Hypothesis of Limiting Fragmentation in High-Energy Collisions

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A hypothesis of limiting fragmentation of the target and of the projectile in a high-energy lepton-hadron or hadron-hadron collision is defined. Arguments are given for the hypothesis. Comparisons with various models and concepts are made. Further speculations are made, including the absence of pionization processes in high-energy collisions and the dependence of multiplicity on the momentum transfer. Experiments are suggested.

INTRODUCTION

In recent years many experiments[1] have been performed on inelastic hadron-hadron collisions and inelastic ep collisions. These experiments, taken together with the droplet interpretation[2,3] of elastic collisions, and with cosmic-ray information, suggest a specific framework in which very-high-energy ep and hadron-hadron collisions could be usefully described. The framework is based on the *hypothesis of limiting fragmentation of each of the two colliding particles* in a high-energy collision which will be defined and discussed in this paper. If the hypothesis is correct, experimentally it suggests that one should measure the limiting fragment distributions. Theoretically, it suggests that the fragmentation process should be a principal subject of study for any model of high-energy collisions.

Our discussion is very much related to the traditional two-fireball model[4] used in cosmic-ray physics. We shall explicitly discuss the points of agreement between the present hypothesis and the two-fireball model, and also the specific points where the two pictures differ. Our discussion is also closely related to the concept of "diffraction dissociation" introduced by Good and Walker.[5]

A dominant feature of multiparticle processes in high-energy collisions is the small value of the transverse momenta of the outgoing particles, a property that is especially difficult to accommodate in the traditional statistical model.[6] On the other hand, longitudinal momenta are usually quite large for some of the emitted particles. The momentum distribution of the outgoing particles, especially when the multiplicity is low, strongly suggests the "persistence" of the longitudinal momentum that resides in the incoming projectile which breaks apart in the collision process. Similarly, the target under the influence of the fast-moving projectile seems, in general, to break up into many pieces. This intuitive picture of a high-energy collision process as two extended objects going through each other, breaking into fragments in the process, is defined precisely in the next section as the hypothesis of limiting fragmentation. Arguments for this hypothesis are then presented and the picture is compared with previously proposed models. Additional speculations, remarks, and suggested experiments are discussed near the end of the paper.

HYPOTHESIS OF LIMITING FRAGMENTATION

1. It has been customary to describe a collision in the c.m. system. We believe that for very-high-energy collisions, the lab system (L) and the projectile system (P, where the incoming projectile is at rest) are to be preferred, because in these systems, some of the outgoing particles *approach limiting distributions*. (Because of the large number of possible kinematic variables involved in a multiparticle process, it is important to use those variables in terms of which the process is most simply described.)

Let us consider the lab system L. In a typical high-energy collision, with incoming energy E, many particles are produced. Some of these particles tend to have increasingly high lab velocity v as E increases. Others have values of $\gamma=(1-v^2)^{-1/2}$ that remain finite as E increases. These latter particles, we propose, approach a limiting distribution as $E\to\infty$.

To be more precise, given a d^3p region for the laboratory $\mathbf{p}$ of a specific outgoing particle of mass m (say, a proton, or a pion), the probability that in a high-energy hadron-hadron or electron-hadron collision a particle of that mass will be found in that region

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[1] See *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968); *Proceedings of the Topical Conference on High-Energy Collisions of Hadrons, Geneva, 1968* (Scientific Information Service, Geneva, 1968).

[2] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

[3] N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1966); T. T. Chou and C. N. Yang, in *Proceedings of the Second International Conference on High-Energy Physics and Nuclear Structure, Rehovoth, Israel, 1967*, edited by G. Alexander (North-Holland Publishing Co., Amsterdam, 1967), pp. 348–359; Phys. Rev. **170**, 1591 (1968); Phys. Rev. Letters **20**, 1213 (1968); Phys. Rev. **175**, 1832 (1968).

[4] G. Cocconi, Phys. Rev. **111**, 1699 (1958); K. Niu, Nuovo Cimento **10**, 994 (1958); P. Coik, T. Coghen, J. Gierula, R. Holynski, A. Jurak, M. Miesowicz, T. Saniewska, and J. Pernegr, *ibid.* **10**, 741 (1958). See also R. K. Adair, Phys. Rev. **172**, 1370 (1968).

[5] M. L. Good and W. D. Walker, Phys. Rev. **120**, 1857 (1960).

[6] E. Fermi, Progr. Theoret. Phys. (Kyoto) **5**, 570 (1950); Phys. Rev. **81**, 683 (1951); **92**, 452 (1953); **93**, 1434 (1954).

approaches a limit as $E \to \infty$. In other words, we hypothesize the existence of

$$\lim_{E\to\infty} \text{(partial cross section that a particle of mass } m \text{ is emitted with lab momentum } \mathbf{p}, \text{ other emitted particles being ignored)} = \rho_1(\mathbf{p})d^3p\,, \quad (1)$$

where

$$\rho_1 > 0\,.$$

2. While in (1) we consider the limiting distribution of one particle, we believe a similar limit exists for any configuration in the lab momentum space for n particles, for any fixed $n=1, 2, \ldots$; i.e., the following limit exists:

$$\lim_{E\to\infty} \text{(partial cross section that a particle of mass } m_1 \text{ and momentum } \mathbf{p}_1\text{, and a particle of mass } m_2 \text{ and momentum } \mathbf{p}_2\text{, are emitted, together with any number of other particles)} = \rho_2(\mathbf{p}_1,\mathbf{p}_2)d^3p_1d^3p_2\,, \text{ etc.,} \quad (2)$$

where

$$\rho_2 > 0\,, \text{ etc.}$$

These limits, of course, are in general different for different collisions (ep, pp, πp, etc.).

3. In (1) and (2) above, no reference is made to the rate at which the limit is approached. Experimental data suggest that for high values of n and/or high values of lab momentum, the approach to a limiting distribution is slow.

4. The limiting distributions (1) and (2) discussed above represent *distributions of broken-up fragments of the target*. The fragments from the projectile, on the other hand, move with increasing velocity in the lab system as $E \to \infty$, and do not contribute to any limiting distribution. To study these fragments from the projectile, one must go into the projectile system P. If there should exist additional outgoing particles which are not fragments (as *defined* above) of either the projectile or the target (for example, pions in the "pionization" process which are slow in the c.m. system), they will not contribute to the limiting distribution in either the lab or the projectile system. The possible existence of a pionization process is not inconsistent with the hypothesis we are discussing here. However, for reasons to be explained later, we shall speculate in the beginning of Sec. 16 that at very high energies there are no pionization processes.

5. If limiting distributions (1) and (2) do exist, their integrals are related to average multiplicities for general or restricted types of collisions. For example,

$$\int \rho_1 d^3p = \sigma_0 \times \text{(average multiplicity per collision of particles of mass } m \text{ emitted from the target)}\,, \quad \text{(3a)}$$

where σ_0 = total collision cross section. Similarly,

$$\int \rho_2 d^3p_2 = \rho_1(\mathbf{p}_1) \times \text{(average multiplicity per collision, where a particle of mass } m_1 \text{ and momentum } \mathbf{p}_1 \text{ is emitted from the target, of particles of mass } m_2 \text{ emitted from the target).} \quad \text{(3b)}$$

While the values of the integrals in (3a) and (3b) are not specified in the hypothesis under discussion here, we believe them to be divergent. This belief derives from the fact that at infinite incoming energies all multiplicities seem to become infinity. If, as we believe, there is no pionization, then the average multiplicity from the projectile plus that from the target must add up to infinity. Hence we speculate that each of them is divergent.

It may appear at first sight that a limiting fragmentation of the two colliding particles is inconsistent with increasing multiplicities at higher and higher energies. It is to be emphasized that this is not so: Increasing multiplicities derive from the fact that more and more of the *divergent* integrals (3a), (3b), etc., are made accessible at higher and higher incident energies. It seems that this view *reconciles* rather satisfactorily, for higher and higher incident energy, *the increasing average multiplicity on the one hand, with the observed persistence of cross sections for events with small multiplicities* (cf. Sec. 8.3) *on the other.*

6. We discuss now the kinematic region in which the distribution ρ_1 of (1) is defined. For those processes such as forward elastic scattering $\pi p \to \pi p$, or diffraction excitation $\pi p \to \pi^* p$, i.e., for those cases where the target remains unchanged, the contribution to ρ_1 is confined to the parabola in momentum space,

$$e - p_{\parallel} = M_t\,, \quad (4)$$

where e and $p_{\parallel}$ are, respectively, the lab energy and longitudinal momentum of the particle in question, and M_t is the mass of the target. If the target breaks into more than one particle, the region of momentum space where such processes contribute to ρ_1 is given by

$$e - p_{\parallel} < M_t\,. \quad (5)$$

Equations (4) and (5) will be proved in Appendix A.

Thus ρ_1 is a delta function σ_1 on the parabola (4) plus a distribution function τ_1 in the region (5). The region (5) is bounded by the parabola (4). A similar breakdown of ρ_2, ρ_3, etc., yields

$$\rho_n = \sigma_n + \tau_n\,, \quad (6)$$

where

$$\tau_n = \text{function defined in } R_n\,, \quad (7)$$

$$\sigma_n = \text{delta function defined on } S_n\,. \quad (8)$$

Here,

$$R_n = \text{the region} \sum_{i=1}^{n} (e-p_{\parallel})_i < M_t \tag{9}$$

and

$$S_n = \text{boundary of } R_n$$
$$= \text{the surface defined by} \sum_{i=1}^{n} (e-p_{\parallel})_i = M_t. \tag{10}$$

These formulas will be proved in Appendix A, with some properties of (10).

Physically, σ_n is the cross section for processes in which the target breaks up into exactly the n particles specified in its argument; τ_n, for those in which the target breaks up into *more* than n particles. Obviously,

$\sigma_1, \sigma_2, \ldots =$ limiting partial cross section of various possible fragmentations of the target under the impact of a projectile at infinite energies. [The particles in σ_n have momenta satisfying (10).] (11)

[If the target can be broken into two fragments in different ways, e.g., $p \to \pi^0 p$ and $p \to K^0\Sigma^+$, then several σ_2's must be included in the left-hand side of (11).]

Because of (11), *the distributions* τ_n *can be obtained from a superposition of* $\sigma_{n+1}, \sigma_{n+2}, \ldots$, etc., *after the redundant coordinates are integrated out.*

7. The discussion above about the fragmentation of the target in the laboratory system L has, of course, its counterpart for the fragmentation process of the projectile in the projectile system P. We must replace M_t by the mass of the projectile M_{inc}. (The case of *ep* collisions is an exception since the electron as the projectile does not break up.)

It is important to notice that a particle which has a finite energy in the projectile system P has a very large energy in the laboratory system. In fact, its laboratory energy is proportional to the energy of the incoming particle.

Viewed in the c.m. system, the fragments of the target form a jet in the backward hemisphere. It is important to recognize that if $\mathbf{p}^*$ is the center-of-mass momentum for the collision, then a particle with a finite momentum in the laboratory system would have a c.m. energy proportional to $\mathbf{p}^*$ when $\mathbf{p}^* \to \infty$. In other words, at very high energies, *almost all of the backward hemisphere in momentum space in the c.m. system represent finite momenta in the laboratory system.* A similar statement holds for the forward hemisphere when one replaces the laboratory by the projectile system. These statements can be put in a concise form for the limit $E \to \infty$, as exhibited in Table I.

Table I. Order of magnitude of momenta as incident energy $E \to \infty$. M_t = mass of target.

	Lab momenta	c.m. momenta
(1) Fragments of target	$O(1)$	$-O(E^{1/2})$
(2) Pionization products	$O(E^{1/2})$	$O(1)$
(3) Fragments of projectile	ξE $(\xi<1)$	$\xi(\frac{1}{2}M_tE)^{1/2}$ $(\xi<1)$
(4) Intact projectile, with excitation or fragmentation of target	$E-O(1)$	$(\frac{1}{2}M_tE)^{1/2}-O(E^{-1/2})$
(5) Elastically scattered projectile	E	$(\frac{1}{2}M_t/E)^{1/2}$

ARGUMENTS FOR HYPOTHESIS OF LIMITING FRAGMENTATION

8. We now present arguments in support of the hypothesis discussed above.

8.1. The differential cross section of inelastic *ep* scattering processes is written in the usual notation[7] as

$$\frac{d^2\sigma}{dq^2d\nu} = \frac{4\pi\alpha^2}{q^4}\frac{E'}{E}\left[\cos^2(\tfrac{1}{2}\theta)\, W_2(q^2,\nu) + 2\sin^2(\tfrac{1}{2}\theta)\, W_1(q^2,\nu)\right]. \tag{12}$$

When E, $E' \to \infty$ and $\theta \to 0$ such that q^2, ν remain finite, this reduces to

$$\lim \frac{d^2\sigma}{dq^2d\nu} = \frac{4\pi\alpha^2}{q^4} W_2(q^2,\nu). \tag{13}$$

The recoil particles have a total invariant mass M given by

$$M^2 = M_p^2 + 2M_p\nu - q^2. \tag{14}$$

Thus (13) shows that

$$\lim \frac{d^2\sigma}{dq^2dM^2} = \frac{2\pi\alpha^2}{M_pq^4} W_2(q^2,\nu). \tag{15}$$

In other words, the cross section for the excitation of the target to any fixed M with a four-momentum transfer q^2 approaches a limit. In fact, such an excitation leads to a fixed density matrix of the recoil particles[8] in the lab system; i.e., for *ep* collisions the limiting distribution (11) for the fragmentations of the target proton exists. If follows that the limits (1) and (2) also exist. Thus for *ep* scattering, to the order (fine-structure constant)^2, the hypothesis discussed in Secs. 1–7 is proved.

8.2. High-energy elastic scattering experiments seem to indicate that as the incoming energy approaches infinity, the differential cross section $d\sigma/dt$ approaches a limit. Thus for any elastic scattering experiment, the momentum distribution of the recoil target particle approaches a limit on the parabola (4).

[7] W. K. H. Panofsky, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968,* edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), pp. 23–39.

[8] The situation is entirely similar to the corresponding problem in neutrino-nucleon collisions. See T. D. Lee and C. N. Yang, Phys. Rev. **126**, 2239 (1962).

8.3. It was observed experimentally[9] that there exists a class of processes $AB \to CD$ with finite limiting cross sections at high energies, such as $pp \to pp^*$, where the p^* is a low-lying resonance having the same quantum numbers as the proton. Two consequences are the following:

(a) The recoil target particle in $pp \to p^*p$ where the target is not excited would have a limiting distribution on the parabola (4) in laboratory momentum space. (The excitation of the projectile to p^* at any finite excitation energy does not affect the allowed laboratory momenta values of the recoil target particle, in the high-energy limit under consideration.) The limiting distribution σ_1, defined in (8), is thus a sum of the contributions from this process, from the elastic scattering process of Sec. 8.2, and from all other processes where the target does not break up.

(b) In the process $pp \to pp^*$, where the target is excited to p^*, the disintegration of p^* would lead to a limiting distribution for the decay products in the lab system.

8.4. Experiments at Brookhaven and at CERN[10] studied the momentum distribution of single medium fast particles (p, π, or K) produced in pp collisions at 30 and 19.2 BeV/c. [These particles belong to category (3) of Table I.] By performing a Lorentz transformation to the projectile system P, these particles become slow. Owing to the symmetry of the pp system, this is equivalent to the study of the slow particles in the lab system L.

In Fig. 1 we plot the distribution of slow protons in L, obtained in this fashion. We observe that they are not too different at 19 and 30 BeV/c incident energy, indicating that the limit is approached at $\sim$30 BeV/c.

We have also plotted the pion and kaon distributions in the lab system, obtained in a similar fashion, in Figs. 2 and 3. Where there are data for both energies, the π^- distribution seems also to have approached a limit.

In a recent paper[11] the momentum distribution of

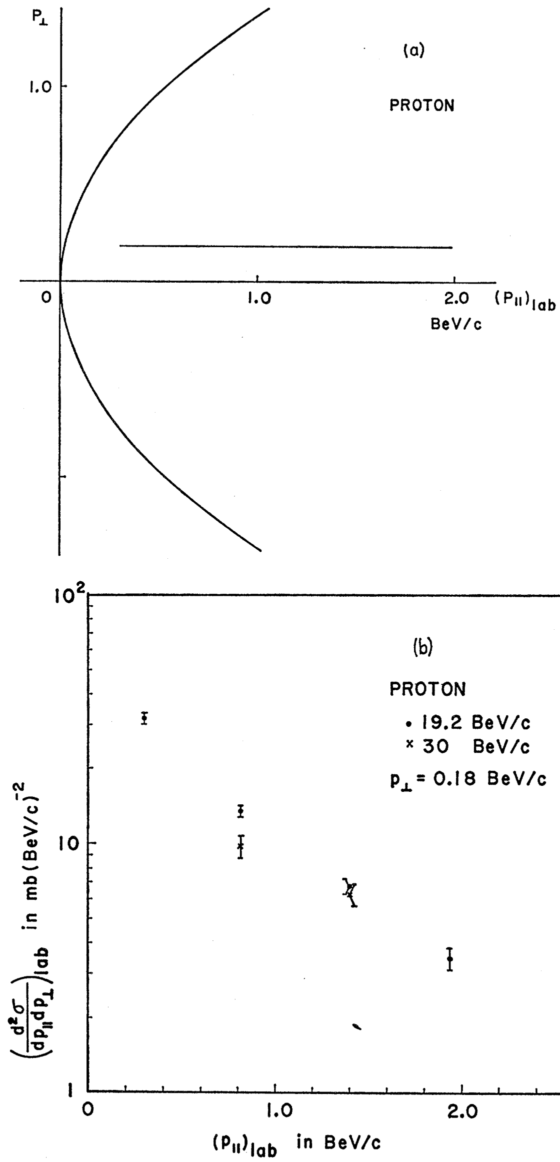

FIG. 1. (a) Kinematic boundary [a parabola, i.e., Eq. (4)] and allowed momentum space [region to the right of parabola, i.e., Eq. (5)] for single proton distribution in pp collisions at infinite energy. Line segment parallel to the $p_\parallel$ axis represents the region where experimental data are available at present energies with $p_\perp = 0.18$ BeV/c. The distribution of proton along this line is plotted in Fig. 1(b). All momenta are in the lab system L. (b) Distribution of proton as fragment of target in the lab system with $p_\perp = 0.18$ BeV/c at 19.2 and 30 BeV/c obtained from Ref. 10 by a procedure described in Sec. 8.4 of the text. The proton distribution at 30 BeV/c can be fitted by $d^2\sigma/dp_\parallel^* dp_\perp = 610\, p_\perp^2 e^{-p_\perp/0.166}$ mb $(\text{BeV}/c)^{-2}$ in c.m. system or $d^2\sigma/dp_\parallel dp_\perp = 2480\, p_\perp^2 e^{-p_\perp/0.166}(1-0.969 p_\parallel/E_{\text{lab}})$ mb $(\text{BeV}/c)^{-2}$ in the lab system, where $p_\perp$ is in BeV/c. For very small values of $(p_\parallel)_{\text{lab}}$, the curve should dip down to zero. Compare Sec. 9.

medium-high-energy secondary π, K, and p were reported up to 70-BeV incident energy. The authors showed that for a fixed p/p_{inc} in the lab system, the

[9] G. Cocconi, A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters **7**, 450 (1961); E. W. Anderson, E. J. Bleser, G. B. Collins, T. Fujii, J. Menes, F. Turkot, R. A. Carrigan, Jr., R. M. Edelstein, N. C. Hien, T. J. McMahon, and I. Nadelhaft, *ibid.* **16**, 855 (1966); K. J. Foley, R. S. Jones, S. J. Lindenbaum, W. A. Love, S. Ozaki, E. D. Platner, C. A. Quarles, and E. H. Willen, *ibid.* **19**, 397 (1967).

[10] E. W. Anderson, E. J. Bleser, G. B. Collins, T. Fujii, J. Menes, F. Turkot, R. A. Carrigan, Jr., R. M. Edelstein, N. C. Hien, T. J. McMahon, and I. Nadelhaft, Phys. Rev. Letters **19**, 198 (1967); J. V. Allaby, F. Binon, A. N. Diddens, P. Duteil, A. Klovning, R. Meunier, J. P. Peigneux, E. J. Sacharidis, K. Schlüpmann, M. Spighel, J. P. Stroot, A. M. Thorndike, and A. M. Wetherell, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[11] Yu. B. Bushnin, S. P. Denisov, S. V. Donskov, A. F. Dunaitsev, Yu. P. Gorin, V. A. Kachanov, Yu. S. Khodirev, V. I. Kotov, V. M. Kutyin, A. I. Petrukhin, Yu. D. Prokoshkin, E. A. Razuvaev, R. S. Shuvalov, D. A. Stoyanova, J. V. Allaby, F. Binon, A. N. Diddens, P. Duteil, G. Giacomelli, R. Meunier, J.-P. Peigneux, K. Schlüpmann, M. Spighel, C. A. Stahlbrandt, J.-P. Stroot, and A. M. Wetherell, Phys. Letters **29B**, 48 (1969), especially Fig. 3(b).

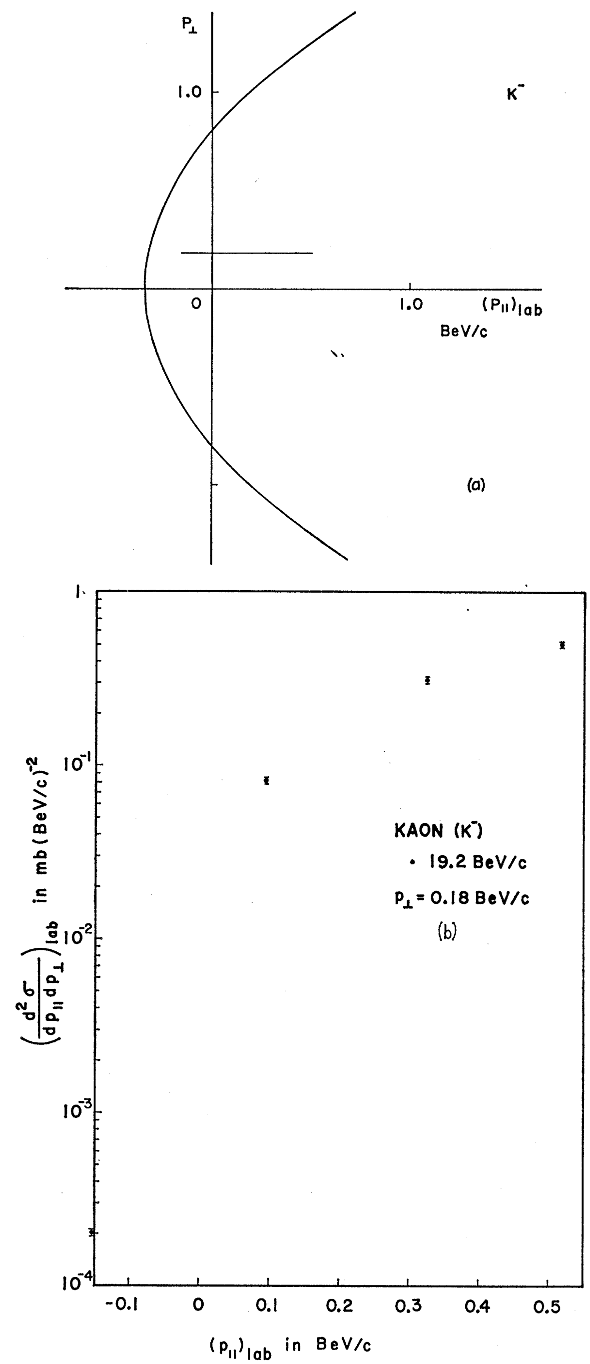

FIG. 2. (a) Kinematic boundary [a parabola, i.e., Eq. (4)] and allowed momentum space [region to the right of parabola, i.e., Eq. (5)] for single kaon (K^-) distribution in pp collisions at infinite energy. Line segment parallel to $p_{\parallel}$ axis represents the region where experimental data are available at present energies with $p_{\perp}=0.18$ BeV/c. The distribution of kaon along this line is plotted in Fig. 2(b). All momenta are in the lab system L. (b) Distribution of kaon as fragment of target in the lab system with $p_{\perp}=0.18$ BeV/c at 19.2 BeV/c obtained from Ref. 10 by a procedure described in Sec. 8.4 of the text.

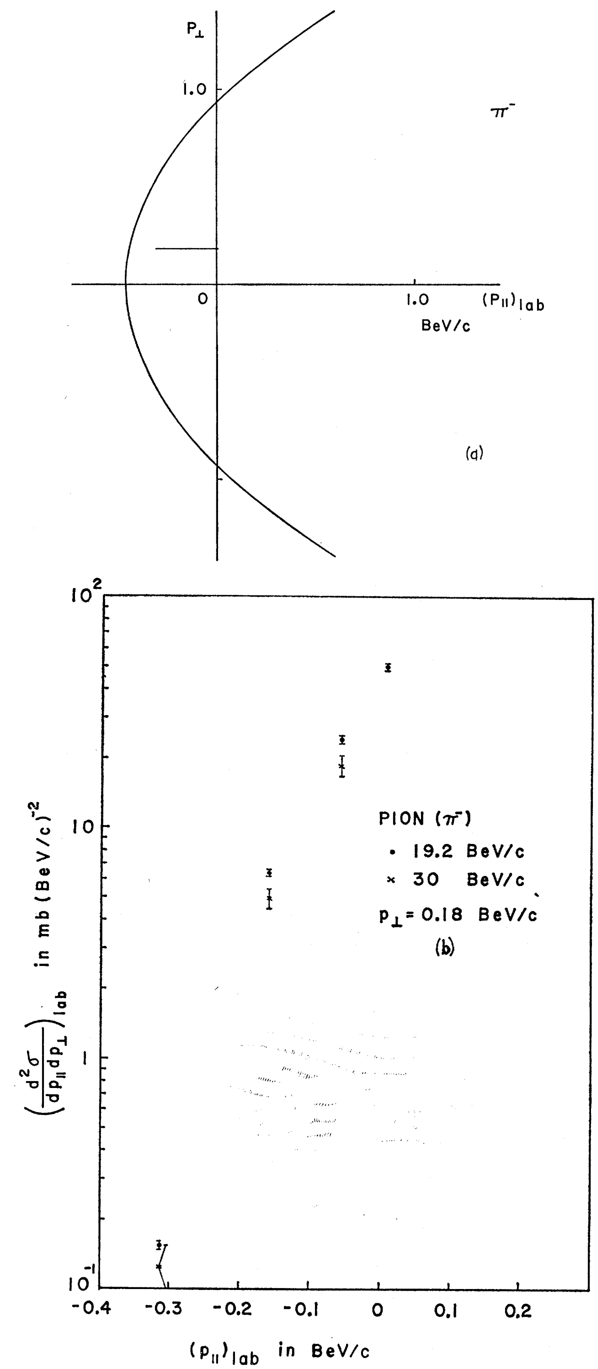

FIG. 3. (a) Kinematic boundary [parabola, i.e., Eq. (4)] and allowed momentum space [region to the right of parabola, i.e., Eq. (5)] for single pion (π^-) distribution in pp collisions at infinite energy. Line segment parallel to $p_{\parallel}$ axis represents the region where experimental data are available at present energies with $p_{\perp}=0.18$ BeV/c. The distribution of pions along this line is plotted in Fig. 3 (b). All momenta are in the lab system L. (b) Distribution of pion as fragment of target in the lab system with $p_{\perp}=0.18$ BeV/c at 19.2 and 30 BeV/c obtained from Ref. 10 by a procedure described in Sec. 8.4 of the text.

particle ratios K^-/π^- and $\bar{p}/\pi^-$ are independent of the incident energy. This independence is easily seen as a *natural consequence* of the hypothesis of a limiting distribution in the projectile system. (Fixed $x=p/p_{\text{inc}}$ in the forward direction means a fixed momentum p^{**} in the rest system of the incoming particle, for high incident energies:

$$p^{**}=\tfrac{1}{2}xM_{\text{inc}}-\tfrac{1}{2}\mu^2/xM_{\text{inc}},$$

where M_{inc}, and μ are the masses of the incoming and the secondary particles.)

8.5. At cosmic-ray energies, an incoming projectile proton emerges in a high-energy collision as an outgoing proton with an "inelasticity" (i.e., fractional energy loss) that seems[12] to approach a limiting distribution. In the projectile system P, the momenta of the outgoing protons would approach a limiting distribution.

8.6. If limiting distributions for the fragmentation on the target and the projectile exist, and if there are no pionization processes, then the transverse momentum of any outgoing particle would approach a limiting distribution, consistent with a dominant feature well known in all high-energy accelerator and cosmic-ray experiments. (See Sec. 18 for a discussion of high-multiplicity events.)

8.7. Elastic scattering at high energies have been described[2,3] in terms of a droplet picture where the target serves as an absorbing medium through which an incoming particle propagates as a wave. In such a picture, the incoming particle in the lab system shrinks into a thin disk by Lorentz contraction at high energies. The target proton has a geometrical extension of the order of 0.7×10^{-13} cm. Passage of the thin disk through the target takes about 2×10^{-24} sec, and the target is excited during this time. The excitation may cause a breakup of the target. What is the effect of higher and higher projectile momentum? The time of passage is essentially fixed, but the disk is further and further compressed (see Fig. 4). The constancy of the total cross section and of the elastic scattering cross section suggest that the momentum and quantum-number transfer process between the "stuff" in the projectile and the "stuff" in the target does not appreciably change when the projectile is further and further compressed. Thus one expects that the excitation and breakup of the target approaches a limiting distribution, which is precisely what (11) asserts.

8.8. Recently Cheng and Wu[13] showed that for certain couplings in field theory, large classes of Feynman diagrams yield, for elastic scatterings $AB\rightarrow AB$, limiting angular distributions $d\sigma/dt$ which can be expressed in terms of impact factors.[14] We believe that extension of their work to inelastic processes $AB\rightarrow CDE\cdots$ would lead to the limiting distributions (1), (2), and (11) discussed above.

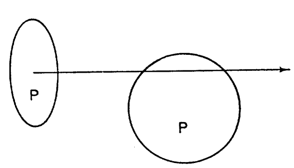

FIG. 4. Passage of Lorentz-contracted projectile through an extended target in the lab system.

DISCUSSIONS

9. Assuming that the hypothesis of limiting fragmentation of the colliding particles is correct, what do we know about the distribution of the fragments? For the single secondary proton distribution, the experiments of Ref. 10 give a rough distribution function for pp collisions as exhibited in the caption of Fig. 1. For single secondary pion and kaon distributions in pp collisions, we have only very rough information, as sketched in Figs. 2 and 3.

The distribution function τ_n satisfies

$$\tau_n=0 \text{ on boundary } S_n \text{ of } R_n. \qquad (16)$$

To see this, we observe that near S_n,

$$\sum_{i=1}^{n}(e-p_{\parallel})_i=M_t-\Delta,$$

where Δ is small. The fragmentation must, besides the n particles in τ_n, yield additional particles for which $\sum(e-p_{\parallel})=\Delta$. Since $e-p_{\parallel}$ is always positive, it must be small for each of the additional particles; i.e., these additional particles must be fast in the lab system. As $\Delta\rightarrow0$, such a fragmentation becomes increasingly unlikely.

Figures 2 and 3 demonstrate (16) very clearly. For the proton distribution in Fig. 1, (16) implies that for smaller $(p_{\parallel})_{\text{lab}}$, the limiting curve should dip to zero.

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[12] M. Koshiba, in *Proceedings of the Tenth International Cosmic Rays Conference, Calgary, Canada, 1967* (to be published).

[13] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969); Phys. Rev. **182**, 1852 (1969); **182**, 1868 (1969); **182**, 1873 (1969); **182**, 1899 (1969).

[14] Cheng and Wu emphasized that their result is not consistent with a straightforward interpretation of the droplet model. We disagree with this emphasis. What seems to us to be most remarkable is, in fact, the general consistency of their results with the spirit of the droplet model. In particular, we cite the following features of their work: (i) the natural formulation in terms of two-dimensional momentum space, (ii) the factorization into two impact factors in a convolution integral, and (iii) the exponentiation, in impact-parameter space, for the transmission coefficient. All of these are characteristic features of the droplet model (Refs. 2 and 3).

Note added in proof. It has recently been shown by B. W. Lee (to be published) that, in fact, the droplet model in q-number formalism proposed by us in Phys. Rev. **175**, 1832 (1968) gives precisely the results of Cheng and Wu. That Cheng and Wu concluded otherwise in their Paper I [*ibid.* **182**, 1852 (1969)] was due to a wrong identification they made.

The main feature of the q-number formalism of the droplet model is the proposal that the S matrix should be given by an exponentiation of a convolution integral in coordinate space of q-number densities. [See Sec. 1 of Phys. Rev. **175**, 1832 (1968).] That this feature seems to be essential is also recently demonstrated by quantum-electrodynamics calculations of S. J. Chang and S. K. Ma, Phys. Rev. Letters **22**, 1334 (1969); S. J. Chang, University of Illinois Report (unpublished); Y. P. Yao, University of Michigan Reports (unpublished).

Experimental information is not at present sufficient to establish the existence of this dip.[15]

The distribution functions τ_n have mesa-like superstructures piled one on top of another. Each such superstructure derives from a process

$$\text{target} \to a^* \qquad (a^* = \text{a resonance}),$$
$$a^* \to (n+1) \text{ particles}.$$

This will be discussed in some detail in Appendix B.

Information about the fragmentation distributions $\sigma_1, \sigma_2, \ldots,$ of (11) are scanty. If we assume no pionization process, i.e., if we assume all final particles to be fragments of either of the two colliding particles, we can divide the pp collision process into four types:

$$pp \to pp, \tag{17a}$$
$$pp \to pp^\dagger, \tag{17b}$$
$$pp \to p^\dagger p, \tag{17c}$$
$$pp \to p^\dagger p^\dagger, \tag{17d}$$

where

$$p^\dagger = \text{excited state } p^*, \text{ or } p\pi, \text{ or any fragmentation of } p. \tag{18}$$

Process (17b) represents those processes in which the projectile remains a proton and the target is excited and/or dissociated into a $p^\dagger$. It would be most interesting to know the relative probabilities of the four processes (17). One knows that (17a) has a cross section of approximately 10 mb. The experiment of Anderson *et al.*[9] shows that $pp \to pp^*$ has a total cross section of the order of 1 mb. Since the total cross section is about 40 mb, we conclude that

$$1 \text{ mb} < \sigma(pp^\dagger) < 15 \text{ mb}, \tag{19}$$
$$0 \leq \sigma(p^\dagger p^\dagger) < 28 \text{ mb}. \tag{20}$$

Lacking detailed information, we find it difficult to estimate $\sigma(pp^\dagger)$ and $\sigma(p^\dagger p^\dagger)$ more precisely. Let us only add that process (17b) leads to a final fast outgoing proton with a finite energy difference compared with the incoming projectile [Table I, category (4)]. Process (17b) would therefore, in cosmic-ray events, be classified as one with an inelasticity of 0.

10. The fragment distributions (11) can, of course, be reclassified according to the total mass M^* of all the fragments of the target

$$(M^*)^2 = (\sum E)^2 - (\sum \mathbf{p})^2. \tag{21}$$

Also, it can be reclassified according to the total four-momentum transfer t to the target:

$$t = (\sum \mathbf{p})^2 - (\sum E - M_t)^2 = (\sum \mathbf{p}_\perp)^2. \tag{22}$$

The summations in (21) and (22) extend over all n fragments of the target in the distributions σ_n of (11). $t = (\sum \mathbf{p}_\perp)^2$ follows from (10). The last identity can be restated as follows: *The total transverse momentum transfer is equal in magnitude to the total four-momentum transfer.*

Keeping M^* and t fixed, one can integrate over all redundant variables in σ_n and sum over n. The resultant cross section

$$\sigma(M^{*2},t)d(M^{*2})dt \tag{23}$$

is then the partial cross section for fragmentation of the target, at infinite incident energy, into fragments with given values of M^*, at the momentum transfer t. For ep collisions, (15) gives

$$\sigma(M^{*2},t) = (2\pi\alpha^2/M_p t^2)W_2(t,\nu), \tag{24}$$

where

$$2M_p\nu = M^{*2} - M_p^2 + t. \tag{25}$$

For pp collisions, Anderson *et al.*[9] have given graphs of $d^2\sigma/dtdM^*$ ($M^* = W$ in their notation) for incident momentum 15.1 BeV/c up to $M^* \sim 2$ BeV. They measured M^* by taking the fast outgoing proton as the projectile after the collision, thereby obtaining the energy and momentum loss of the projectile. For very high energies, their procedure would give the contribution of (17b) to $\sigma(M^{*2},t)$ of (23). [At very high energies, an event where the projectile breaks up would yield another fragment X of the projectile with very high lab energy. In their procedure, X would be included in computation of M^* as a fragment of the target. Thus, M^* would be very large and the event does not contribute to $d^2\sigma/dtdM^*$ for any finite M^*. In the language of Table I, they explored the protons of category (4).]

11. It is well known[7] that in ep scattering, the function $W_2(t,\nu)$ for small t is related to the total γp cross section for an incident photon energy ν. Taking the γp total cross section to be a constant at infinite energy ν, one obtains for small t in ep scattering that

$$\sigma(M^{*2},t) \propto (M^*)^{-2}. \tag{26}$$

It is not likely that (26) is true for hadron-hadron collision, for which one must have

$$\int\int \sigma(M^{*2},t)d(M^{*2})dt = \text{total cross section} \neq \infty. \tag{27}$$

12. A number of very interesting experiments[16] of the type

$$\pi(\text{nucleus}) \to (\pi\pi\pi)(\text{nucleus}) \tag{28}$$

have been performed (or are in progress) in connection

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[15] Figure 1(a) of Phys. Rev. Letters **19**, 198 (1967) may be interpreted as indicating such a dip at 14 ± 2 mrad lab angle, $\sim$25 BeV/c lab momentum. It is to be emphasized that at present accelerator energies, the study of such a dip is necessarily difficult because background due to the breakup of the projectile may fill the dip region. In the language of Table I, the dip occurs between categories 3 and 4.

[16] Berkeley-Milan-Orsay-Saclay Collaboration; in *Proceedings of the Topical Conference on High-Energy Collisions of Hadrons, Geneva, 1968* (Scientific Information Service, Geneva, 1968), pp. 537–555.

with the concept of diffraction dissociation.[5] Viewed in the rest system of the incoming pion, these experiments, under the hypothesis of limiting fragmentation, supply information on the *fragmentation of the pion* into three pions, etc. The mass and momentum-transfer distribution for such fragmentation would therefore directly yield information on the distributions (23) and (11).

13. The hypothesis of limiting fragmentation gives emphasis to the lab and projectile systems. In this it is very different from the statistical[6] model. In the latter model, the two incoming particles collide and *arrest* each other in the c.m. system, the final product of the collision being emitted from this arrested amalgamation of the original particles. Thus the c.m. system is the important reference system in the statistical model.

We now know, through experimental observation of the apparent tendency for the longitudinal momentum to persist, that the two incoming particles *do not, in general, arrest each other*—certainly not completely. Instead, they have a tendency to go through each other and in the process to break into fragments. This observation leads to the hypothesis of limiting fragmentation in which the fragments have finite momenta in either the lab or the projectile system.

Another way to compare the statistical model with the hypothesis of limiting fragmentation may be instructive. If the statistical model is correct, there will be *no* particles of any finite energy in the lab system. Thus all the limiting cross sections $\sigma_1, \sigma_2, \ldots$, of (11) are zero. Also $\rho_1=\rho_2=\cdots=0$.

14. The spirit of the hypothesis of limiting fragmentation is very much the same as that of the two-fireball model,[4] with or without pionization. The differences are as follows:

(a) The hypothesis as discussed in Secs. 1–7 is precisely defined, while the two-fireball model is not. Perhaps because of its lack of precise definition, the two-fireball model has not served as a useful guide for experiments at accelerator energies.

(b) An essential feature of the fireball model is that each fireball is assumed to decay more or less spherically symmetrically. This is not likely to be correct in the hypothesis of limiting fragmentation, as can be seen from the following argument: In ep collisions, only the proton can break up, i.e., there is only one fireball. For the case with a small q^2 [see (14)], the proton is essentially hit by a real photon of lab energy ν. For large ν one knows from high-energy γp experiments that the angular distribution in the c.m. system of γp is very much forward-backward, and bears no resemblance to spherical symmetry. If this is fitted to a fireball model, the fireball rest system being necessarily the c.m. system of γp, the fireball decay could not give rise to anything close to spherical symmetry in its rest system.

It is useful to observe that the fragments of the target are *more clearly separated* from pionization products and from fragments of the projectile in lab momentum space than they are in the c.m. momentum space. For example, in Fig. 1, the proton fragment exhibits a peaked differential cross section which drops off as one goes to high values of lab momenta. In contrast, this same curve in the c.m. system is flat versus the c.m. longitudinal momentum[10] and exhibits no tendency of separating the fragments of the target from other particles. In the light of this observation, it seems to us that it is better to think of *the two fireballs as limiting fragment distributions in the lab and projectile system, rather than as separated concentrations of particles in the c.m. momentum space.*

It is obvious that the hypothesis of limiting fragmentation is also very much similar in spirit to the isobar model.[17] The main difference is that under our hypothesis while the fragments may be the decay product of an isobar, they also may not be. For example, in pp collisions the target proton may become p^*, but it may also become a nonresonant "background" πp. In fact, for large momentum transfers, the latter dominates over the former (cf. Sec. 18).

15. The spirit of the hypothesis of limiting fragmentation is also very much the same as that of diffraction dissociation.[5] In fact, in the hypothesis of limiting fragmentation as formulated in (11), if one assumes that

$$\text{the total } G,\ I^2,\ I_z,\ N, \text{ and charge of the particles in } \sigma_n = \text{that of the target}, \quad (29)$$

one would have a more restricted hypothesis which can be considered as a precise statement of diffraction dissociation. We believe this restricted hypothesis is likely to be correct.[18] [The above discussion refers to hadron-hadron collisions. For lepton-hadron collisions, one must exercise caution in drawing specific conclusions. For example, in the collision $\nu n \rightarrow \mu^-$ plus hadrons, charge is transferred from the lepton to the hadron. See a discussion on pp. 515–516 in *High Energy Collisions*, edited by C. N. Yang *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1969).]

ADDITIONAL SPECULATIONS AND REMARKS

16. In cosmic-ray experiment, one often discusses the process of pionization[12] in which slow pions are supposed to be emitted more or less isotropically in the c.m. system. The need for the pionization process arises mostly from the increasing multiplicity observed at higher energies. If the hypothesis of limiting fragmentation is correct, we have already discussed before in Sec. 5 how to accommodate phenomena of increasing multiplicity. No pionization process is needed. If pioni-

[17] S. J. Lindenbaum and R. M. Sternheimer, Phys. Rev. **105**, 1874 (1957).

[18] For some recent experiments, see W. E. Ellis, T. W. Morris, R. S. Panvini, and A. M. Thorndike, in Proceedings of the Lund International Conference on Elementary Particles, Lund, Sweden, 1969 (unpublished).

zation processes are absent, then

$$\sum_{n=1}^{\infty}\int \sigma_n d^3p_1\cdots d^3p_n = \text{total cross section.}$$

In the following two paragraphs, we give additional arguments against the pionization process.

The pionization process implies, in the c.m. system, the arresting of the colliding particles with subsequent evaporation of slow pions. Such a picture is very far removed from a model of two extended objects *going through* each other as semitransparent bodies, a model that underlies the droplet interpretation of high-energy elastic scattering.[2,3]

The c.m. system in pp or πp collisions has no intrinsic significance once one emphasizes the hadrons as extended objects with many internal degrees of freedom. To illustrate the point, let us consider π-Pb collision. The c.m. system of such a collision is of great physical importance at low energies, when, for example, one wants to know the threshold energy for the production of another π meson. However, at high energies it is well known that the π-Pb c.m. system has no great physical importance. For the same reason, once we accept, as we must, the thesis that hadrons are extended objects with many internal degrees of freedom, the c.m. system in πp collisions loses its particular physical significance. (To be more concrete, e.g., in the quark model, if the pion is supposed to be made up of two quarks and the proton of three, then the πp c.m. system is *not* the same as the quark-quark c.m. system.)

17. In (23) we discussed the fragmentation of a particle into total mass M^* at a momentum transfer t. If there is no pionization process, there would be a corresponding fragmentation of the other particle into fragments at the same momentum transfer t. The cross section will be defined to be

$$\sigma(M_1^{*2},M_2^{*2},t)d(M_1^{*2})d(M_2^{*2})dt. \qquad (30)$$

This combined distribution may or may not factorize. If it does,

$$\sigma(M_1^{*2},M_2^{*2},t) = (\text{const})\sigma_{\text{target}}(M_1^{*2},t)\sigma_{\text{projectile}}(M_2^{*2},t). \qquad (31)$$

We rather believe that factorization (31) is not quantitatively valid because different processes (projectile $\rightarrow M_2^*$) should probably in general imply different excitations of the target. For example, in

$$ep \rightarrow ep^\dagger, \qquad (32)$$

$$pp \rightarrow pp^\dagger, \qquad (33)$$

$$pp \rightarrow p^*p^\dagger, \qquad (34)$$

the distribution of the different states $p^\dagger$ is probably qualitatively similar but quantitatively different for the three processes. In particular, for fixed t and M^*, the average number of hadrons that is contained in the fragment $p^\dagger$ in (32) and (33) are approximately the *same. This is an experimentally testable conclusion.* [On the other hand, the t dependence of (32) and (33) are expected to be quite *different*, as will be discussed in Sec. 18.]

18. In the fragmentation concept, the rapid decrease of elastic cross sections for large $t=q^2$ is a consequence of the idea that for large momentum transfers t, the hadron[2] breaks up in general into fragments. Consistent with this idea, it is to be expected that *for larger values of the momentum transfer t, the breakup process favors larger multiplicities of hadrons* (at fixed M^*). This particular point can be qualitatively tested in ep, μp, or hadron-hadron collisions, although exactly how the average multiplicity of the fragmentation process depends on t cannot be quantitatively predicted without a detailed model. (A great difficulty in formulating such a model lies in the following fact: Consider, say, ep collisions. *What absorbs the momentum transfer from the electron does not, in general, come out simply as one of the outgoing fragments of the proton.* Instead, it rapidly dissipates its energy-momentum to neighboring space-time points in the proton before the final fragmentation takes place. It seems that various models can be proposed to describe such a dissipation process, and one must look for guidance from future experiments.)

In fact, experimental data both from ep collisions[7] and hadron-hadron collisions[9] already give support to the speculation that the average multiplicity increases with increasing $t=q^2$ at a fixed M^*: For example, in ep collisions, the cross section (12) shows ridgelike structures when plotted against q^2 and M^*. The ridges are due to

$$ep \rightarrow ep^*, \qquad (35)$$

and the background under the ridges is due to

$$ep \rightarrow ep^\dagger, \text{ where } p^\dagger = p\pi \text{ or } p\pi\pi, \text{ etc.}, \neq p^*. \qquad (36)$$

In (35), the hadronic matter remains one piece (i.e., multiplicity of hadrons $=1$), while in (36), the hadronic matter breaks up into two or more pieces (i.e., multiplicity of hadrons $\geqq 2$). Experimentally at a fixed M^*, with increasing q^2, (35) becomes rapidly insignificant[7] compared with (36). The same is true in pp and πp collisions.[9] *We regard this behavior as one of the most striking features of the inelastic data, a feature confirming the fragmentation concept.*

The concept of breaking up under large momentum transfers t suggests that the *transverse momentum for each outgoing particle may not be large even though t is large.* This is a testable proposition in hadron-hadron and lepton-hadron collisions. One can draw additional qualitative conclusions from the breakup concept. Consider an ep collision at a fixed M^* of excitation of the proton [cf. (14)] and very large t. The multiplicity is limited by the value of M^*. Thus the individual transverse momenta cannot be all small. The net result is that the cross section would be small.

Speculations along this line cannot be made more precise lacking a complete theory of the fragmentation process. [Simple heuristic arguments lead, for ep and μp collisions, to

$$W_2 = A\exp(-\alpha q^2/\langle n\rangle), \tag{37}$$

where A and α are constants, and $\langle n\rangle$ is the average number of hadrons for the given q^2 and ν.]

The concept of fragmentation leads to a number of further testable qualitative features which we shall discuss in the rest of this section.

It is interesting to compare $\sigma(M^{*2},t)$ for ep collisions and pp or πp collisions. For hadron-hadron collisions, the experimental quantity easy to measure is not $\sigma(M^{*2},t)$, but a part of it, $\sigma_{pp\to pp^\dagger}(M^{*2},t)$, which represents those parts of σ where the incoming particle does not break up. This quantity, which we shall call σ in the rest of this paragraph, was measured by Anderson *et al.*[9] For the case that the target does not break up, i.e., $M^*=M_p$, σ represents elastic scattering and decreases with increasing t extremely fast. Our interpretation of this follows that of Ref. 2, namely, high-t elastic pp scattering is rare, because it is difficult to keep *two* protons intact: σ falls like F^4, the fourth power of the proton form factor. The same applies for the ridges in σ at M^* = resonance masses. For values of M^* in between resonances or beyond the resonance region, σ is expected to fall with increasing t like the product of F^2 (= the square of the form factor, since only one proton needs to be kept intact in this case) and a factor like (37). On the other hand, $\sigma_{ep\to ep^\dagger}(M^{*2},t)$ falls with t simply like (37).

19. Although throughout this paper we speak of limits at infinite incident energy, that is only to clarify, for an idealized case, the precise concepts under discussion. In practice, infinite energy is, of course, unattainable. Furthermore, it is quite possible that all cross sections, total or partial, have some dependence on the incoming energy through factors such as $(\ln E)^\beta$ with positive or negative values of β, or E^β with small fractional value for β. If that should turn out to be the case, the discussion of this paper may be taken to cover, at high incident energies, wide energy ranges in which the energy dependence is negligible.

SOME SUGGESTED EXPERIMENTS

20. It is very desirable to have more complete lab momentum distributions of various slow particles, p, π, K, $\bar{p}$, etc., in πp, pp, Kp, etc., collisions at high energies. In each case one wants to test whether the partial cross sections approach limits as defined in (1), (2), and (11).

One could study the same problem by measuring the momentum distribution of fast particles in the lab system and then transform to the projectile system (see Sec. 8.4). Especially interesting are the experiments of Ref. 16 mentioned in Sec. 12.

In this connection it is perhaps useful to point out some obvious kinematic facts: (a) Fast laboratory forward π and K in pp collisions are fragments of the projectile emitted *backwards* against the direction in which the projectile is hit by the target. (b) If the projectile does not break up, it will lose in the laboratory, in general, only a few BeV or less, to cause the excitation or breakup of the target [category (4), Table I]. However, if the projectile does break up, it will lose in the lab system a large amount of energy [category (3), Table I]. For example, consider a pp collision at 70 BeV. If the projectile breaks up into $p\pi$, or $p\pi\pi$, the proton will most likely lose in the lab system an energy of the order of $M_\pi/(M_\pi+M_p)$ (70 BeV) = 9 BeV or more.

21. Of particular interest, among the measurements mentioned above, are those relating to the dip (Sec. 9) and the values of $\sigma(pp^\dagger)$, $\sigma(p^\dagger p^\dagger)$, etc.

22. For ep and μp collisions, especially interesting experiments are (a) to measure the average multiplicity versus the four-momentum transfer t from the lepton to the hadrons, and (b) to measure the transverse momentum (i.e., perpendicular to incident momentum) of individual outgoing hadrons for the case of large t. The significance of these experiments was discussed at the end of Sec. 17 and in Sec. 18.

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APPENDIX A

We prove here formulas (9) and (10). [(4) and (5) are special cases of these formulas.]

By conservation of energy and momentum,

$$\sum(e-p_{\parallel}) = M_t + (E-p_{\rm inc}). \tag{A1}$$

At high energies, this becomes[19]

$$\sum(e-p_{\parallel}) = M_t. \tag{A2}$$

Consider a high-energy collision where the projectile and target undergo fragmentation, with or without additional pionization process yielding slow evaporation pions in the c.m. system. At very high energies, only the fragments of the target have finite energies in the lab system (compare Table I). Thus, only these particles contribute to the sum in (A2); hence we have (10). Since $e-p_{\parallel}$ is always positive but can be arbitrarily small, omitting some fragments of the target would immediately give (9).

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[19] Formula (A2) has been used in cosmic-ray experiments in connection with a concept called effective target mass. See N. G. Birger and Yu. A. Smorodin, Zh. Eksperim. i Teor. Fiz. **36**, 1159 (1959); **37**, 1355 (1959) [English transls.: Soviet Phys.—JETP **9**, 823 (1959); **10**, 964 (1960)].

Next we shall state the following simple theorem which gives a restatement of (10):

Theorem: In the breakup process of the target

$$\text{target} \to abcd \ldots ,$$

the quantity $\sum(e-p_{\parallel})$ is conserved:

$$\sum(e-p_{\parallel})=\text{const.} \tag{A3}$$

Furthermore, this conservation law is invariant in any Lorentz frame that moves with a velocity $<c$ along the longitudinal direction.

According to (22), the total transverse momentum of the fragments in (A3) has a magnitude of $\sqrt{t}$.

For the process

$$p \to p\pi\pi\pi \ldots , \tag{A4}$$

the quantity $M_p^{-1}(e-p_{\parallel})$ for the outgoing proton in the rest system of the original proton is the "elasticity" of the fragmenting projectile proton.

APPENDIX B

We prove here the statement in Sec. 9 about mesa-like superstructures for τ_n. Take $n=1$. The physical process is

$$\text{target} \to a^* \quad (a^*=\text{a resonance}), \tag{B1}$$

$$a^* \to bc. \tag{B2}$$

The components of lab momenta of a^* will be denoted by $p_{a\perp}$, $p_{a\parallel}$; its mass, by M^*. Then

$$M^{*2}=M_t^2+2M_t p_{a\parallel}-p_{a\perp}^2. \tag{B3}$$

[This is proved like (A2).] Now assume a^* to have zero width. For each $\mathbf{p}_a$, the lab momentum $\mathbf{p}_b$ of b from the decay (B2) lies *on* an ellipsoid. As $\mathbf{p}_a$ ranges over the paraboloid (B3), these ellipsoidal surfaces sweep over a region in $\mathbf{p}_b$ space (cf. Fig. 5). This region

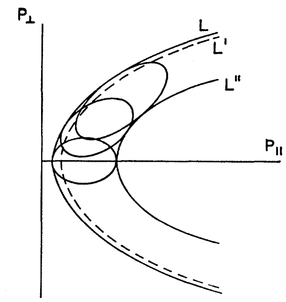

FIG. 5. Allowed values of lab momenta for particle b in processes (B1) and (B2).

becomes a mesa-like structure in the plot of $\tau_1(\mathbf{p}_b)$ versus $p_{b\parallel}$ and $p_{b\perp}$. To see this, consider that portion X of the ellipsoids between the envelope L and the surface L' to its right at a distance $dp_{\parallel}$ from L. The fractional area of each ellipsoid included in X is evidently proportional to $dp_{\parallel}$. Hence the partial cross section for $\mathbf{p}_b$ to lie between L and L' is $Ydp_{\parallel}$, where $Y\neq 0$. Thus between L and L'', (B1)-(B2) contributes a finite value to $\tau_1(\mathbf{p}_b)$, and this contribution does not vanish as one approaches L or L''; i.e., (B1) and (B2) contribute a mesa-like superstructure to $\tau_1(\mathbf{p}_b)$.

If a^* has a finite width, the bluffs of the mesa-like structure are rounded off.

Existence of mesa-like superstructures for $\tau_2, \tau_3, \ldots,$ can be shown in a similar way.

Contributions to τ_1 due to

$$\text{target} \to a^*,\quad a^* \to bcd$$

or

$$\text{target} \to a^*f,\quad a^* \to bc$$

can be discussed similarly.