ldots,n+1$, by requiring as in (j,r) the vector Q_j to have no energy component and the z axis to lie along $\mathbf{Q}_j$, but now $\mathbf{p}_{j-1}$ is parallel to the z axis and the y axis is defined to lie along $\mathbf{p}_{j-2}\times\mathbf{Q}_j$ (Fig. 3).

Evidently, for $j=2,\ldots,n+1$, the system (j,l) is obtained from the system (j,r) by a succession of two Lorentz transformations, the first being an x boost $B_x(\xi_j)$ and the second a rotation $R_z(\mu_j)$ around the z axis. It is straightforward to deduce

$$\cosh\xi_j=\frac{1}{\Delta_2(Q_j,p_{j-1})^{1/2}\Delta_2(Q_j,p_j)^{1/2}}\begin{bmatrix}Q_j & p_{j-1}\\ Q_j & p_j\end{bmatrix}, \tag{2.8a}$$

$$\sinh\xi_j=\frac{(-t_j)^{1/2}[\Delta_3(Q_j,p_{j-1},p_j)]^{1/2}}{\Delta_2(Q_j,p_{j-1})^{1/2}\Delta_2(Q_j,p_j)^{1/2}}\quad \text{for } j=1,2,\ldots,n+1, \tag{2.8b}$$

$$\cos\mu_j=\frac{-1}{\Delta_3(Q_j,p_{j-1},p_{j-2})^{1/2}\Delta_3(Q_j,p_{j-1},p_j)^{1/2}}\times\begin{bmatrix}Q_j & p_{j-1} & p_{j-2}\\ Q_j & p_{j-1} & p_j\end{bmatrix}, \tag{2.9a}$$

$$\sin\mu_j=\frac{\epsilon(Q_j,p_{j-1},p_j,p_{j-2})[\Delta_2(Q_j,p_{j-1})]^{1/2}}{\Delta_3(Q_j,p_{j-1},p_{j-2})^{1/2}\Delta_3(Q_j,p_{j-1},p_j)^{1/2}}\quad \text{for } j=2,\ldots,n+1; \tag{2.9b}$$

$$\mu_1=0. \tag{2.9c}$$

Here we have again used the notation for determinants introduced in Ref. 1 (see also the Appendix). Moreover, we have anticipated the immediate conclusion that the results above also apply to the case $j=1$, only that $\mu_1=0$.

Note that our choice of three-dimensional reference frames implies that, in the CD notation, $\nu_j=0$, for $j=1,2,\ldots,n+1$, so that μ_j is just the "Toller angle"; see Sec. V for further comments.

Furthermore, it follows that the Lorentz system (j,r) is obtained from the system $(j+1,l)$ by a z boost $B_z(q_j)$, with

$$\cosh q_j=(-Q_jQ_{j+1})/[(-t_j)^{1/2}(-t_{j+1})^{1/2}], \tag{2.10a}$$

$$\sinh q_j=\Delta_2(Q_j,Q_{j+1})^{1/2}/[(-t_j)^{1/2}(-t_{j+1})^{1/2}]\quad \text{for}\quad j=1,2,\ldots,n. \tag{2.10b}$$

It remains only to define the special rest frame b_b for particle b to have its z axis along $\mathbf{Q}_{n+1}=-\mathbf{p}_{n+1}$ and the y axis parallel to $\mathbf{p}_n\times\mathbf{Q}_{n+1}$ (Fig. 4). In the CD language this means choosing $r_b=1$. Clearly, the system $(n+1,r)$ is obtained from b_b by a z boost $B_z(q_{n+1})$, where

$$\cosh q_{n+1}=\Delta_2(Q_{n+1},p_b)^{1/2}/[(-t_{n+1})^{1/2}M_b], \tag{2.11a}$$

$$\sinh q_{n+1}=(-Q_{n+1}p_b)/[(-t_{n+1})^{1/2}M_b]. \tag{2.11b}$$

B. Definition of State Vectors

The rest frames b_j, for $j=a,b,1,\ldots,n+1$, as introduced above, are chosen to be the frames in which the rest states $|\mathbf{0},m_j\rangle$ are defined; m_j denotes the magnetic quantum number, and the usual Condon-Shortley phase conventions are understood.

In an arbitrary reference system ("the lab system") the state vectors $|\mathbf{p}_j,m_j\rangle$ are, following BCP and CD, ob-

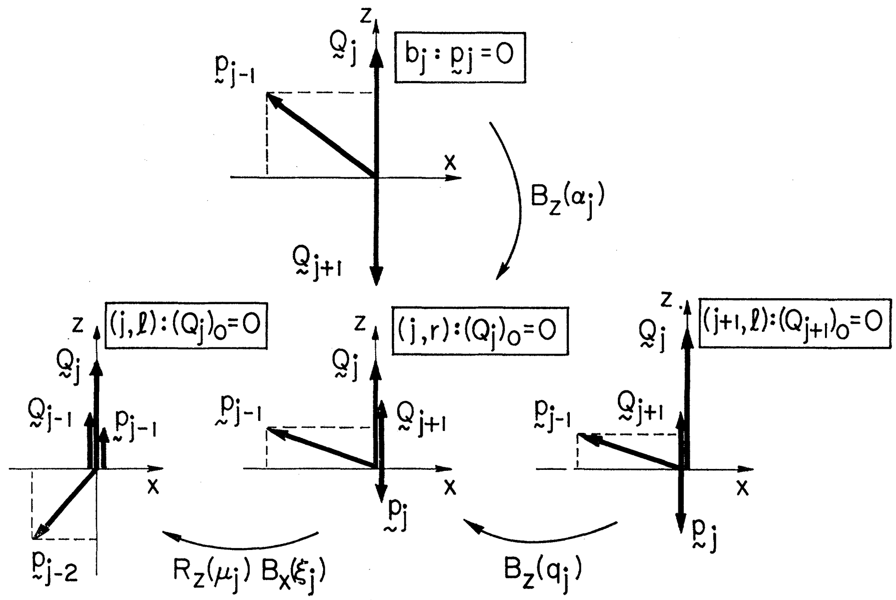

FIG. 3. Lorentz systems associated with an internal vertex j, $j=2, 3, \ldots, n$, of the multiperipheral chain. Only the xz planes are shown; the three vectors drawn have no y components.

tained by applying the series of Lorentz transformations that take a vector from the system b_j, over the system (j,r), and down through the multiperipheral chain to the system b_a, followed finally by a Lorentz transformation from b_a to the lab system. The reaction amplitude is independent of this last transformation.[3,6,7,9]

It is convenient for our purposes to use the freedom in the choice of the lab system to specify it differently for each particular application. As an example, if the lab system is taken as the system b_a, one has

$$|\mathbf{p}_a,m_a\rangle = |\mathbf{0},m_a\rangle\,, \qquad (2.12a)$$

$$|\mathbf{p}_0,m_0\rangle = B_z(\alpha_0)|\mathbf{0},m_0\rangle\,, \qquad (2.12b)$$

$$|\mathbf{p}_1,m_1\rangle = B_z(q_0)B_x(\xi_1)B_z(\alpha_1)|\mathbf{0},m_1\rangle\,, \qquad (2.12c)$$

$$|\mathbf{p}_k,m_k\rangle = B_z(q_0)B_x(\xi_1)B_z(q_1)R_z(\mu_2)B_x(\xi_2)B_z(q_2) \cdots B_z(q_{k-1})R_z(\mu_k)B_x(\xi_k)B_z(\alpha_k)|\mathbf{0},m_k\rangle \quad \text{for} \quad k=2,\ldots,n+1\,, \qquad (2.12d)$$

$$|\mathbf{p}_b,m_b\rangle = B_z(q_0)B_x(\xi_1)\cdots R_z(\mu_{n+1}) \times B_x(\xi_{n+1})B_z(q_{n+1})|\mathbf{0},m_b\rangle\,. \qquad (2.12e)$$

In terms of the generators J_k and K_k (for $k=x, y, z$) for rotations and boosts, respectively, the operators defining the states read

$$B_k(u) = e^{-iuK_k}\,, \qquad (2.13a)$$

$$R_k(v) = e^{-ivJ_k} \quad \text{for} \quad k=x, y, z. \qquad (2.13b)$$

C. Definition of BCP Amplitudes

The reaction amplitudes, being the expectation value of the T matrix between the initial state

$$|i\rangle = |\mathbf{p}_a,m_a\rangle|\mathbf{p}_b,m_b\rangle \equiv |\{\mathbf{p}_i\},\{m_i\}\rangle \qquad (2.14a)$$

and the final state

$$|f\rangle = \prod_{k=0}^{n+1} |p_k,m_k\rangle \equiv |\{\mathbf{p}_f\},\{m_f\}\rangle\,, \qquad (2.14b)$$

are given by

$$T_{\{m\}} = \langle\{m_f\},\{\mathbf{p}_f\}|T|\{\mathbf{p}_i\},\{m_i\}\rangle\,. \qquad (2.15)$$

Here, the state vectors in Eqs. (2.14) are of course the BCP ones defined above. Moreover, $\{m\}$ stands collectively for all magnetic quantum numbers, while $\{m_i\}$ and $\{m_f\}$ stand for those of the initial and final state, respectively; similarly, $\{\mathbf{p}_i\}$ and $\{\mathbf{p}_f\}$ denote collectively the momenta.

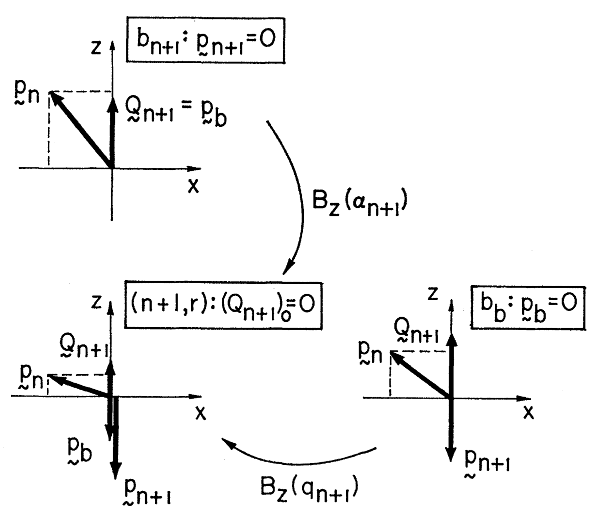

FIG. 4. Lorentz systems associated with the rightmost particles of the multiperipheral chain. Only the xz planes are shown; the three vectors drawn have no y components.

D. Variables and Analyticity

The amplitudes $T_{\{m\}}$ are now considered as functions of all the scalar products p_jp_k, $j>k$ (for j, $k=a$, b, 1, ..., $n+1$), subjected to those restrictions that arise from energy-momentum conservation and from mass-shell conditions. Some of these questions were treated in Ref. 1, where additional references are quoted, and we do not discuss them further here. The same applies to the fact that the reaction amplitudes are also functions of the pseudoscalar products that can be formed from four linearly independent four-vectors.

As is natural from the way in which the BCP amplitudes are defined, we shall most often use those combinations of the scalar products given by t_j, Q_jp_j, Q_jp_{j-1}, etc. It should be kept in mind, though, that not all of these are independent.

Following Refs. 1 and 2, the assumption is now that kinematic singularities of the BCP amplitudes occur whenever the Lorentz-group parameters $(\alpha_j,q_j,\xi_j,\mu_j)$ considered as functions of the scalar products are singular, and that this is the only source of kinematic singularities in the amplitudes. If follows from Eqs. (2.4)–(2.11) that kinematic singularities could occur on the surfaces

$$t_j=0 \quad \text{for} \quad j=1,2,\ldots,n+1\,, \qquad (2.16a)$$

$$\Delta_2(Q_j,p_j)=0 \quad \text{for} \quad j=1,2,\ldots,n+1\,, \qquad (2.16b)$$

$$\Delta_2(Q_1,p_a)=0\,, \qquad (2.16c)$$

$$\Delta_3(Q_j,p_{j-1},p_j)=0 \quad \text{for} \quad j=1,2,\ldots,n+1\,. \qquad (2.16d)$$

We have not listed here the singularities due to the vanishing of the above-mentioned pseudoscalar products; they can be treated by the method used for helicity amplitudes in Ref. 1.

It is convenient to have the following concepts and notations, introduced in Ref. 1. An amplitude $T_{\{m\}}$ is said to be *b-analytic* if it is analytic, except possibly for dynamical singularities and for the occurrence of the pseudoscalar variables. Moreover, the notation

$$g(Z)\sim_+ f(Z) \quad \text{at} \quad Z=Z_0 \qquad (2.17a)$$

means that $g(Z)-f(Z)$ is b-analytic at $Z=Z_0$, and

$$g(Z)\sim_\times f(Z) \quad \text{at} \quad Z=Z_0 \qquad (2.17b)$$

means that $g(Z)/f(Z)$ is b-analytic at $Z=Z_0$.

III. KINEMATIC SINGULARITIES

Each of the singularity surfaces (2.16) is now treated one at a time, by use of the general procedure described in Ref. 1. As we shall see, special attention must be paid to those surfaces pertaining to the left and right ends of the multiperipheral chain of Fig. 1.

In the treatment, we assume that whenever one of the Gram determinants (2.16) vanishes, all the others do not. As was discussed in Ref. 1, the problem of "coinciding singularities" is really a harmless one in the sense that the combination of the separate singularity structures on two (or more) of the surfaces (2.16) gives the structure of the coinciding singularities also.

A. $t_j=0$ Singularity for $j=1, 2, \ldots, n+1$

If the masses obey $M_a\neq M_0$ and $M_{n+1}\neq M_b$, the treatment in this section applies without restrictions. The case when these inequalities are not true is treated in Secs. III B and III C.

At $t_j=0$ we find from Eqs. (2.5)–(2.8) that α_j, q_{j-1}, q_j, and ξ_j are singular. In fact, using the notation (2.17), we find[12] at $t_j=0$,

$$\alpha_j\sim_+[\pm\tfrac{1}{2}\ln(-t_j)] \quad \text{if} \quad \mathrm{Re}(p_jQ_j)\gtrless 0 \quad \text{for} \quad j=1,2,\ldots,n+1\,, \qquad (3.1a)$$

$$q_j\sim_+[\pm\tfrac{1}{2}\ln(-t_j)] \quad \text{if} \quad \mathrm{Re}(Q_jQ_{j+1})\gtrless 0 \quad \text{for} \quad j=1,2,\ldots,n\,, \qquad (3.1b)$$

$$q_{j-1}\sim_+[\pm\tfrac{1}{2}\ln(-t_j)] \quad \text{if} \quad \mathrm{Re}(Q_jQ_{j-1})\gtrless 0 \quad \text{for} \quad j=2,\ldots,n+1\,, \qquad (3.1c)$$

$$\xi_j\sim_\times(-t_j)^{1/2} \quad \text{if} \quad \mathrm{Re}(Q_jp_j)\,\mathrm{Re}(Q_jp_{j-1})>0\,,$$
$$\xi_j-i\pi\sim_\times(-t_j)^{1/2} \quad \text{if} \quad \mathrm{Re}(Q_jp_j)\,\mathrm{Re}(Q_jp_{j-1})<0 \quad \text{for} \quad j=1,2,\ldots,n+1\,, \qquad (3.1d)$$

while those cases not covered by the general formulas are

$$q_0\sim_+[\pm\tfrac{1}{2}\ln(-t_1)] \quad \text{at} \quad t_1=0 \quad \text{if} \quad \mathrm{Re}(p_aQ_1)\gtrless 0\,, \qquad (3.1e)$$

$$q_{n+1}\sim_+[\pm\tfrac{1}{2}\ln(-t_{n+1})] \quad \text{at} \quad t_{n+1}=0 \quad \text{if} \quad \mathrm{Re}(Q_{n+1}p_b)\gtrless 0\,. \qquad (3.1f)$$

Now, since $\mathrm{Re}(Q_jQ_{j+1})$ equals $\mathrm{Re}(Q_jp_j)$ at $t_j=0$, with similar relations for the other real parts entering in Eq. (3.1), it follows immediately that

$$q_{j-1}+q_j \text{ is analytic at } t_j=0 \quad \text{if} \quad \mathrm{Re}(Q_jp_j)\,\mathrm{Re}(Q_jp_{j-1})>0\,, \qquad (3.2a)$$

$$q_{j-1}-q_j \text{ is analytic at } t_j=0 \quad \text{if} \quad \mathrm{Re}(Q_jp_j)\,\mathrm{Re}(Q_jp_{j-1})<0 \quad \text{for} \quad j=1,2,\ldots,n+1; \qquad (3.2b)$$

note in particular the correlation with Eq. (3.1d).

If, in the definition of the state vectors, the lab system is chosen as the system b_a, it follows from Eqs. (2.12) that the potential singularity at $t_j=0$ for $j=1$, 2, ..., $n+1$, will occur only in the states $|\mathbf{p}_k,m_k\rangle$ for $k=j$, $j+1$, ..., $n+1$, b. If here $k\neq j$, one may use commutation rules for Lorentz-group generators[1] to deduce

$$B_z(q_{j-1})B_x(\xi_j)B_z(q_j)=\exp[-i(q_{j-1}+q_j)K_z]\times\exp[-i\xi_j(K_x\cosh q_j-J_y\sinh q_j)]\,, \qquad (3.3a)$$

[12] For a function $u=f(Z)$ that is positive in the physical region the continuation of $u^{1/2}$ is throughout this paper defined to have a positive real part or, if $u<0$ is real, to have a positive imaginary part. Therefore, we shall always in a statement referring to the sign of $\mathrm{Re}(u^{1/2})$ include the implication that if $\mathrm{Re}(u^{1/2})=0$ but $u\neq 0$, we mean the sign of $\mathrm{Im}(u^{1/2})$.

which from the relation (3.2a) shows that this product of boost operators is analytic at $t_j=0$ if $\mathrm{Re}(Q_jp_j)\times\mathrm{Re}(Q_jp_{j-1})$ is positive. If it is negative, one must instead write

$$B_z(q_{j-1})B_x(\xi_j)B_z(q_j) = \exp[-i(q_{j-1}-q_j)K_z]\exp[-i(i\pi)K_x] \times\exp[-i(\xi_j-i\pi)(K_x\cosh q_j-J_y\sinh q_j)] \quad (3.3b)$$

to obtain an analytic expression.

For the state $|\mathbf{p}_j,m_j\rangle$ the only modification is that the boost operator $B_z(\alpha_j)$ replaces $B_z(q_j)$ in Eqs. (3.3); the conclusion again is that there are no singularities.

In summary, there are no $t_j=0$ singularities in the state vectors (2.12). By assumption, there are therefore no kinematic singularities in the BCP amplitudes at $t_j=0$. This conclusion applies at $t_1=0$ only for $M_a\neq M_0$ and at $t_{n+1}=0$ only for $M_{n+1}\neq M_b$. If these inequalities are not fulfilled, there will be singularities, as discussed in Secs. III B and III C.

B. $t_1=0$ Singularity for $M_a=M_0$

In this mass configuration we have

$$\Delta_2(p_a,p_0)=\Delta_2(p_a,Q_1)=\Delta_2(p_0,Q_1)=(-t_1)(M_0^2-\tfrac{1}{4}t_1), \quad (3.4)$$

so that Eqs. (2.5) and (2.6) imply, in the notation (2.17),

$$q_0\sim_\times(-t_1)^{1/2} \quad \text{at} \quad t_1=0, \quad (3.5a)$$

$$\alpha_0-q_0\sim_\times(-t_1)^{1/2} \quad \text{at} \quad t_1=0, \quad (3.5b)$$

while α_1 and q_1 have the behavior given by Eqs. (3.1a) and (3.1b) for $j=1$. Finally,

$$\sinh\xi_1=i \quad \text{at} \quad t_1=0, \quad (3.6a)$$

$$\cosh\xi_1\sim_\times(-t_1)^{1/2} \quad \text{at} \quad t_1=0, \quad (3.6b)$$

from which we conclude

$$\xi_1+\tfrac{1}{2}i\pi\sim_\times(-t_1)^{1/2} \quad \text{at} \quad t_1=0. \quad (3.6c)$$

No other group parameters are singular at $t_1=0$.

It is here convenient to identify the lab system with the system $(2,l)$, so that

$$|\mathbf{p}_a,m_a\rangle=B_z(-q_1)B_x(-\xi_1)B_z(-q_0)|\mathbf{0},m_a\rangle, \quad (3.7a)$$

$$|\mathbf{p}_0,m_0\rangle=B_z(-q_1)B_x(-\xi_1)B_z(\alpha_0-q_0)|\mathbf{0},m_0\rangle, \quad (3.7b)$$

$$|\mathbf{p}_1,m_1\rangle=B_z(\alpha_1-q_1)|\mathbf{0},m_1\rangle. \quad (3.7c)$$

No other states have parameters singular at $t_1=0$. Note also that α_1-q_1 is analytic at $t_1=0$, implying that $|\mathbf{p}_1,m_1\rangle$ has no singularity either.

Now, the commutation relations for Lorentz-group generators allow us to write

$$B_z(-q_1)B_x(-\xi_1) =\exp[i(\xi_1+\tfrac{1}{2}i\pi)(K_x\cosh q_1-J_y\sinh q_1)] \times\exp[-i(\tfrac{1}{2}i\pi)K_x]\exp[i(iq_1)J_y], \quad (3.8a)$$

and, furthermore,

$$\exp[i(iq_1)J_y]B_z(-q_0) =\exp[iq_0(K_z\cosh q_1+iK_x\sinh q_1)]\exp[i(iq_1)J_y]. \quad (3.8b)$$

Therefore, from Eq. (3.7a), we have

$$|\mathbf{p}_a,m_a\rangle=(\cdots)\exp[i(iq_1)J_y]|\mathbf{0},m_a\rangle =(\cdots)\sum_{m_a'}|\mathbf{0},m_a'\rangle d_{m_a'm_a}{}^{\sigma a}(-iq_1), \quad (3.9)$$

where the dots indicate factors that are analytic at $t_1=0$.

In an analogous fashion, we find

$$|\mathbf{p}_0,m_0\rangle=(\cdots)\exp[i(iq_1)J_y]|\mathbf{0},m_0\rangle =(\cdots)\sum_{m_0'}|\mathbf{0},m_0'\rangle d_{m_0'm_0}{}^{\sigma 0}(-iq_1). \quad (3.10)$$

As a consequence, the amplitudes may be written

$$T_{\{m\}}=A_{\{m_0',\ldots;\,m_a',m_b\}}d_{m_a'm_a}{}^{\sigma a}(-iq_1)d_{-m_0'-m_0}{}^{\sigma 0}(-iq_1), \quad (3.11)$$

where the functions $A_{\{m\}}$ are all b-analytic at $t_1=0$, and where a sum over m_a' and m_0' is understood. The kinematic singularity of the BCP amplitudes $T_{\{m\}}$ at $t_1=0$ for $M_a=M_0$ is therefore contained in the two d functions of Eq. (3.11). They may be coupled, and the resulting d function explicitly expanded in powers of $(-t_1)^{1/2}$ in a manner used many times in Refs. 1 and 2. We give this expression in the summary, Eq. (4.1).

C. $t_{n+1}=0$ Singularity for $M_{n+1}=M_b$

An argument paralleling that of Sec. III B shows that the amplitudes have the representation

$$T_{\{m\}}=B_{\{\ldots,\,m_{n+1}';\,m_a,m_b'\}}d_{m_b'm_b}{}^{\sigma b}(-iq_n) \times d_{-m_{n+1}'-m_{n+1}}{}^{\sigma n+1}(-iq_n), \quad (3.12)$$

where the functions $B_{\{m\}}$ are all b-analytic at $t_{n+1}=0$ if $M_{n+1}=M_b$, and the kinematic singularities of the amplitudes therefore are contained in the d functions with the singular argument given by Eq. (3.1c) for $j=n+1$.

D. $\Delta_2(Q_j,p_j)=0$ Singularities for $j=1, 2, \ldots, n$

For convenience, the two cases $\Delta_2(Q_1,p_a)=0$ and $\Delta_2(Q_{n+1},p_{n+1})=0$, affecting as they do the two extreme vertices in the multiperipheral chain, are treated separately in Secs. III E and III F.

Since

$$\Delta_2(Q_j,p_j)=\Delta_2(Q_j,Q_{j+1})=\Delta_2(Q_{j+1},p_j), \quad (3.13)$$

it follows from Eqs. (2.7)–(2.9) that α_j, q_j, ξ_j, ξ_{j+1}, and μ_{j+1} are singular at $\Delta_2(Q_j,p_j)=0$ for $j=1, 2, \ldots, n$. In particular, when the notation (2.17) is used,

$$\alpha_j \mp \tfrac{1}{2}i\pi \sim_\times [D_2^{(\pm)}(Q_j,p_j)]^{1/2} \text{ at } D_2^{(\pm)}(Q_j,p_j)=0\,, \quad (3.14a)$$

where the determinant $\Delta_2(Q_j,p_j)$ has been factorized into the two functions[1]

$$D_2^{(\pm)}(Q_j,p_j)=Q_jp_j\pm iM_j(-t_j)^{1/2}. \quad (3.14b)$$

Similarly,

$$q_j \sim_\times [D_2^{(+)}(Q_j,Q_{j+1})]^{1/2} \quad \text{at} \quad D_2^{(+)}(Q_j,Q_{j+1})=0\,, \quad (3.15a)$$

$$q_j - i\pi \sim_\times [D_2^{(-)}(Q_j,Q_{j+1})]^{1/2} \text{ at } D_2^{(-)}(Q_j,Q_{j+1})=0\,, \quad (3.15b)$$

where we introduce

$$D_2^{(\pm)}(Q_j,Q_{j+1})=Q_jQ_{j+1}\pm(-t_j)^{1/2}(-t_{j+1})^{1/2}. \quad (3.15c)$$

Moreover, for the x-boost parameters we find, at $\Delta_2(Q_j,Q_{j+1})=0$,

$$\xi_j \sim_+ [\pm\tfrac{1}{2}\ln\Delta_2(Q_j,Q_{j+1})]$$

$$\text{for } \operatorname{Re}\{\cdots\}_j \equiv \operatorname{Re}\left\{(-t_j)^{-1/2}\begin{bmatrix} Q_j & p_{j-1} \\ Q_j & p_j \end{bmatrix}\right\} \gtrless 0, \quad (3.16a)$$

$$\xi_{j+1} \sim_+ [\pm\tfrac{1}{2}\ln\Delta_2(Q_j,Q_{j+1})]$$

$$\text{for } \operatorname{Re}\{\cdots\}_{j+1} \equiv \operatorname{Re}\left\{(-t_{j+1})^{-1/2}\begin{bmatrix} Q_{j+1} & p_j \\ Q_{j+1} & p_{j+1} \end{bmatrix}\right\} \gtrless 0. \quad (3.16b)$$

Here, the notation for Gram determinants follows Ref. 1; see also the Appendix.

Finally, for the z-rotation angle one has

$$\sin\mu_{j+1} \sim_\times [\Delta_2(Q_j,Q_{j+1})]^{1/2} \quad \text{at} \quad \Delta_2(Q_j,Q_{j+1})=0\,, \quad (3.17a)$$

while a short calculation using in particular the determinantal identity (A5) of Ref. 1, leads to

$$\cos\mu_{j+1}=\kappa\delta \quad \text{at} \quad \Delta_2(Q_j,Q_{j+1})=0\,, \quad (3.17b)$$

$$\kappa=\pm1 \quad \text{if} \quad \operatorname{Re}\{\cdots\}_j\operatorname{Re}\{\cdots\}_{j+1}\gtrless 0\,, \quad (3.17c)$$

$$\delta=\pm1 \quad \text{if} \quad D_2^{(\pm)}(Q_j,Q_{j+1})=0\,, \quad (3.17d)$$

where the notation introduced in Eqs. (3.15c) and (3.16) has been used.

The states (2.12) affected by $\Delta_2(Q_j,p_j)=0$, $j=1,\ldots,n$, are $|\mathbf{p}_k,m_k\rangle$, $k=j, j+1, \ldots, n+1, b$. For all except $k=j$, the relevant operators $B_x(\xi_j)$, $B_z(q_j)$, $R_z(\mu_{j+1})$, and $B_x(\xi_{j+1})$ are treated as follows. First, commute to obtain

$$R_z(\mu_{j+1})B_x(\xi_{j+1})=\exp(-i\xi_{j+1}K_x)$$
$$\times\exp[-i\mu_{j+1}(J_z\cosh\xi_{j+1}-K_y\sinh\xi_{j+1})]. \quad (3.18a)$$

Next, use

$$\exp(-iq_jK_z)\exp(-i\xi_{j+1}K_x)=\exp(-i\xi_{j+1}K_x)$$
$$\times\exp[-iq_j(K_z\cosh\xi_{j+1}+J_y\sinh\xi_{j+1})] \quad (3.18b)$$

to find that the two operators $B_x(\xi_j)$ and $B_x(\xi_{j+1})$ have been brought to adjacent positions in the operator product.

Now, provided $D_2^{(+)}(Q_j,Q_{j+1})=0$ and $\operatorname{Re}\{\cdots\}_j \times\operatorname{Re}\{\cdots\}_{j+1}$ is negative, our being able to handle the operators as we did, together with the behavior (3.15a), (3.16), and (3.17), shows that the operator product has no singularity here. For the remaining three combinations of the signs κ and δ in Eqs. (3.17c) and (3.17d), an appropriately modified procedure shows that in no case have the states $|\mathbf{p}_k,m_k\rangle$ (for $k=j+1,\ldots,n+1,b$) any singularities at $\Delta_2(Q_j,p_j)=0$ (for $j=1,2,\ldots,n$).

It remains to consider the state $|\mathbf{p}_j,m_j\rangle$. Here, the singularity is contained in

$$\begin{aligned} B_x(\xi_j)B_z(\alpha_j)|\mathbf{0},m_j\rangle &= \exp[-i(\alpha_j\mp\tfrac{1}{2}i\pi)(K_z\cosh\xi_j-J_y\sinh\xi_j)] \\ &\quad\times\exp[\mp i(\tfrac{1}{2}i\pi)K_z]\exp[\pm i(i\xi_j)J_y]|\mathbf{0},m_j\rangle \\ &=(\cdots)\sum_{m_j'}|\mathbf{0},m_j'\rangle d_{m_j'm_j}{}^{\sigma_j}(\mp i\xi_j)\,, \end{aligned} \quad (3.19)$$

where the dots indicate factors that are analytic at, respectively, $D_2^{(\pm)}(Q_j,p_j)=0$.

In summary, the BCP amplitudes may be written

$$T_{\{m\}}=C_{\{\ldots,m_j',\ldots;m_a,m_b\}}{}^{(\pm)(j)}d_{m_j'm_j}{}^{\sigma_j}(\pm i\xi_j)\,, \quad (3.20)$$

where the functions $C_{\{m\}}{}^{(\pm)(j)}$ are all b-analytic at, respectively, $D_2^{(\pm)}(Q_j,p_j)=0$ for $j=1,2,\ldots,n$, so that the kinematic singularities of the amplitudes are contained in the d functions. The explicit representation of the amplitudes in powers of $D_2^{(\pm)}(Q_j,p_j)^{1/2}$ is given in Eq. (4.4).

E. $\Delta_2(Q_1,p_a)=0$ Singularity

The group parameters, being singular, are now α_0, q_0, and ξ_1. It is therefore convenient to identify the lab system with the system $(1,r)$, in which

$$|\mathbf{p}_a,m_a\rangle=B_x(-\xi_1)B_z(-q_0)|\mathbf{0},m_a\rangle\,, \quad (3.21a)$$

$$|\mathbf{p}_0,m_0\rangle=B_x(-\xi_1)B_z(\alpha_0-q_0)|\mathbf{0},m_0\rangle\,, \quad (3.21b)$$

and no other states contain any singular parameters.

If $\Delta_2(p_a,p_0)$ is factorized into

$$D_2^{(\pm)}(p_a,p_0)=p_ap_0\pm M_aM_0\,, \quad (3.22)$$

the usual arguments lead, in the notation (2.17), to

$$q_0+\tfrac{1}{2}i\pi \sim_\times [D_2^{(+)}(p_a,p_0)]^{1/2} \quad \text{at} \quad D_2^{(+)}(p_a,p_0)=0\,, \quad (3.23a)$$

$$q_0\pm\tfrac{1}{2}i\pi \sim_\times [D_2^{(-)}(p_a,p_0)]^{1/2} \quad \text{at} \quad D_2^{(-)}(p_a,p_0)=0 \quad \text{if} \quad M_a\gtrless M_0. \quad (3.23b)$$

The case $M_a=M_0$, already treated in Sec. III B, is neglected in Eq. (3.23b). Similarly,

$$q_0-\alpha_0-\tfrac{1}{2}i\pi\sim_\times [D_2^{(+)}(p_a,p_0)]^{1/2} \quad \text{at} \quad D_2^{(+)}(p_a,p_0)=0\,, \qquad (3.24a)$$

$$q_0-\alpha_0\pm\tfrac{1}{2}i\pi\sim_\times [D_2^{(-)}(p_a,p_0)]^{1/2} \quad \text{at} \quad D_2^{(-)}(p_a,p_0)=0 \quad \text{if} \quad M_a\gtrless M_0\,. \qquad (3.24b)$$

Finally, as in Sec. III D,

$$\left.\begin{matrix}\cosh\xi_1\\ \sinh\xi_1\end{matrix}\right\}\sim_\times \{\Delta_2(Q_1,p_a)\}^{-1/2} \quad \text{at} \quad \Delta_2(Q_1,p_a)=0\,. \qquad (3.25)$$

Consequently, the same technique as used in Eqs. (3.19) and (3.20) leads to the representations

$$T_{\{m\}}=E_{\{m_0',\ldots;m_a',m_b\}}d_{m_a'm_a}{}^{\sigma a}(i\xi_1)d_{m_0'm_0}{}^{\sigma 0}(i\xi_1)\,, \qquad (3.26a)$$

where the functions $E_{\{m\}}$ are all b-analytic at

$$D_2^{(+)}(p_a,p_0)=0\,,$$

and

$$T_{\{m\}}=F_{\{m_0',\ldots;m_a',m_b\}}{}^{(\pm)}d_{m_a'm_a}{}^{\sigma a}(\pm i\xi_1)d_{m_0'm_0}{}^{\sigma 0}(\mp i\xi_1)\,, \qquad (3.26b)$$

where the functions $F_{\{m\}}{}^{(\pm)}$ are all b-analytic at $D_2^{(-)}(p_a,p_0)=0$ provided $M_a\gtrless M_0$; the signs in Eq. (3.26b) correlate with this inequality. Again, the explicit dependence on $[D_2^{(\pm)}(p_a,p_0)]^{1/2}$ is given in the summary, Eqs. (4.2) and (4.3).

F. $\Delta_2(Q_{n+1},p_b)=0$ Singularity

Arguments similar to those in Sec. III E show that the BCP amplitudes have the representations

$$T_{\{m\}}=G_{\{\ldots,m_{n+1}';m_a,m_b'\}}d_{m_b'm_b}{}^{\sigma b}(i\xi_{n+1}) \times d_{m_{n+1}'m_{n+1}}{}^{\sigma n+1}(i\xi_{n+1})\,, \qquad (3.27a)$$

where the functions $G_{\{m\}}$ are all b-analytic at $D_2^{(+)}(p_b,p_{n+1})=0$, and

$$T_{\{m\}}=H_{\{\ldots,m_{n+1}';m_a,m_b'\}}{}^{(\pm)}d_{m_b'm_b}{}^{\sigma b}(\pm i\xi_{n+1}) \times d_{m_{n+1}'m_{n+1}}{}^{\sigma n+1}(\mp i\xi_{n+1})\,, \qquad (3.27b)$$

where the functions $H_{\{m\}}{}^{(\pm)}$ are all b-analytic at $D_2^{(-)}(p_b,p_{n+1})=0$ if $M_b\gtrless M_{n+1}$; the signs in Eq. (3.27b) correlate with this inequality.

G. $\Delta_3(Q_j,p_{j-1},p_j)=0$ Singularities for $j=2, 3, \ldots, n$

The cases $j=1$ and $n+1$ are slightly more complicated and are treated separately in the two following sections.

It is clear from Sec. II A that only ξ_j, μ_j, and μ_{j+1} are singular at $\Delta_3(Q_j,p_{j-1},p_j)=0$, for $j=2, 3, \ldots, n$.

In particular, using the notation (2.17),

$$\xi_j\sim_\times [D_3^{(+)}(Q_j;\,p_{j-1},p_j)]^{1/2} \quad \text{at} \quad D_3^{(+)}(Q_j;\,p_{j-1},p_j)=0\,, \qquad (3.28a)$$

$$\xi_j-i\pi\sim_\times [D_3^{(-)}(Q_j;\,p_{j-1},p_j)]^{1/2} \quad \text{at} \quad D_3^{(-)}(Q_j;\,p_{j-1},p_j)=0\,, \qquad (3.28b)$$

where $\Delta_3(Q_j,p_{j-1},p_j)$ have been factorized into the two functions[1]

$$D_3^{(\pm)}(Q_j;\,p_{j-1},p_j)=[\Delta_2(Q_j,p_{j-1})]^{1/2}[\Delta_2(Q_j,p_j)]^{1/2} \pm\begin{bmatrix}Q_j & p_{j-1}\\ Q_j & p_j\end{bmatrix}. \qquad (3.28c)$$

Moreover, both sine and cosine of μ_j and μ_{j+1} behave as $[\Delta_3(Q_j,p_{j-1},p_j)]^{-1/2}$, and a short calculation shows

$$\tan\mu_{j+1}=\mp\tan\mu_j \quad \text{at} \quad D_3^{(\pm)}(Q_j;\,p_{j-1},p_j)=0\,. \qquad (3.29)$$

It follows immediately that

$$\mu_j\pm\mu_{j+1} \quad \text{is analytic at} \quad D_3^{(\pm)}(Q_j;\,p_{j-1},p_j)=0\,. \qquad (3.30)$$

In the state vectors $|\mathbf{p}_k,m_k\rangle$ (for $k=j+1,\ldots, n+1, b$), we find, for the relevant operators,

$$R_z(\mu_j)B_x(\xi_j)R_z(\mu_{j+1})=R_z(\mu_j+\mu_{j+1}) \times\exp[-i\xi_j(K_x\cos\mu_{j+1}+K_y\sin\mu_{j+1})]\,, \qquad (3.31)$$

so that this operator product is analytic at $D_3^{(+)}(Q_j;\,p_{j-1},p_j)=0$. A similar argument using instead Eq. (3.28b) leads to the conclusion that the product is analytic also at $D_3^{(-)}(Q_j;\,p_{j-1},p_j)=0$.

Concerning the state $|\mathbf{p}_j,m_j\rangle$, the appropriate procedure is to write

$$R_z(\mu_j)B_x(\xi_j)=\exp[-i\xi_j(K_x\cos\mu_j-K_y\sin\mu_j)] \times\exp(-i\mu_jJ_z) \qquad (3.32)$$

in order to find that the singularity of this state occurs only in a factor $\exp(-i\mu_jm_j)$ at $D_3^{(+)}(Q_j;\,p_{j-1},p_j)=0$. By a similar argument, the singularity at $D_3^{(-)}(Q_j;\,p_{j-1},p_j)=0$ may be isolated in a factor $\exp(i\mu_jm_j)$.

As a consequence, the kinematic singularities of the amplitudes are given by the representations

$$T_{\{m\}}=N_{\{m\}}{}^{(\pm)(j)}\exp(\pm i\mu_jm_j)\,, \qquad (3.33)$$

where the functions $N_{\{m\}}{}^{(\pm)(j)}$ are b-analytic at, respectively, $D_3^{(\pm)}(Q_j;\,p_{j-1},p_j)=0$ for $j=2,\ldots,n$; the notation (3.28c) is used. The angle μ_j is given by Eqs. (2.9).

Because the angle μ_j depends on the pseudoscalar product $\epsilon(Q_j,p_{j-1},p_j,p_{j-2})$, the singularity (3.33) cannot be written unambiguously in powers of $[D_3^{(\pm)}(Q_j;\,p_{j-1},p_j)]^{1/2}$; this circumstance is discussed in relation to Eq. (4.50) of Ref. 1.

H. $\Delta_3(Q_1,p_0,p_1)=0$ Singularity

Here, ξ_1 and μ_2 are singular; in particular, Eqs. (3.28) apply also in this case with $j=1$. It is therefore convenient to identify the lab system with the one obtained from $(1,r)$ by a z rotation $R_z(-\mu_2)$, in which system the states containing any singular parameters are

$$|\mathbf{p}_a,m_a\rangle=R_z(-\mu_2)B_x(-\xi_1)B_z(-q_0)|\mathbf{0},m_a\rangle, \quad (3.34a)$$

$$|\mathbf{p}_0,m_0\rangle=R_z(-\mu_2)B_x(-\xi_1)B_z(\alpha_0-q_0)|\mathbf{0},m_0\rangle, \quad (3.34b)$$

$$|\mathbf{p}_1,m_1\rangle=R_z(-\mu_2)B_z(\alpha_1)|\mathbf{0},m_1\rangle. \quad (3.34c)$$

The usual technique therefore allows us to write

$$T_{\{m\}}=N_{\{m\}}{}^{(\pm)(1)}\exp[-i\mu_2(\mp m_a\pm m_0+m_1)], \quad (3.35)$$

where the functions $N_{\{m\}}{}^{(\pm)(1)}$ are b-analytic at, respectively, $D_3{}^{(\pm)}(Q_1;\ p_0,p_1)=0$ in the notation (3.28c). The singularity of the amplitudes is thus contained in the exponential factor, with μ_2 given by Eqs. (2.9) for $j=2$.

I. $\Delta_3(Q_{n+1},p_n,p_{n+1})=0$ Singularity

In a manner analogous to the treatment in Sec. III H one deduces the representation

$$T_{\{m\}}=N_{\{m\}}{}^{(\pm)(n+1)}\exp[\mp i\mu_{n+1}(m_b-m_{n+1})], \quad (3.36)$$

where the functions $N_{\{m\}}{}^{(\pm)(n+1)}$ are b-analytic at, respectively, $D_3{}^{(\pm)}(Q_{n+1};\ p_n,p_{n+1})=0$, and μ_{n+1} is given by Eqs. (2.9) for $j=n+1$.

IV. SUMMARY OF RESULTS

In summarizing the findings of the detailed investigations in Secs. III A through III I, it is convenient to appeal to the multiperipheral picture, Fig. 1. Indeed, our results show that the kinematic singularities of the BCP amplitudes, apart from the $\Delta_3(Q_1,p_0,p_1)=0$ singularity, are always associated with a particular *vertex* in the multiperipheral chain. Of course, this is just a reflection of the way the BCP amplitudes are defined. We now proceed to exhibit our results in the concepts and notations laid down in Fig. 1.

A. Leftmost $(p_a-p_0-Q_1)$ Vertex

There is a $t_1=0$ singularity only if the masses are equal, $M_a=M_0$. In that case, invoking a method applied several times in Refs. 1 and 2, one concludes from Eq. (3.11) that the amplitudes have the representation

$$T_{\{m\}}=\sum_{J=0}\sum_{p=0}\begin{pmatrix}\sigma_a & \sigma_0 & J\\ m_a & -m_0 & m\end{pmatrix}\frac{\exp[\frac{1}{2}i\pi(J+m)]}{[(J+m)!(J-m)!]^{1/2}}\times(-t_1)^{-J/2}[m(-t_1)^{1/2}]^p a_{J,\{m\}}{}^{(p)}, \quad (4.1)$$

where the standard notation for Wigner's 3-j symbols[13] is used, and where the functions $a_{J,\{m\}}{}^{(p)}$ have no kinematic singularities[14] at $t_1=0$ provided $M_a=M_0$; they are, moreover, independent of the magnetic quantum numbers m_a and m_0.

Note that the singularity structure (4.1) is very similar to that of the 2-to-2 particle helicity amplitudes as given in, e.g., Eq. (4.5) of Ref. 2. As there, suitable linear combinations of the amplitudes may be formed which have still simpler kinematic singularities; viz., they equal a power of $(-t_1)^{1/2}$ times a function kinematically regular at $t_1=0$.

At the incoming-state threshold $t_1=(M_a+M_0)^2$ for the t_1 channel, i.e., the reaction $a+\bar{0}\to 1+2+\cdots+(n+1)+\bar{b}$, one may, from Eq. (3.26a), deduce the representation

$$T_{\{m\}}=\sum_{J=0}\sum_{p=0}\begin{pmatrix}\sigma_a & \sigma_0 & J\\ m_a & m_0 & m\end{pmatrix}\frac{\exp[\frac{1}{2}i\pi(J-m)]}{[(J+m)!(J-m)!]^{1/2}}\times[-t_1+(M_a+M_0)^2]^{-J/2}\times\{m[-t_1+(M_a+M_0)^2]^{1/2}\}^p e_{J,\{m\}}{}^{(p)}, \quad (4.2)$$

with all the functions $e_{J,\{m\}}{}^{(p)}$ kinematically regular at $t_1=(M_a+M_0)^2$, and independent of m_a and m_0. The analogy to the 2-to-2 particle helicity amplitudes, Eq. (2.14) of Ref. 2, is evident.

Similarly, at $t_1=(M_a-M_0)^2$, and depending on whether $M_a>M_0$ or $M_a<M_0$, Eq. (3.26b) may be transformed into the representations

$$T_{\{m\}}=\sum_{J=0}\sum_{p=0}\begin{pmatrix}\sigma_a & \sigma_0 & J\\ m_a & -m_0 & m\end{pmatrix}\frac{\exp[\frac{1}{2}i\pi(J\mp m)]}{[(J+m)!(J-m)!]^{1/2}}\times[-t_1+(M_a-M_0)^2]^{-J/2}\times\{m[-t_1+(M_a-M_0)^2]^{1/2}\}^p f_{J,\{m\}}{}^{(p)(\pm)}, \quad (4.3)$$

where the functions $f_{J,\{m\}}{}^{(p)(\pm)}$, being independent of m_a and m_0, are without kinematic singularities at the t_1-channel pseudothreshold $t_1=(M_a-M_0)^2$, and the signs correlate to the mass inequality $M_a\gtrless M_0$. The 2-to-2 particle analog is now Eq. (2.21) of Ref. 2.

The "Toller-angle singularity" at $\Delta_3(Q_1,p_0,p_1)=0$ is exhibited in Eq. (3.35) and further commented upon in Sec. V.

B. Rightmost $(p_b-p_{n-1}-Q_{n+1})$ Vertex

By Eqs. (3.12) and (3.27), the singularities involving only the variable t_{n+1} are obtained from the results of Sec. IV A if the substitutions $a\to b$, $0\to(n+1)$, and $t_1\to t_{n+1}$ are made throughout.

The Toller-angle singularity at $\Delta_3(Q_{n+1},p_n,p_{n+1})=0$ is exhibited in Eq. (3.36).

[13] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957), pp. 46 ff.

[14] In these summary sections we use phrases like "have no kinematic singularity" to mean "be b-analytic" as given prior to Eq. (2.17), and in Ref. 1.

C. An Internal $(Q_j-p_j-Q_{j+1})$ Vertex for $j=1, 2, \ldots, n$

From Sec. III A, the BCP amplitudes have no kinematic singularities at $t_j=0$, with the exceptions already covered by Eq. (4.1) and its rightmost vertex analog.

Furthermore, the results of Sec. III D, in particular Eq. (3.20), imply

$$T_{\{m\}}=\frac{\exp[\frac{1}{2}i\pi(\sigma_j\mp m_j)]}{[(\sigma_j+m_j)!(\sigma_j-m_j)!]^{1/2}} \times\{t_{j+1}+[(-t_j)^{1/2}\pm iM_j]^2\}^{-\sigma_j/2} \times\sum_{p=0}\{m_j[t_{j+1}+((-t_j)^{1/2}\pm iM_j)^2]^{1/2}\}^p \times c_{\{m\}}{}^{(\pm)(j)(p)}, \quad (4.4)$$

where the functions $c_{\{m\}}{}^{(\pm)(j)(p)}$ are independent of m_j and have no kinematic singularities at the "threshold parabola"[7] $t_{j+1}=(t_j^{1/2}\pm M_j)^2$, respectively, for each $j=1, 2, \ldots, n$.

Finally, the Toller-angle singularity at

$$\Delta_3(Q_j,p_{j-1},p_j)=0$$

is exhibited in Eq. (3.33).

V. CONCLUDING REMARKS

We have in this paper investigated the kinematic singularities of the BCP amplitudes as functions of the invariant variables. In its choice of variables our approach generalizes the conventional lines followed for 2-to-2 particle reactions, where a knowledge of the kinematic singularities in terms of invariant variables[15] is essential in understanding the kinematic constraints that any model, in particular the Regge-pole model, must obey.

Toller and his collaborators[5–8,10] have taken another approach. They consider the amplitudes as functions of the momentum transfers squared and of certain Lorentz-group parameters which are similar, although not identical, to those specified by BCP[3] and CD,[9] and proceed to show that, with proper conventions, there are no kinematic singularities in these variables. Moreover, as a consequence of having "too many" variables in this group-theory approach, the amplitudes obey certain covariance conditions.

The amplitudes in the BCP and CD conventions, being a particular realization of the Toller amplitudes, are still not unambiguously defined, since they leave open the choice of the y axis in the definition of the rest system b_j for each of the reacting particles; it goes without saying that this is not a defect of their conventions. In the CD language, it means that the two z-rotation angles ν_j and μ_j are not uniquely defined. However, the "Toller angle" $\omega_j=\nu_j+\mu_{j+1}$ is unique once the other conventions have been accepted.[3]

In order to have a unique set of amplitudes in our approach, we have had to specify unambiguously the rest systems b_j, amounting to a more or less arbitrary definition of the angles ν_j and μ_j. However, from the fact that ω_j is independent of this definition it follows that the only place where these conventions are of any importance is in the $\Delta_3=0$ singularities exhibited in Eqs. (3.34), (3.36), and (3.37). Namely, independent of the choice of y axes, if there is a kinematic singularity at a surface $\Delta_3=0$, it will occur in (a product of) factors $e^{im_j\psi}$, where the sine and cosine of the angle ψ are proportional to $(\Delta_3)^{-1/2}$; we have not specified the arguments in the Δ_3 Gram determinant here, since they may depend on the conventions. Moreover, the *a priori* possibility exists that some convention could be found for which there are no $\Delta_3=0$ singularities at all. We have not been able, however, to find such a convention, at least not without introducing other singularities.

Apart from these circumstances, related to the choice of z-rotation angles, the kinematic singularities of the BCP amplitudes involve only the momentum transfers squared in a way which seems useful for a subsequent multi-Regge-pole analysis incorporating problems arising from spin.

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APPENDIX: NOTATION FOR DETERMINANTS

For convenience we repeat here the determinantal notation of Ref. 1.

The Gram determinant between two sets of n four-vectors $\{q_j\}$ and $\{r_j\}$ is denoted

$$\det(q_jr_k)\equiv\begin{bmatrix} q_1 & q_2 & \cdots & q_n \\ r_1 & r_2 & \cdots & r_n \end{bmatrix}, \quad (A1)$$

and the symmetric ones

$$\Delta_n(q_1,q_2,\ldots,q_n)=(-)^{n+1}\begin{bmatrix} q_1 & q_2 & \cdots & q_n \\ q_1 & q_2 & \cdots & q_n \end{bmatrix}. \quad (A2)$$

Finally,

$$\epsilon(q_1,q_2,q_3,q_4)=\epsilon_{\kappa\lambda\mu\nu}q_1{}^\kappa q_2{}^\lambda q_3{}^\mu q_4{}^\nu, \quad (A3)$$

where $\epsilon_{\kappa\lambda\mu\nu}$ is the completely antisymmetric isotropic tensor with $\epsilon_{0123}=1$.

[15] See, e.g., Ref. 1 for a list of references.