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Noncausality and Other Defects of Interaction Lagrangians for Particles with Spin One and Higher*

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We analyze critically what happens when various interaction terms are added to a Lagrangian describing a free particle with spin $\geqslant 1$. Good behavior results when the charged spin-one particle is coupled minimally to an external electromagnetic field or via a magnetic dipole moment. However, with an arbitrary electric quadrupole moment, the spin-one particle propagates noncausally ($v>c$) in an electrostatic field. Noncausal behavior is also found for the neutral vector field with self-coupling $\lambda(W_\mu W^\mu)^2$. Another kind of disease appears when the spin-two particle is given a charge: A constraint is converted into an equation of motion, so that there are six degrees of freedom instead of the desired five.

1. INTRODUCTION

During the last thirty years, it has become a popular technique among theoretical physicists to construct Lagrangians for free higher-spin particles,[1] which yield both the equations of motion and the constraints. The method of higher-spin Lagrangians was originated by Fierz and Pauli[2] to avoid the immediate algebraic inconsistencies that arise, in the presence of interactions, when the constraints are postulated independently of the equations of motion.

However, the result of the analysis presented here, which deals with several examples of interacting higher-spin equations ($s\geqslant 1$), is that the Lagrangian device by itself does not automatically provide satisfactory wave equations. The bad features displayed by these equations include noncausal propagation ($v>c$), loss of constraints, or failure to propagate.[3] Most of the cases concern the four-vector field representing a spin-one particle.[4] It exhibits good behavior in an external electromagnetic field when coupled minimally and with anomalous magnetic dipole moment. But with an arbitrary anomalous electric quadrupole moment, it propagates noncausally in an electrostatic field and is nonpropagating (exponential instead of oscillatory behavior) in a sufficiently strong magnetostatic field. Noncausal behavior is again found when the spin-one particle is coupled to an external symmetric tensor field and when the neutral vector field has the self-interaction $\lambda(W_\mu W^\mu)^2$. Because the noncausality of the minimally coupled Rarita-Schwinger equation for spin $\frac{3}{2}$ has been discussed earlier,[5] we turn, for our last example, to the minimally coupled spin-two equation.[6] It presents the new feature that, in the region of nonvanishing external field, two of the constraints become equations of motion.[7]

To understand the origin of the difficulties, let us briefly review how wave fields may be described mathematically. Wave propagation is usually associated with hyperbolic systems of partial differential equations.[8] Such equations allow an initial value problem to be posed on a class of surfaces, called "spacelike" with respect to the equations, and they possess solutions with wave fronts that travel along rays at finite velocities. The rays through any point form a ray cone that is entirely determined by the coefficients of the highest derivatives. Thus, for hyperbolic systems, when coupling occurs only in lower derivatives, the ray cone is the same in the interacting and free case. The free Klein-Gordon and Dirac equations are familiar examples of hyperbolic systems, and so, when they are coupled through lower-order derivatives, the ray cone remains the light cone.

On the other hand, for spin greater than one-half, the free Lagrangian equations are not hyperbolic, but constitute instead a degenerate system because they imply constraints. However, it may be shown that they are equivalent to a system of hyperbolic equations, which describe the wave propagation, supplemented by constraints that are conserved in time. But it is not true that, if any low or nonderivative coupling term is added to the free higher-spin Lagrangian, the resulting Lagrangian equations remain equivalent to a hyperbolic

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[1] See, e.g., S. J. Chang, Phys. Rev. **161**, 1308 (1967). Other references may be found here.
[2] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).
[3] K. Johnson and E. C. G. Sudarshan [Ann. Phys. (N. Y.) **13**, 126 (1961)] first observed that something goes wrong with the minimally coupled spin-$\frac{3}{2}$ equation. A. S. Wightman [in *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1968*, edited by A. Perlmutter, C. Hurst, and A. Kursunoglu (W. A. Benjamin, Inc., New York, 1968)] gives a general discussion of the higher-spin problem, insisting on the importance of the stability of the representations of the Poincaré group.
[4] A. Proca, Compt. Rend. **202**, 1420 (1936); G. Wentzel, *Quantum Theory of Fields* (Wiley-Interscience, Inc., New York, 1969), p. 90.
[5] G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1969).
[6] G. Wentzel, Ref. 4, p. 205.
[7] Other cases of noncausal behavior, unrelated to the problem of higher-spin particles, have been observed recently. See S. Bludman and M. Ruderman, Phys. Rev. **170**, 1176 (1968); M. Ruderman, *ibid.* **172**, 1286 (1968); Y. Aharonov, A. Komar, and L. Susskind, *ibid.* **182**, 1400 (1969).
[8] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley-Interscience, Inc., New York, 1962), Vol. 2, Chap. VI.

system with the light cone as ray cone, supplemented by the same number of constraints. In fact, our examples show instead that the defective behavior mentioned above does indeed occur.

In view of these results, the situation for higher-spin particles has become acute. There is at present no known example of a satisfactory equation with interaction for spin greater than 1. The case of spin one is marginal; some interactions appear to lead to satisfactory equations, but others are unacceptable.

2. SIMPLE EXAMPLES WITH VECTOR PARTICLES IN EXTERNAL FIELDS

As a first example, in which the Lagrangian method is successful, we consider the charged vector particle with minimal electromagnetic coupling. For this purpose we take the Proca Lagrangian with minimal coupling[4]

$$L=-\tfrac{1}{4}G_{\mu\nu}{}^{\dagger}G^{\mu\nu}+\tfrac{1}{2}m^2W_{\mu}{}^{\dagger}W^{\mu}, \qquad (2.1)$$

where[9]

$$G_{\mu\nu}=\pi_\mu W_\nu-\pi_\nu W_\mu,$$
$$\pi_\mu=i\partial_\mu+eA_\mu,$$

and $A_\mu(x)$ is a given external electromagnetic potential. The Lagrangian equations,

$$\pi_\mu(\pi^\mu W^\nu-\pi^\nu W^\mu)-m^2W^\nu=0, \qquad (2.2)$$

are not true equations of motion, because they appear to be of second order but the second time derivative of W^0 never occurs. [More technically, the system (2.2) has the feature that every surface in space-time is a characteristic surface.[10]] The zeroth component of this equation is, in fact, a primary constraint, and a secondary constraint is obtained by taking the divergence of Eq. (2.2),

$$ieF^{\mu\nu}\pi_\mu W^\nu+m^2\pi\cdot W=0, \qquad (2.3)$$

where we have used

$$[\pi_\mu,\pi_\nu]=ie(\partial_\mu A_\nu-\partial_\nu A_\mu)=ieF_{\mu\nu}. \qquad (2.4)$$

Substitution of Eq. (2.3) into Eq. (2.2) yields

$$\pi^2W_\nu+iem^{-2}\pi_\nu F^{\lambda\mu}\pi_\lambda W_\mu+ieF_\nu{}^\mu W_\mu-m^2W_\nu=0, \quad (2.5)$$

which is a true equation of motion because one may solve for the second time derivative of every component of W_μ. To find the normals n_μ to its characteristic surfaces,[10] we replace $i\partial_\mu$ by n_μ in the highest derivatives and calculate the determinant $D(n)$ of the resulting coefficient matrix, which we call characteristic determinant:

$$D(n)=|n^2g_{\mu\nu}+iem^{-2}n_\mu F_{\lambda\nu}n^\lambda| \qquad (2.6)$$

or

$$D(n)=(n^2)^4. \qquad (2.7)$$

The equation of motion (2.5) will be hyperbolic if the solutions n^0 to $D(n)=0$ are real for any $\mathbf{n}$, which is obviously true. The characteristic surfaces are the same in the free and interacting cases, and the ray cone remains the light cone.

One easily verifies that if a magnetic moment interaction term $i\mu W_\nu F^{\nu\lambda}W_\lambda$ is added to the Lagrangian (2.1), propagation remains causal. On the other hand, if the vector particle is coupled to an external symmetric tensor field $T^{\nu\lambda}$ by the same interaction Lagrangian

$$\lambda W_\nu T^{\nu\lambda}W_\lambda,$$

noncausal propagation occurs. To see this, consider the Lagrangian

$$L=-\tfrac{1}{4}G_{\mu\nu}{}^{\dagger}G^{\mu\nu}+\tfrac{1}{2}m^2W_\mu{}^{\dagger}W^\mu+\tfrac{1}{2}\lambda W_\mu{}^{\dagger}T^{\mu\nu}W_\nu, \quad (2.8)$$

where

$$G_{\mu\nu}=p_\mu W_\nu-p_\nu W_\mu, \quad p_\mu=i\partial_\mu \qquad (2.9)$$

and $T^{\mu\nu}(x)$ is an arbitrary external symmetric tensor field. As before, the zeroth component of the Lagrangian equation

$$p_\mu(p^\mu W^\nu-p^\nu W^\mu)-m^2W^\mu-\lambda T^{\mu\nu}W_\nu=0 \qquad (2.10)$$

constitutes a primary constraint,

$$(-\mathbf{p}^2W^0+p^0\mathbf{p}\cdot\mathbf{W})-m^2W^0-\lambda T^{0\nu}W_\nu=0, \quad (2.11)$$

and the secondary constraint is obtained by taking the divergence of Eq. (2.10),

$$m^2p\cdot W+\lambda p\cdot T\cdot W=0. \qquad (2.12)$$

A true equation of motion results when Eq. (2.12) is substituted back into Eq. (2.10):

$$p^2W^\nu+\lambda m^{-2}p^\nu p\cdot T\cdot W-m^2W^\nu-\lambda T^{\nu\mu}W_\mu=0. \quad (2.13)$$

Up to now we have seen that every solution of the Lagrangian equation (2.10) also satisfies the constraints (2.11) and (2.12) and the new equation of motion (2.13). Conversely, every solution of Eq. (2.13) which satisfies the constraints (2.11) and (2.12) at a given time, satisfies them at all times and satisfies the original Lagrangian equation (2.10) as well. The proof of this statement is omitted because it is very similar to the discussion given in Appendixes A and B for other cases. To determine the nature of the propagation, we compute the characteristic determinant of Eq. (2.13):

$$D(n)=|n^2g^{\mu\nu}+\lambda m^{-2}n^\mu(n\cdot T)^\nu|. \qquad (2.14)$$

The normals n_μ to the characteristic surfaces are determined by equating this to zero:

$$D(n)=(n^2)^3(n^2+\lambda m^{-2}n\cdot T\cdot n)=0. \qquad (2.15)$$

[Note that if $T^{\mu\nu}$ were replaced by the antisymmetric tensor $F^{\mu\nu}$, then the characteristic surfaces would be determined by $(n^2)^4=0$, so that a magnetic moment coupling does not disturb the causal propagation, as asserted above.] To see the kind of behavior that may occur, suppose that $T^{\mu\nu}$ has only the component T^{00} different from zero. In this case, the last factor of Eq.

[9] Our conventions are $\hbar=c=1$, $g^{\mu\nu}=(1,-1,-1,-1)$.

[10] Reference 8, pp. 590 and 596.

(2.15) determines a characteristic surface with normals n_μ satisfying

$$(n^0)^2 = (1+\lambda m^{-2}T^{00})^{-1}\mathbf{n}^2. \qquad (2.16)$$

Therefore, if $\lambda m^{-2}T^{00}$ lies between 0 and -1, the equation of motion is hyperbolic, but has spacelike characteristic surfaces and is therefore noncausal. Using the method of Ref. 5, Appendix B, one easily verifies that the noncausal ray is not eliminated by the constraints.

3. NONCAUSAL PROPAGATION WITH ELECTRIC QUADRUPOLE COUPLING

Just as the neutron and proton are sometimes described phenomenologically by introducing a point Pauli magnetic moment, it is natural to attempt to describe a spin-one particle, such as the deuteron, by a corresponding point electric quadrupole moment. Such a description will now be shown to contradict causality.

Consider the Lagrangian

$$L = -\tfrac{1}{4}G_{\mu\nu}{}^\dagger G^{\mu\nu} + \tfrac{1}{2}m^2 W_\mu{}^\dagger W^\mu + \tfrac{1}{2}q[W_\lambda{}^\dagger Q^\lambda{}_{\mu\nu}p^\mu W^\nu + (p^\mu W^\nu)^\dagger Q^\lambda{}_{\mu\nu}W_\lambda], \qquad (3.1)$$

where

$$G_{\mu\nu} = p_\mu W_\nu - p_\nu W_\mu, \quad p_\mu = i\partial_\mu, \quad Q_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu}, \qquad (3.2)$$

and $F_{\mu\nu}(x)$ is a given external electromagnetic field. Here q is a real constant, with dimensions (charge) $\times$(length)^2, that measures the strength of the electric quadrupole coupling. For simplicity, we omit electric charge and magnetic dipole moment. Variation with respect to $W^\dagger$ yields

$$p_\mu(p^\mu W^\nu - p^\nu W^\mu) - m^2 W^\nu - q(Q^\nu{}_{\mu\lambda}p^\mu W^\lambda + p^\mu Q_{\lambda\mu}{}^\nu W^\lambda) = 0. \qquad (3.3)$$

As before, the zeroth component is a constraint. However, upon contracting this equation with p_ν, we find, instead of a first-order constraint, the second-order equation[11]

$$m^2 p\cdot W + q p^\nu Q_{\nu\mu\lambda}p^\mu W^\lambda = 0. \qquad (3.4)$$

If we were to substitute this equation back into Eq. (3.3), corresponding to what was done before, a third-order equation would result. Consequently, a different procedure must be employed.

We decompose the solution W^μ into a transverse vector field V^μ, with

$$p\cdot V = 0, \qquad (3.5)$$

and the gradient of a scalar field B:

$$W^\mu = V^\mu + p^\mu B. \qquad (3.6)$$

Here V^μ and B are determined to within the gauge transformation

$$V^\mu \to V^\mu + p^\mu\Lambda, \qquad (3.7a)$$

$$B \to B - \Lambda, \qquad (3.7b)$$

with Λ an arbitrary scalar solution of the wave equation

$$p^2\Lambda = 0. \qquad (3.8)$$

Upon substitution of Eq. (3.6) into Eqs. (3.3) and (3.4) one obtains, using Eq. (3.5),

$$p^2 V^\nu - m^2(V^\nu + p^\nu B) - q[Q^\nu{}_{\mu\lambda}p^\mu V^\lambda + p_\mu Q_\lambda{}^{\mu\nu}(V^\lambda + p^\lambda B)] = 0, \qquad (3.9)$$

$$m^2 p^2 B + q p_\nu Q^\nu{}_{\mu\lambda}p^\mu v^\lambda = 0. \qquad (3.10)$$

We take V^μ and B to be new dynamical variables for which Eqs. (3.9) and (3.10) provide equations of motion. Because Eqs. (3.9) and (3.10) are invariant under the gauge transformation (3.7) and (3.8), the number of independent variables is reduced to four. A further reduction to three independent variables, as required for a spin-one particle, is provided by the constraint (3.5) and the zeroth component of Eq. (3.3) written in terms of V^μ and B:

$$p_i(p^i V^0 - p^0 V^i) - m^2(V^0 + p^0 B) - q[Q^0{}_{\mu\lambda}p^\mu V^\lambda + p^\mu Q_{\lambda\mu}{}^0(V^\lambda + p^\lambda B)] = 0. \qquad (3.11)$$

[The two constraints (3.5) and (3.11) are required to eliminate one component, because the equations of motion are of second order.]

Thus far we have established that every solution W^μ of the Lagrangian equations (3.3) determines a solution V^μ and B of Eqs. (3.9) and (3.10) that is unique modulo the gauge transformation (3.7), (3.8) and that also satisfies the constraints (3.5) and (3.11). Conversely, we show the following in Appendix A:

(a) Every solution of Eqs. (3.9) and (3.10) that satisfies the constraints (3.5) and (3.11) at a given time also satisfies them for all time.

(b) Every solution of Eqs. (3.9) and (3.10) that satisfies the constraints (3.5) and (3.11) provides a unique solution W^μ, given by Eq. (3.6), of the Lagrangian equations (3.3).

To see whether propagation is causal we evaluate the characteristic determinant of the system (3.9) and (3.10):

$$D(n) = \begin{vmatrix} m^2n^2 & qn_\nu Q^\nu{}_{\mu 0}n^\mu & qn_\nu Q^\nu{}_{\mu 1}n^\mu & qn_\nu Q^\nu{}_{\mu 2}n^\mu & qn_\nu Q^\nu{}_{\mu 3}n^\mu \\ -qn_\mu Q_\lambda{}^{\mu 0}n^\lambda & n^2 & 0 & 0 & 0 \\ -qn_\mu Q_\lambda{}^{\mu 1}n^\lambda & 0 & n^2 & 0 & 0 \\ -qn_\mu Q_\lambda{}^{\mu 2}n^\lambda & 0 & 0 & n^2 & 0 \\ -qn_\mu Q_\lambda{}^{\mu 3}n^\lambda & 0 & 0 & 0 & n^2 \end{vmatrix}.$$

[11] This equation contains the second time derivative of the spatial components W^i only. It may be eliminated using Eq. (3.3), giving rise to a true secondary constraint.

The normals n_μ to the characteristic surfaces are given by

$$D(n)=(n^2)^3[(n^2)^2m^2+q^2(n\cdot Q\cdot n)^\sigma(n\cdot Q\cdot n)_\sigma]=0, \quad (3.12)$$

with

$$(n\cdot Q\cdot n)^\sigma \equiv n^\mu Q_\mu{}^{\nu\sigma}n_\nu \equiv n^\mu \partial_\mu F^{\nu\sigma}n_\nu. \quad (3.13)$$

We now analyze the last factor of Eq. (3.12) in the simplest case of static electric field $\mathbf{E}(\mathbf{x})=F_{0i}(\mathbf{x})$, $F_{ij}=0$, when it becomes

$$(n_0^2-\mathbf{n}^2)^2m^2+q^2[(\mathbf{n}\cdot\nabla\mathbf{n}\cdot\mathbf{E})^2-n_0^2(\mathbf{n}\cdot\nabla\mathbf{E})^2]=0. \quad (3.14)$$

One easily sees that, for any given $\mathbf{n}$, all solutions n_0 of this equation are real, which establishes that the system (3.9), (3.10) is hyperbolic for any electrostatic field $\mathbf{E}$. In addition, one may verify that Eq. (3.14) possesses solutions n_μ that are timelike vectors. Hence we conclude that the Lagrangian equation (3.3) does imply noncausal propagation. The maximum velocity of propagation, according to Eq. (3.14), is, to lowest order in $(q/m)\nabla\cdot\mathbf{E}$, given by

$$v/c=n_0/|\mathbf{n}|=1+\tfrac{1}{2}(q/m)|\hat{n}\cdot\nabla\,\hat{n}\times\mathbf{E}|. \quad (3.15)$$

If one were to attempt a phenomenological description of the deuteron by a vector field, one would find that, in the electrostatic field of a typical crystal,

$$(v/c-1)\sim 10^{-19}.$$

On the other hand, if, in Eq. (3.12), the external field is taken to be magnetostatic, one obtains complex solutions n_0 for sufficiently strong magnetic field. This means that the system (3.9) and (3.10) is no longer hyperbolic and so is not appropriate to describe wave propagation.

4. NONCAUSALITY OF SELF-COUPLED NEUTRAL VECTOR FIELD

In this section, we give a simple example in which the noncausality is not produced by an external potential but occurs in the closed system of the self-coupled neutral vector field. We will make use of the simplest possible self-interaction, with Lagrangian

$$L=-\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}+\tfrac{1}{2}m^2W^2+\tfrac{1}{4}\lambda(W^2)^2, \quad (4.1)$$

where

$$W_\mu=W_\mu{}^\dagger, \quad W^2=W_\mu W^\mu, \quad G_{\mu\nu}=\partial_\mu W_\nu-\partial_\nu W_\mu.$$

The zeroth component of the Lagrangian equation

$$\partial_\mu(\partial^\mu W^\nu-\partial^\nu W^\mu)+m^2W^\nu+\lambda W^2W^\nu=0 \quad (4.2)$$

is a primary constraint, and its divergence yields the secondary constraint

$$m^2\partial\cdot W+\lambda W^2\partial\cdot W+2\lambda W^\mu W^\nu\partial_\mu W_\nu=0. \quad (4.3)$$

Upon substitution of this constraint into Eq. (4.2), one finds the equation of motion

$$\partial^2W^\nu+\lambda m^{-2}\partial^\nu(W^2\partial\cdot W+2W^\mu W^\lambda\partial_\mu W_\lambda) +m^2W^\nu+\lambda W^2W^\nu=0. \quad (4.4)$$

The normals n_μ to its characteristic surfaces are determined by

$$D(n)=|n^2g^{\mu\nu}+\lambda m^{-2}n^\mu(W^2n^\nu+2n\cdot WW^\nu)| =(n^2)^3\{n^2+\lambda m^{-2}[n^2W^2+2(n\cdot W)^2]\}=0. \quad (4.5)$$

The last factor of this equation determines characteristic surfaces with normals n_μ satisfying

$$(1+\lambda m^{-2}W^2)n^2=-2\lambda m^{-2}(n\cdot W)^2. \quad (4.6)$$

Note that in the present case the characteristic surfaces are not a property of the equation alone, but they depend on the particular solution W_μ. We see by inspection of Eq. (4.6) that if $\lambda m^{-2}W^2>-1$, which will certainly hold for $|W^2|$ sufficiently small, then n_μ will be spacelike for $\lambda>0$ and timelike for $\lambda<0$. Therefore, with an initial value of W^2 sufficiently small (consistent with the constraints), the initial propagation will be noncausal for $\lambda<0$. It is worth remarking that with W^2 sufficiently small, the energy remains positive, as in the free case.

5. PATHOLOGY OF MINIMALLY COUPLED SPIN-TWO EQUATION

At present we leave the vector particle and turn our attention to higher spin. The minimally coupled Rarita-Schwinger equation for spin $\frac{3}{2}$ has been discussed elsewhere.[5] Here we consider a minimally coupled spin-two equation[12] which turns out to exhibit a much worse pathology. We shall show that, as compared to the free case, a constraint is lost when the external field is different from zero. The spin-two Lagrangian

$$L=(\pi_\lambda\psi_{\mu\nu})^\dagger\pi^\lambda\psi^{\mu\nu}-2(\pi_\lambda\psi_{\mu\nu})^\dagger\pi^\lambda\psi^{\mu\nu}+(\pi^\mu\psi_{\mu\nu})^\dagger\pi^\nu\psi +(\pi_\mu\psi)^\dagger\pi_\lambda\psi^{\mu\lambda}-(\pi_\mu\psi)^\dagger\pi^\mu\psi -m^2[(\psi_{\mu\nu})^\dagger\psi^{\mu\nu}-\psi^\dagger\psi] \quad (5.1)$$

yields the equation

$$L^{\mu\nu}\equiv\pi^2\psi^{\mu\nu}-\pi_\lambda(\pi^\mu\psi^{\lambda\nu}+\pi^\nu\psi^{\lambda\mu})+\tfrac{1}{2}(\pi^\mu\pi^\nu+\pi^\nu\pi^\mu)\psi +g^{\mu\nu}(\pi_\sigma\pi_\lambda\psi^{\sigma\lambda}-\pi^2\psi)-m^2(\psi^{\mu\nu}-g^{\mu\nu}\psi)=0, \quad (5.2)$$

where $\psi^{\mu\nu}$ is a symmetric tensor,

$$\psi\equiv\psi_\mu{}^\mu, \quad (5.3)$$

$\pi_\mu=i\partial_\mu+eA_\mu$, and $[\pi_\mu,\pi_\nu]=ieF_{\mu\nu}$. Examination of Eq. (5.2) shows that no second time derivative appears when μ (or ν) is zero. These components constitute four primary constraint equations. Secondary constraints

[12] Our Lagrangian (5.1) is obtained from the Lagrangian of G. Wentzel (Ref. 4, p. 205) by the substitution $i\partial_\mu\rightarrow i\partial_\mu+eA_\mu$. It differs from the Lagrangian (14) of Ref. 2 by a magnetic dipole term.

result from the divergence of Eq. (5.2):

$$C^\nu \equiv (\pi_\mu\psi^{\mu\nu} - \pi^\nu\psi) - iem^{-2}[F_{\mu\lambda}\pi^\lambda\psi^{\mu\nu} + (F_\mu{}^\nu\pi_\lambda + \pi_\lambda F_\mu{}^\nu)\psi^{\mu\lambda} + (\tfrac{1}{2}\pi_\mu F^{\mu\nu} + F^{\mu\nu}\pi_\mu)\psi] = 0. \quad (5.4)$$

The difference between the free and interacting cases will now become apparent, for, when A_μ vanishes, the further constraint $\psi=0$ follows upon comparing the trace of Eq. (5.2) with the divergence of Eq. (5.4). This reduces the number of independent components to five, as required to describe a spin-two particle ($\partial_\mu\psi^{\mu\nu}=0$ and $\psi_\mu{}^\mu=0$). However, in the region where $F_{\mu\nu}$ is different from zero, no further constraints beyond Eq. (5.4) exist. To see this, we combine Eqs. (5.2) and (5.4) and obtain a new equation of motion which may be written

$$L^{\mu\nu} + \pi^\mu C^\nu + \pi^\nu C^\mu = 0, \quad (5.5)$$

where $L^{\mu\nu}$ is the left-hand side of Eq. (5.2) and

$$C^\nu = -m^{-2}\pi_\mu L^{\mu\nu} \quad (5.6)$$

is the left-hand side of Eq. (5.4). In Appendix B it is established that, if the new equation (5.5) holds and the constraints $L^{\mu 0}=0$ and $C^\mu=0$ are satisfied at a given time, the Lagrangian equations $L^{\mu\nu}=0$ are satisfied at all times. Let us examine the characteristic determinant of Eq. (5.5). We do not write it out because it involves a 10×10 matrix; however, the computation is not very long and yields

$$D(n) = (e^2/m^4)(n^2)^9[(n\cdot F)^2 + (4e^2/m^4)n^2(\mathbf{E}\cdot\mathbf{B})^2]. \quad (5.7)$$

We see that, in the region where the field is different from zero, $D(n)$ does not vanish identically and (as long as the characteristic surfaces are avoided) Eq. (5.5) provides, at least locally, a unique solution $\psi^{\mu\nu}$ for arbitrary initial $\psi^{\mu\nu}$ and $\dot\psi^{\mu\nu}$. Furthermore, we know that, whenever $\psi^{\mu\nu}$ and $\dot\psi^{\mu\nu}$ satisfy $L^{\mu 0}=0$ and $C^\mu=0$ initially, then $\psi^{\mu\nu}$ is also a solution of the Lagrangian equations. But this means that six initial components of $\psi^{\mu\nu}$ and six of $\dot\psi^{\mu\nu}$ may be specified arbitrarily and still the Lagrangian equations are satisfied, whereas only five independent components are allowed for a spin-two particle. Therefore Eq. (5.2), which provides the required additional constraints $\psi_\mu{}^\mu=0$, $\dot\psi_\mu{}^\mu=0$ where $F_{\mu\nu}$ vanishes, fails to do so when the field is turned on, and so it cannot be a satisfactory description of a charged spin-two particle. We regard the loss of constraint observed here as a more serious disease than the noncausal behavior of the preceding examples and of the Rarita-Schwinger equation for spin $\frac{3}{2}$.

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APPENDIX A

We will prove assertions (a) and (b) of the paragraph following Eq. (3.11). Let us denote by L^ν the left-hand side of Eq. (3.3), with W^μ replaced by $V^\mu + p^\mu B$, and similarly let us denote the left-hand side of Eq. (3.9) by S^ν and of Eq. (3.10) by S. In this language, Eqs. (3.9) and (3.10) may be written

$$S^\nu = L^\nu + p^\nu p\cdot V = 0, \quad (A1)$$

$$S = 0. \quad (A2)$$

We assume that these equations are satisfied at all times and that the constraints

$$p\cdot V = 0, \quad (A3)$$

$$L^0 = 0 \quad (A4)$$

are satisfied at a given initial time. From Eq. (A4) and the zeroth component of Eq. (A1), we obtain

$$p^0 p\cdot V = 0 \quad (A5)$$

at the initial time. Furthermore, because of the identity

$$p_\nu S^\nu + S \equiv (p^2 - m^2)p\cdot V, \quad (A6)$$

we conclude that $p\cdot V$ vanishes at all times because it satisfies the Klein-Gordon equation at all times and vanishes, together with its first derivative, at the initial time. Hence, from Eq. (A1) we deduce that

$$L^\nu = 0 \quad (A7)$$

holds at all times. Thus the constraints $p\cdot V=0$ and $L^0=0$ are preserved and the original Lagrangian equations are satisfied at all times.

APPENDIX B

We prove that the constraints $L^{\mu 0}=0$ and $C^\mu=0$ are preserved by Eq. (5.5), and that Eq. (5.2) is a consequence of Eq. (5.5) and these constraints. We suppose that $L^{\mu 0}$ and C^μ vanish at $t=0$. The $\mu=0$ components of Eq. (5.5) give

$$\pi^0 C^\nu = 0 \quad (B1)$$

at $t=0$, and hence, by Eq. (5.5), one has, at $t=0$,

$$L^{\mu\nu} = 0 \quad (B2)$$

for all μ and ν. From Eq. (5.6) and the vanishing of C^ν, one obtains

$$\pi_0 L^{0\nu} = 0 \quad (B3)$$

at $t=0$. Contraction of Eq. (5.5) with π_μ yields a second-order equation for C^μ. Hence, because C^μ and $\pi_0 C^\mu$ vanish at $t=0$, C^μ vanishes identically. From Eq. (5.5) we conclude that $L^{\mu\nu}$ vanishes identically.