Quasi-Unitary Three-Particle Approximations*

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The three-particle discontinuity equations are expressed in forms convenient for the study of unitarity violations. As an application, the constraints on the two-particle transition operators imposed by requiring consistency with three-particle unitarity are deduced. This provides a framework for some recent proposals for generating approximate, finite-rank, two-particle transition operators which do not satisfy off-shell unitarity. Also, the general features of some quasi-unitary impulse approximations are discussed in a unified fashion in order to clarify the conditions for their validity as well as for higher-order quasi-unitary approximations.

1. INTRODUCTION

It has been known for a long time within the usual off-shell formulations of three-particle scattering that two-particle (off-shell) unitarity implies three-particle unitarity. However, in view of some recent proposals[1] for approximating the two-particle dynamics by transition operators which violate off-shell unitarity, it is perhaps relevant to inquire as to what constraints are imposed on the two-particle operators by demanding a certain minimal consistency with three-particle unitarity. Our first objective will be to determine these constraints and thus provide general criteria for determining such transition operators which automatically generate quasi-unitary three-particle theories.

Another sort of quasi-unitary approximation has been proposed by Sloan[2] and Finkel and Rosenberg[3] which involves an intrinsic mulitation of the three-particle rather than the two-particle dynamics. This scheme may prove to be of considerable practical importance for performing those calculations for which the customary numerical methods appear to be ill suited.[4–6] Some general features of this technique will be discussed as an application of the analysis of the three-particle discontinuity equations given in the first Secs. 2 and 3 of this paper. This will constitute, first of all, the embedding of the somewhat different realizations of this scheme obtained in Refs. 2 and 3 within a single unified formalism in order to explicate the approximations which are actually being made as well as the alterations in the singularity structure which are thereby induced. Secondly, this will clarify the conditions for higher-order approximations which are also quasi-unitary.

2. THREE-PARTICLE DISCONTINUITY RELATIONS

We will employ throughout this paper the elegant formulation of the three-particle scattering problem found by Alt *et al.*[7–10] in which the scattering operators $U(z)$ satisfy

$$\begin{aligned} U(z) &= \bar{U}(z) + \bar{U}(z)\kappa^s(z)U(z) \\ &= \bar{U}(z) + U(z)\kappa^s(z)\bar{U}(z)\,, \end{aligned} \tag{2.1}$$

where the effective potential[11] $\bar{U}(z)$ is defined as the solution of

$$\begin{aligned} \bar{U}(z) &= \zeta(z) + \zeta(z)\bar{\kappa}(z)\bar{U}(z) \\ &= \zeta(z) + \bar{U}(z)\bar{\kappa}(z)\zeta(z)\,. \end{aligned} \tag{2.2}$$

The two-particle transition operator $t(z)$ has been decomposed into the arbitrary sum[12]

$$t = t^s + \bar{t}\,, \tag{2.3}$$

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[1] A. E. A. Warburton and M. S Stern, Nuovo Cimento **60A**, 131 (1969); W. Bierter and K. Dietrich, Z. Physik. **202** 74 (1967); Nuovo Cimento **52A**, 1209 (1967); D. D. Brayshaw, Phys. Rev. **182**, 1658 (1969).

[2] I. H. Sloan, Phys. Rev. **165**, 1587 (1968).

[3] R. W. Finkel and L. Rosenberg, Phys. Rev. **168**, 1841 (1968).

[4] For nucleon-deuteron scattering, for example, we have in mind scattering at moderate energies (say, above 10 MeV) including spin effects. The essential point is that the final, Heitler-type equations obtained in Refs. 2 and 3 can, conceivably, be solved without a partial-wave decomposition; they are two-dimensional integral equations but with (angular) variables with finite domains. Of course, one still has the old and difficult problem of determining the average two-body amplitudes (see Ref. 5). Nonetheless, the now customary procedures of integrating exactly various one-dimensional versions of the Faddeev equations in given states of total angular momentum appear hopelessly unwieldly when confronted with the physical problem just mentioned. Even at very low energies these latter techniques have difficulty reproducing the forward diffraction peaks (see Ref. 6 for a review of these calculations). The origin of this circumstance, for the *N-d* case, seems to be that the so-called exact integration procedures do not appear to exploit in a very efficient manner the special physical simplicities which follow as a consequence of the weak deuteron binding.

[5] K. L. Kowalski and D. Feldman, Phys. Rev. **130**, 276 (1963); H. Kottler and K. L. Kowalski, *ibid.* **138**, B619 (1965); M. L'Huillier, Nucl. Phys. **A122**, 667 (1968).

[6] I. Duck, *Advances in Nuclear Physics*, edited by M. Baranger and E. Vogt (Plenum Press, Inc., New York, 1968), Vol. I, p. 343.

[7] E O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 181 (1967).

[8] See also Refs. 2 and 9.

[9] L. Rosenberg, Phys. Rev. **168**, 1756 (1968).

[10] Our rephrasing of the work of Ref. 7 is sufficiently different to require some clarification. A very brief review of this is given in the Appendix.

[11] This terminology is due to L. Rosenberg, Ref. 9.

[12] The unique practicality of the formalism arises, however, when t^s is chosen to be a finite-rank operator on the appropriate two-particle spaces. We recall that t and t^s are operators in the three-particle Hilbert space.

and

$$\kappa^s(z)=G_0(z)t^s(z)G_0(z)\,,$$
$$\bar{\kappa}(z)=G_0(z)\bar{t}(z)G_0(z)\,,$$
$$\zeta(z)=\bar{\delta}G_0(z)^{-1}\,,$$

where

$$G_0(z)=(z-H_0)^{-1}$$

is the free-particle Green's function and z denotes the (complex) parametric energy.

In writing the preceding equations we have employed the usual matrix notation with respect to the channel indices $\alpha=0,1,2,3$, where 0 refers to the three-free-particles channel and $\alpha=2$, for example, refers to the channel in which particle 2 is asymptotically free. That is, $U(z)$ represents the 4×4 matrix whose elements are the operators $U_{\beta\alpha}(z)$, $t(z)$ is a diagonal matrix with elements $t_\alpha(z)$, $t_0\equiv0$, and $\bar{\delta}$ is the matrix with elements $1-\delta_{\beta\alpha}$.

Let us employ the notation

$$\Delta U\equiv U(+)-U(-)$$

for the discontinuity of the quantity across the unitary cut in the z plane, where

$$U(\pm)=[U(z)]_{z=E\pm i\epsilon}\,,$$

with E real and $\epsilon>0$. The technique of Freedman *et al.*[13] when applied to Eqs. (2.1) and (2.2) yields the following discontinuity relations for the scattering operators and the effective potential:

$$\Delta U=U(\pm)\Delta\kappa^s U(\mp)+[1+U(\pm)\kappa^s(\pm)]\Delta\bar{U}[1+\kappa^s(\mp)U(\mp)]\,,\quad(2.4)$$

$$\Delta\bar{U}=\bar{U}(\pm)\Delta\bar{\kappa}\bar{U}(\mp)+[1+\bar{U}(\pm)\bar{\kappa}(\pm)]\Delta\zeta[1+\bar{\kappa}(\mp)\bar{U}(\mp)]\,.\quad(2.5)$$

Since (2.4) and (2.5) hold for any decomposition (2.3), and since U is necessarily independent (in contrast to $\bar{U}$) of how t is split up, we find, in general, that

$$\Delta U=U(\pm)\Delta\kappa U(\mp)+[1+U(\pm)\kappa(\pm)]\Delta\zeta[1+\kappa(\mp)U(\mp)]\,,\quad(2.6)$$

where

$$\kappa(z)=G_0(z)t(z)G_0(z)\,.$$

It is useful to note that

$$U(\pm)\Delta\kappa U(\mp)=[U(\pm)G_0(\pm)]\Delta t[G_0(\mp)U(\mp)]-[U(\pm)\kappa(\pm)]\Delta\zeta[1+\kappa(\mp)U(\mp)]-[1+U(\pm)\kappa(\pm)]\Delta\zeta[\kappa(\mp)U(\mp)]\,.\quad(2.7)$$

Thus, (2.6) can be rewritten as

$$\Delta U=[U(\pm)G_0(\pm)]\Delta t[G_0(\mp)U(\mp)]+\Delta\zeta-[U(\pm)\kappa(\pm)]\Delta\zeta[\kappa(\mp)U(\mp)]\,.\quad(2.8)$$

Since an identity similar to (2.7) also obtains for $\bar{U}(\pm)\times\Delta\bar{\kappa}\bar{U}(\pm)$, Eq. (2.5) becomes

$$\Delta\bar{U}=[\bar{U}(\pm)G_0(\pm)]\Delta\bar{t}[G_0(\mp)\bar{U}(\mp)]+\Delta\zeta-[\bar{U}(\pm)\bar{\kappa}(\pm)]\Delta\zeta[\bar{\kappa}(\mp)\bar{U}(\mp)]\,.\quad(2.9)$$

Thus far we have not specified anything concerning the two-particle transition operator $t(z)$. Normally, $t(z)$ satisfies the reflection property

$$t(z)^\dagger=t(z)^*\,,\quad(2.10)$$

which, in turn, along with Eqs. (2.1), implies that[14]

$$U(z)^\dagger=U(z^*)\,.\quad(2.11)$$

In addition, $t(z)$ satisfies the two-particle off-shell unitarity relation

$$\Delta t=t(\pm)gt(\mp)+\Delta t_b\,,\qquad g=-2\pi i\delta(E-H_0)\,,\quad(2.12)$$

where Δt_b represents the contribution to Δt arising solely from the two-particle bound-state poles. Both these properties (2.10) and (2.12) are essential in obtaining the usual statement of physical three-particle unitarity from the preceding discontinuity relations. We will reserve the demonstration of this (well-known) result until Sec. 3.

Similar discontinuity relations can be easily formed using the same method,[13] for the operators

$$\begin{aligned}F(z)&\equiv G_0(z)U(z)G_0(z)\\&=\bar{F}(z)+\bar{F}(z)t^s(z)F(z)\\&=\bar{F}(z)+F(z)t^s(z)\bar{F}(z)\,,\end{aligned}\quad(2.13)$$

$$\begin{aligned}\bar{F}(z)&\equiv G_0(z)\bar{U}(z)G_0(z)\\&=\bar{\delta}G_0(z)+\bar{\delta}G_0(z)\bar{t}(z)\bar{F}(z)\\&=\bar{\delta}G_0(z)+\bar{F}(z)\bar{t}(z)G_0(z)\bar{\delta}\,,\end{aligned}\quad(2.14)$$

which are used along with the supposition of a finite-rank form[12] for the components of $t^s(z)$ to derive quasi-two-particle scattering integral equations.[7] These expressions for ΔF and $\Delta\bar{F}$ are not of any direct interest except in the special case in which the "vertex states" appearing in the assumed t^s are independent of z. In the general circumstance where this is not the case, it is better simply to derive discontinuity relations directly for the quasi-two-particle amplitudes from their defining integral equations as is done in Ref. 13. In general, however, these amplitude discontinuity equations are not very informative, because the connection between the physical and quasi-two-particle amplitudes is usually rather remote. Fortunately, we will never have any need to consider such equations.

3. APPROXIMATE TWO-PARTICLE UNITARITY

Instead of supposing that $t(z)$ satisfies the two-particle off-shell relation (2.12), let us assume that

$$\Delta t=\Delta t_c+\Delta t_b\,,\quad(3.1)$$

[13] D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento 43A, 258 (1966).

[14] Since we are using a matrix notation, the adjoint operation includes a matrix transposition with respect to the channel indices. For the individual components $U_{\beta\alpha}(z)$ of these matrices, (2.11) implies that $U_{\beta\alpha}(z)^\dagger=U_{\alpha\beta}(z^*)$.

where, roughly speaking, Δt_c is the continuum contribution to Δt, and Δt_b has the same interpretation as (but need not be identical to) the corresponding quantity in Eq. (2.12).

Before we employ (3.1) in the three-particle discontinuity equations some clarifying remarks are in order. We are assuming that in place of a $t(z)$ which is connected to the usual manner to the interparticle potentials, we are inserting into Eqs. (2.1) and (2.2) an operator for which this connection no longer holds. Obviously, this cannot be done in an arbitrary fashion without completely losing the physical interpretation of these equations. Therefore, it is necessary to confine ourselves to a suitably restricted class of t's.[15]

In writing (3.1) we have implicitly supposed that $t(z)$ has essentially the same behavior with respect to the parameter z as does a normal two-particle transition operator. Specifically, we assume, first, the reflection property (2.10), which along with Eqs. (2.1) implies (2.11). The remaining properties are most conveniently phrased in terms of the operator $\hat{t}_\alpha(\hat{z})$, corresponding to $t_\alpha(z)$ on the appropriate relative two-particle subspace. Namely, we suppose $\hat{t}_\alpha(\hat{z})$ is discontinuous across the real $\hat{z}$ line from 0 to $+\infty$ which implies that

$$\Delta t_c = 0$$

for z real and below the three-particle threshold; finally, we assume that $\hat{t}_\alpha(\hat{z})$ has at most pole singularities when $\hat{z}<0$ (with the usual factorizable residues) which give rise to the discontinuity Δt_b in (3.1).

In practice, an approximation to some presumably "exact" transition operator will not yield the "correct" pole positions or residues. Corresponding to this, $\Delta\hat{t}_b$ in (3.1) is not supposed to represent, necessarily, the same discontinuity as that possessed by this "ideal" t. In point of fact, the approximation constitutes the introduction of a new model, namely, the specification of new ("approximate") channel states. The retention of the factorizability of the residues at the poles is quite essential if the three-particle discontinuity equations are to have any resemblance to the physical unitarity relations.

Now inserting (3.1) into (2.8), we find that

$$\Delta U = [U(\pm)G_0(\pm)]\Delta t_b[G_0(\mp)U(\mp)] + \Gamma + \gamma + \Delta\zeta, \quad (3.2)$$

where

$$\Gamma \equiv [U(\pm)G_0(\pm)t(\pm)](1+\bar{\delta})g \times [t(\mp)G_0(\mp)U(\mp)], \quad (3.3)$$

$$\gamma \equiv [U(\pm)G_0(\pm)][\Delta t_c - t(\pm)g(\mp)] \times [G_0(\mp)U(\mp)]. \quad (3.4)$$

[15] No artificial degree of generality is being sought here. We assume, for example, that t_α has all the usual symmetry properties expected of it, particularly invariance with respect to spatial translations of the system as a whole and of the αth particle as well as time-reversal invariance. The only changes of interest are evidently those involving its behavior as a function of z.

If two-particle off-shell unitarity [Eq. (2.12)] were satisfied, γ would vanish identically. We will next demonstrate that (3.2) without γ gives rise to the ordinary unitarity conditions so that γ is the entire measure of the unitarity violation arising from a nonunitary $t(z)$. It is clear that γ will vanish below the three-particle threshold.

Next let us note that a typical element $\Gamma_{\beta\alpha}$ of Γ can be written as

$$\begin{aligned}\Gamma_{\beta\alpha} &= [\sum_\lambda U_{\beta\lambda}(\pm)G_0(\pm)t_\lambda(\pm)] \times g[\sum_\gamma t_\gamma(\mp)G_0(\mp)U_{\gamma\alpha}(\mp)] \\ &= U_{\beta 0}(\pm)gU_{0\alpha}(\mp) - \bar{\delta}_{\beta 0}G_0^{-1}(\pm)gU_{0\alpha}(\mp) \\ &\quad - U_{\beta 0}(\pm)gG_0^{-1}(\mp)\bar{\delta}_{0\alpha} + \bar{\delta}_{\beta 0}\bar{\delta}_{0\alpha}G_0^{-1}(\pm)gG_0^{-1}(\mp).\end{aligned} \quad (3.5)$$

Also for $\alpha \neq 0$,

$$G_0(\pm)[\Delta t_b]_\alpha G_0(\mp) = -2iD_\alpha, \quad (3.6)$$

where

$$D_\alpha \equiv \pi\sum |\phi_\alpha(E_\alpha)\rangle\delta(E-E_\alpha)\langle\phi_\alpha(E_\alpha)|,$$

for all α, and $|\phi_\alpha(E_\alpha)\rangle$ denotes the α-channel state. Then referring back to (3.2), we recover the usual form[7] of the discontinuity relation for U:

$$\Delta U = -2iU(\pm)DU(\mp). \quad (3.7)$$

In (3.7) we have omitted, in addition to γ, the G_0^{-1}-dependent terms which appear in (3.2) and (3.5) with the understanding that only matrix elements with respect to ϵ-independent vectors are to be considered. The rather trivial issue here is that for the off-shell extensions actually employed in the work of Alt *et al.*[7] [cf. Eqs. (2.13) and (2.14)] this is not the case, and Eq. (3.7), which was derived by these authors in a different manner, is quite incorrect within the context of their paper.[16] Of course, these extra terms cannot contribute when one is interested in the on-shell physical statement of unitarity which follows from (3.7) with the aid of (2.11) and the original physical significance assigned to U.[7]

Let us define the imaginary part $\mathrm{Im}[\mathcal{O}]$ of an operator as

$$\mathrm{Im}[\mathcal{O}] \equiv (2i)^{-1}[\mathcal{O} - \mathcal{O}^\dagger].$$

Since by (2.11) we have

$$\mathrm{Im}[U(+)] = (2i)^{-1}\Delta U,$$

(3.7) can be rewritten as

$$-\mathrm{Im}[U(+)] = U(\pm)DU(\pm)^\dagger, \quad (3.8)$$

which explicitly demonstrates, upon recalling the definition of D, that $-\mathrm{Im}[U(+)]$ is a positive-semidefinite operator in the case when three-particle unitarity ob-

[16] To put this in another way, one cannot derive the proper discontinuity relations for F from (3.7).

tains and we denote this formally as[17]

$$-\mathrm{Im}[U(+)]\geqslant 0. \qquad (3.9)$$

If we decompose the matrix D into a part D_b containing only the two-particle bound-state contributions and D_c which is related to ΔG_0,[18]

$$D=D_b+D_c,$$

we see from (3.7) that

$$-\{\mathrm{Im}[U(+)]+U(\pm)D_bU(\pm)^\dagger\}=U(\pm)D_cU(\pm)^\dagger,$$

which implies that

$$-\{\mathrm{Im}[U(+)]+U(\pm)D_bU(\pm)^\dagger\}\geqslant 0. \qquad (3.10)$$

The on-shell version of the positivity condition (3.10), confined to the α, $\beta\neq 0$ submatrix, is the usual form of the minimal constraint imposed by unitarity in the case where there are two-particle bound states.[2,3] In this context, namely, as applied to two-particle multichannel amplitudes, this condition is very well known and will be exploited as such in Sec. 4.

It will be necessary to go beyond the preceding type of constraint in order to include three-particle problems for which there exist no bound states in the two-particle subsystems. In such a case the $U_{\beta\alpha}$ for β, $\alpha\neq 0$ have significance only as auxiliary, but, nonetheless, exceedingly useful mathematical entities somewhat akin in their role to the two-particle transition operators. This example emphasizes our intent to regard (3.10) as a (generally) off-shell condition on the $U_{\beta\alpha}$ for all values of α and β and under any physical circumstances. In the case without two-particle bound states, (3.10) reduces to (3.9), which is certainly the minimal constraint one could impose upon the $U_{\beta\alpha}$; if there are two-particle bound states, (3.10) includes all the usual unitary constraints.

Let us explore the consequences of (3.10) in the case where we have a unitarity violation as a result of a non-off-shell unitary $t(z)$. First of all, the essence of our assumptions concerning the two-particle bound-state poles is that (3.6) still obtains, with, of course, some appropriately modified channel states. Thus ignoring the $\Delta\zeta$ term, (3.2) is

$$\mathrm{Im}[U(+)]=-[U(\pm)G_0(\pm)]D_b[G_0(\mp)U(\mp)] + (2i)^{-1}(\Gamma+\gamma),$$

and so the unitarity constrant (3.10) reduces to

$$-(2i)^{-1}(\Gamma+\gamma)\geqslant 0, \qquad (3.11)$$

and in this connection we note from the explicit form (3.5) that

$$-(2i)^{-1}\Gamma\geqslant 0. \qquad (3.12)$$

[17] This has strict meaning, of course, only where matrix elements with respect to normalizable states are understood.

[18] That is, D_b, for example, is equal to the diagonal matrix D with the $\alpha=0$ element set equal to zero.

If we refer to the definitions (3.3) and (3.4), we see that (3.11) is equivalent to the condition

$$-\mathrm{Im}t_c+t(\pm)\bar{\delta}\bar{D}_0t(\mp)\geqslant 0, \qquad (3.13)$$

where we have written

$$\mathrm{Im}t_c=(2i)^{-1}\Delta t_c, \quad \bar{D}_0=-(2i)^{-1}g.$$

The positivity condition (3.13) is our general constraint on the two-particle transition operators. However, the only way we have found of satisfying this condition which is consistent with the independence of the t_α for different α is

$$-[\mathrm{Im}t_c+t(\pm)\bar{D}_0t(\mp)]\geqslant 0, \qquad (3.14)$$

which is equivalent to satisfying (3.11) by having

$$-(2i)^{-1}\gamma\geqslant 0. \qquad (3.15)$$

In partial-wave form (3.14) becomes, on the relative two-particle subspace,

$$\mathrm{Im}\hat{t}_l(p,p;k^2)\leqslant -\tfrac{1}{2}\pi k|\hat{t}_l(p,k;k^2)|^2, \qquad (3.16)$$

where $k^2\geqslant 0$, $p\geqslant 0$, and where the equality would hold if the amplitude were off-shell unitary.[19]

The mode (3.15) of satisfying (3.11) can be justified in another way by a (rough) symmetry argument. The point is that if (3.15) holds, then in addition to (3.10) we also have

$$-\{\mathrm{Im}[U(+)]+U(\pm)D_cU(\pm)^\dagger\}\geqslant 0,$$

which is a constraint just as legitimate as (3.10), except perhaps in the absence of two-particle bound states.

The preceding arguments are general and quite independent of any subsequent manipulations which are always needed to obtain practical integral equations provided, of course, no new approximations are introduced in the process.[20] In particular, our conclusions are independent of the decomposition (2.3) that one might eventually choose in exploiting the effective-potential formalism.

One final point should be made concerning the disconnected structure in the discontinuity equation for U_{00} in the case when two-particle off-shell unitarity is not satisfied. It follows from (3.2) that all the disconnected parts will all cancel out quite independently of the validity of off-shell unitarity, and this equation will reduce to a discontinuity equation for the connected part of U_{00}.

4. QUASI-UNITARY IMPULSE APPROXIMATION

As an application of some of the work of Secs. 2 and 3, we will discuss the quasi-unitary approximations of Sloan[2] and of Finkel and Rosenberg.[3] Although the final

[19] The equality in (3.16) of course, would, still not imply the amplitude was off-shell unitary.

[20] Obviously, truncations in total angular momentum or in the number of partial waves contributing to the two-particle amplitudes are approximations which do not alter our conclusions.

integral equations used by these authors for practical calculations are nearly identical, the formalisms from which they are derived are quite distinct. Our primary objective is to clarify the differences between these two approaches within the unified context of the formalism of Alt *et al.*[21] In the course of this study we will establish some apparently previously unnoticed features of the Sloan procedure which renders it especially appealing, both physically and mathematically, particularly with regard to the question of higher-order corrections.

The fundamental difference between the two methods resides in the choice of $t^s(z)$ in the decomposition (2.3). Sloan chooses[22]

$$t_\alpha{}^s(z)=\tfrac{1}{2}V_\alpha P_\alpha[G_\alpha(z)-G_\alpha(z)^\dagger]P_\alpha V_\alpha\,,\quad \alpha\neq 0 \qquad (4.1)$$

where P_α is the projection onto the α-channel subspace and

$$G_\alpha(z)=(z-H_0-V_\alpha)^{-1}.$$

Thus

$$\Delta t^s=\Delta t_b\,, \qquad (4.2)$$

namely, $t^s(z)$ contains all the discontinuities of the two-particle bound-state poles. The nontrivial consequence of the choice (4.1) is that $\bar{t}(z)$ then satisfies a legitimate off-shell unitarity relation[23]

$$\Delta\bar{t}=\bar{t}(\pm)g\bar{t}(\mp)\,. \qquad (4.3)$$

The proof of (4.3) follows from the elementary fact that since

$$(E-H_0)|\phi_\alpha(E)\rangle=V_\alpha|\phi_\alpha(E)\rangle\,,$$

we have

$$\delta(E-H_0)V_\alpha|\phi_\alpha(E)\rangle=0\,,$$

so that with the choice (4.1),

$$t^s(\pm)gt^s(\pm)=0\,.$$

Since (4.1) also implies that

$$\kappa^s(\pm)=\mp iD_b\,,$$

the scattering integral equations (2.1) become

$$\begin{aligned}U&=\bar{U}-i\bar{U}D_bU\\&=\bar{U}-iUD_b\bar{U}\,.\end{aligned} \qquad (4.4)$$

The $\beta,\alpha\neq 0$ submatrix of Eqs. (4.4) is easily transformed into a Heitler-type integral equation for the physical on-shell amplitudes $\langle\phi_\beta(E)|U_{\beta\alpha}(E)|\phi_\alpha(E)\rangle$.[24] The source terms in these equations are evidently the matrix elements $\langle\phi_\beta(E)|\bar{U}_{\beta\alpha}(E)|\phi_\alpha(E)\rangle$.

We need be concerned only with the on-shell discontinuity equation for U which is

$$\begin{aligned}-\mathrm{Im}[U(+)]&=U(\pm)D_bU(\mp)+[1\mp iU(\pm)D_b]\\&\quad\times[-\mathrm{Im}\bar{U}(+)][1\pm iD_bU(\mp)]\,.\end{aligned} \qquad (4.5)$$

The unitary constraint (3.10) will be satisfied if $\bar{U}$ is such that the last term in (4.5) is positive-semidefinite and this is certainly the case for the exact $\bar{U}$, as follows from

$$\begin{aligned}-\mathrm{Im}[\bar{U}(+)]&=\bar{U}(\pm)G_0(\pm)\bar{t}(\pm)(1+\bar{\delta})\bar{D}_0\bar{t}(\mp)\\&\quad\times G_0(\mp)\bar{t}(\mp)\bar{U}(\mp)-(2i)^{-1}\Delta\zeta\,.\end{aligned}$$

The unique feature of the Sloan formalism should now be apparent. It defines a clean division of the problem of scattering from a bound target into the solution of a scattering problem for a three-particle system,[25] for which there exists no two-particle bound states, and the solution of a relatively trivial Heitler-type equation which properly accounts for the bound-state scattering portion of the unitary cut.

In this context the actual form of the impulse approximation proposed in Ref. 2, namely,

$$\bar{U}(z)\simeq\zeta(z)+\bar{\delta}\bar{t}(z)\bar{\delta}\,, \qquad (4.6)$$

which is the inhomogeneous term of the once-iterated equation for $\bar{U}$ and for which

$$-\mathrm{Im}\bar{U}(+)\geqslant 0$$

in (4.5), has a simple interpretation. Clearly, (4.6) amounts to representing the 3-3 transition amplitude of the subsidiary problem generated by $\bar{t}(z)$ by only its disconnected parts. Consequently, the rescattering pole is the principal feature which is omitted from the singularity structure for both this auxiliary problem and the actual three-particle scattering. The kinematical circumstances for which this produces a negligible effect on the scattering from a bound target may be regarded as one of the conditions for the validity of the impulse approximation.

Finkel and Rosenberg choose, instead of (4.1),

$$t_\alpha{}^s(z)=V_\alpha P_\alpha G_\alpha(z)P_\alpha V_\alpha\,, \qquad (4.7)$$

which also satisfies (4.2). Equation (4.7) corresponds to the identification of $t^s(z)$ with the complete bound-state pole contribution to $t(z)$. Thus, for the choice (4.7),

$$\Delta\bar{t}=t(\pm)gt(\mp)\,.$$

Thus, the problem represented by $\bar{U}(z)$ [or $\bar{F}(z)$] in this case falls into the category studied at length in Sec. 3.[26]

[21] The Finkel-Rosenberg formalism is already of the Alt *et al.* type, although it is not phrased in terms of the same notation.

[22] We confine ourselves to three-particle systems for which there exists at least one bound state in one of the two-particle subsystems.

[23] This point apparently was not noticed in Ref. 2.

[24] In the event that not all subsystems contain bound states, the order of the relevant matrix equations reduces accordingly.

[25] This differs from a true physical problem only in the sense that some of the usual analyticity properties with respect to z are necessarily lost. This is a consequence of the fact that in $\bar{t}(z)$ one has a bound-state contribution in the nonanalytic form $G_\alpha(z)+G_\alpha(z^*)$.

[26] Actually, Finkel and Rosenberg use $F(z)$ and $\bar{F}(z)$ in the terminology of the present paper. With their choice [(4.6)] of $t^s(z)$ it is evident from (2.13) that they are involved with off-shell quasi-two-particle equations. With the Sloan's choice [(4.1)], on the other hand, the use of (2.13) and (2.14) is entirely equivalent to the direct employment of $U(z)$ and $\bar{U}(z)$.

Namely, one has a two-particle transition operator $\bar{t}(z)$ which is *not* off-shell unitary. This complicates considerably the discussion of questions of consistency with unitarity as well as the question (at least in principle) of higher-order corrections to an approximation like (4.6). This is in marked contrast to the model of Ref. 2.

Note added in proof. In connection with Ref. 4 it should be mentioned that the method of Y. Avishai, W. Ebenhöh, and A. S. Rinat-Reiner [Phys. Letters **29B**, 638 (1969); Ann. Phys. (N.Y.) **55**, 341 (1969)] appears to be equally practical. Also, explicit calculations of the type outlined in Ref. 4 have now been completed [J. Krauss and K. L. Kowalski, Phys. Letters (to be published)] and have indicated that the solution of the modified Heitler equations using a partial-wave decomposition is by far the most efficient technique, at least for relatively low energies (11–40 MeV). We would like to thank Professor I. Sloan for pointing out a serious ambiguity in the original version of this paper. Namely, if in Eq. (2.12) one were to employ ΔG_0 rather than g, a consistent interpretation of the $\epsilon\to 0$ limit would lead to a double counting of Δt_b (see Ref. 2). On the other hand, $t(\pm)\bar{\delta}\Delta G_0 t(\mp)=t(\pm)\;gt(\mp)$.

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APPENDIX

We being from the proof by Alt *et al.*[7] that the operator

$$U_{\beta\alpha}(z)=\bar{\delta}_{\beta\alpha}G_0^{-1}(z)+V-V_\alpha-V_\beta+\delta_{\beta\alpha}V_\alpha+V^\beta G(z)V^\alpha\,, \quad \text{(A1)}$$

$\alpha,\beta=0,1,2,3$, whose matrix elements with respect to the appropriate channel states are equal to the physical scattering amplitudes, satisfies the (matrix) integral equations

$$U(z)=\zeta(z)+\zeta(z)\kappa(z)U(z) \quad \text{(A2a)}$$

$$=\zeta(z)+U(z)\kappa(z)\zeta(z)\,. \quad \text{(A2b)}$$

In Eq. (A1), V_α is the interaction between particles $\beta,\gamma\neq\alpha$, $V_0=0$,

$$V^\beta=\sum_\gamma \bar{\delta}_{\beta\gamma}V_\gamma\,,$$

$$V=\sum_\gamma V_\gamma\,,$$

and

$$G(z)=(z-H_0-V)^{-1}\,,$$

where H_0 is the kinetic-energy operator. As is customary,[27] it is formally convenient to include in our matrix notation quantities with indices referring to the three-free-particles channel $\alpha=0$, although the $U_{\beta\alpha}$ operators, for instance, for these index values are really entirely determined by the $\beta,\alpha\neq 0$ operators. With the decomposition (2.3), (A2a), for example, can be rewritten as

$$U(z)=\zeta(z)[1+\kappa^s(z)U(z)]+\zeta(z)\bar{\kappa}(z)U(z)\,.$$

Comparing this with Eq. (2.2), we immediately infer Eqs. (2.1).

[27] C. Lovelace, Phys. Rev. **135**, B1225 (1964).