Persistence of the Castillejo-Dalitz-Dyson Ambiguity in Relativistic Crossing-Symmetric Amplitudes*

D. ATKINSON†
Argonne National Laboratory, Argonne, Illinois 60439

AND

ROBERT LEE WARNOCK
Illinois Institute of Technology, Chicago, Illinois 60616 and Argonne National Laboratory, Argonne, Illinois 60439

(Received 11 August 1969)

The relativistic equations expressing analyticity, crossing, and unitarity are analyzed without approximations. An infinite family of solutions is constructed, corresponding to a Castillejo-Dalitz-Dyson (CDD) ambiguity in the s wave. This ambiguity is in addition to the one resulting from arbitrary inelastic functions. The amplitudes constructed have nonvanishing single spectral functions, and this implies that a Kronecker δ is present in the angular-momentum plane. This relation between CDD poles and the Kronecker δ is proved only within a certain limited range of the coupling strength. A computational program for reaching the interesting domain of large couplings is outlined. In the latter domain it is not expected that all CDD poles entail Kronecker δ's in the l plane.

I. INTRODUCTION

A CONSTRUCTIVE proof has recently been given[1] of the existence of functions that satisfy the following conditions:

(1) A Mandelstam representation holds with no subtractions.

(2) The crossing symmetry appropriate to pion-pion scattering is observed.

(3) The elastic unitarity condition holds below the four-pion threshold.

(4) The inelastic unitarity constraints are satisfied above the four-pion threshold.

It was shown that there exists an infinite number of functions that satisfy these conditions, corresponding to an infinite number of allowed input inelastic functions $v(s,t)$.

In the present work, it will be shown that there is a further infinity of functions, corresponding to the Castillejo-Dalitz-Dyson (CDD) ambiguity. For a given $v(s,t)$, we prove the existence of an infinite family of functions that satisfy conditions (1)–(4). This family is parametrized by the positions and residues of the CDD poles in the s wave.

It is expected that a similar analysis should be applicable to a function that satisfies a Mandelstam representation with, say, n subtractions, and that one would then be free to add CDD poles to the $n+1$ lowest partial waves. The CDD pole-free equations have been treated, for general n, in Ref. 2, but so far without condition (4) above, while Kupsch[3] has given an existence proof for the case $n=1$, with condition (4).

It is interesting to review briefly the history of the CDD ambiguity. Since the early work of Poincaré, Hilbert, and others, it has been known that singular integral equations with Cauchy kernels possess, in general, infinite families of solutions. It is clear from the book of Muskhelishvili[4] that an understanding of the linear Hilbert problem is a key to most of the problems of the theory of linear singular equations. The Hilbert problem is known to have an infinite family of solutions; an arbitrary rational function occurs as a factor in the general solution.

In 1956, Castillejo *et al.*[5] obtained explicitly the general solution of the Low equation for certain static models with simple crossing properties. Their solutions can be understood in terms of an auxiliary, linear Hilbert problem, and their celebrated ambiguity is exactly the ambiguity in solving this Hilbert problem. In the general static-model problem where the crossing matrix is arbitrary, one has a nonlinear Hilbert problem. No solution of the latter has been obtained in closed form, but for a small coupling strength solutions have been constructed by a convergent iterative procedure. Because of the particular form of nonlinearity arising from unitarity, a portion of the nonlinear problem is linearized by the N/D method. The D function is a solution of the linear Hilbert problem

$$D(s-i0)=e^{2i\delta(s)}D(s+i0)\,, \tag{1.1}$$

where $\delta(s)$ is the phase shift. In Refs. 6 and 7 it was shown that the ambiguity in the solution of this linear problem makes itself felt in the solution of the full nonlinear problem of either the static model or the Chew-Mandelstam equation. This was shown to be true in a

* Work partially supported by the National Science Foundation, and performed in part under the auspices of the U. S. Atomic Energy Commission.

† Present address: CERN, Geneva.

[1] D. Atkinson, Nucl. Phys. **B7**, 375 (1968); **B8**, 377 (1968).

[2] D. Atkinson, Nucl. Phys. **B13**, 415 (1969).

[3] J. Kupsch, Nucl. Phys. **B11**, 573 (1969); **B12**, 155 (1969).

[4] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

[5] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[6] D. Atkinson, J. Math. Phys. **8**, 2281 (1967).

[7] H. McDaniel and R. L. Warnock, Phys. Rev. **180**, 1433 (1969).

domain of small coupling constants. In this domain the nonlinearity due to crossing and unitarity is essentially irrelevant for existence of the CDD ambiguity. We cannot say what happens at large couplings (near the physical values) but at present we have no reason to expect the disappearance of the ambiguity.

By means of heuristic arguments and examination of soluble models, CDD poles have been associated with elementary particles in Lagrangian field theory.[8,9] An elementary particle is understood as being represented by a definite field operator in the free Lagrangian. A composite particle, on the other hand, does not enter the free Lagrangian, but appears as a bound state or a scattering resonance when the interaction is present.

It has been argued plausibly by Mandelstam[9] that an elementary field of spin σ will give rise to a Kronecker δ in the angular-momentum plane at $l=\sigma$ (unless special cancellations occur).[10] In that case, elementary fields may be ruled out by a principle of "maximum analyticity in the l plane,"[11] which would forbid Kronecker δ's. By the same principle we rule out any CDD poles which are due to elementary particles.

Within the class of amplitudes constructed in this paper, a CDD pole in the s wave does indeed give rise to a Kronecker δ at $l=0$. This is wholly consistent with the above picture, but this consistency may be traced to the fact that we work in a domain of weak couplings. When the couplings become strong, it is known that there can, in general, be another kind of CDD pole which is associated with important many-channel effects, and not necessarily with elementary particles at all. We expect that such a pole would not lead to a Kronecker δ.

The contents of the paper are arranged as follows. In Sec. II we state the equations which express unitarity, crossing, and analyticity. The s wave is treated explicitly by an N/D equation with only one CDD pole, but it would be a trivial extension to allow any finite number of CDD poles. The equations are interpreted as nonlinear operator equations in an appropriate function space. In Sec. III we analyze the N/D sector of the problem, while Sec. IV is devoted to the unitarity-crossing equations for the double-spectral functions. The results of Secs. III and IV together imply that the equations constitute a contraction mapping of a subspace of our function space into itself. Consequently, there is a unique solution in that subspace, and this may be constructed by iteration. In Sec. V we describe a program for extending our solutions into the region of large coupling strengths. We suggest stepwise applications of the Newton-Kantorovich iteration, for larger and larger values of the coupling. We show that our weak-coupling solutions depend analytically on a parameter multiplying the inelastic input function, and this procedure will amount to an analytic continuation in that parameter.

II. FORMULATIONS OF EQUATIONS

The equations for the double-spectral function and for the s-wave amplitude will be written down as a double mapping. In the notation of Ref. 2,

$$\rho(s,t)=[\bar{\rho}(t,s)+v(t,s)]+\beta[\bar{\rho}(s,t)+v(s,t)],\tag{2.1}$$

$$\bar{\rho}_I'(s,t)=\gamma_I{}^{M,N}\int_4 dt_1\int_4 dt_2 \times K(s;t,t_1,t_2)d_M{}^*(s,t_1)d_N(s,t_2),\tag{2.2}$$

where the upper limit of the double integration is defined by the first zero of K^{-2}, and

$$d(s,t)=\frac{1}{\pi}\int_4^\infty ds'\left[\frac{1}{s'-s}+\frac{\eta}{s'-u}-\frac{1+\eta}{t-4}\ln\left(1+\frac{t-4}{s'}\right)\right]\rho(t,s')+\sigma(t),\tag{2.3}$$

with $u=4-s-t$;

$$\sigma'(s)=\frac{1}{2q(s)}\times\frac{[1-\eta(s)][\mathrm{Re}D(s)]^2+[1+\eta(s)][q(s)n(s)]^2}{[\mathrm{Re}D(s)]^2+[q(s)n(s)]^2}.\tag{2.4}$$

Here

$$q(s)=[(s-4)/s]^{1/2},\tag{2.5}$$

and $n(s)$ satisfies the following linear integral equation[12]:

$$\eta(s)n(s)=B(s)+c_1\frac{B(s)-B(s_1)}{s-s_1}+\frac{1}{\pi}\int_4^\infty ds'\frac{B(s)-B(s')}{s-s'}q(s')n(s'),\tag{2.6}$$

where

$$B(s)=B_R(s)+B_L(s).\tag{2.7}$$

Here

$$B_R(s)=\frac{P}{\pi}\int_{16}^\infty\frac{ds'}{s'-s}\frac{1-\eta(s')}{2q(s')}\tag{2.8}$$

and

$$B_L(s)=\frac{1}{\pi}\int_{-\infty}^0\frac{ds'}{s'-s}\Delta(s'),\tag{2.9}$$

[8] See, e.g., M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

[9] S. Mandelstam, Phys. Rev. **137**, B949 (1965).

[10] M. Gell-Mann, M. L. Goldberger, F. E. Low, and F. Zachariasen, Phys. Letters **4**, 265 (1963); M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964); M. Gell-Mann, M. L. Goldberger, F. E. Low, V. Singh, and F. Zachariasen, *ibid.* **133**, B161 (1964).

[11] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); see also G. F. Chew, *The Analytic S-Matrix* (W. A. Benjamin, Inc., New York, 1966).

[12] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

where

$$\Delta(s)=\frac{2\beta}{4-s}\int_4^{4-s} dt\,\mathrm{Re}d(s,t)\,. \qquad (2.10)$$

In Eq. (2.4), the function $\mathrm{Re}D(s)$ is defined in terms of the solution of Eq. (2.6) by

$$\mathrm{Re}D(s)=1+\frac{c_1}{s-s_1}-\frac{P}{\pi}\int_4^\infty \frac{ds'}{s'-s}q(s')n(s')\,. \qquad (2.11)$$

In Eq. (2.1), $\bar{\rho}(s,t)$ is the elastic part of the double-spectral function, which is to be determined, while $v(s,t)$ is part of the inelastic contribution, and is assumed to be given. The isospin matrices β, η, and $\gamma_I{}^{M,N}$ have been given in Ref. 2, as has the well-known unitarity kernel $K(s;t,t_1,t_2)$. The subtraction term $\sigma(t)$ in Eq. (2.3) is the s-wave absorptive part of the amplitude. The elasticity parameter $\eta(s)$ is regarded as given. The CDD pole is located at the point $s_1>4$, at which the residue of the D function is c_1.

Equations (2.1)–(2.11) may be represented compactly as follows:

$$\bar{\rho}'=M(\bar{\rho},\sigma)\,, \qquad (2.12)$$

$$\sigma'=N(\bar{\rho},\sigma)\,. \qquad (2.13)$$

Our object is to find a fixed point of this double mapping: $\bar{\rho}'=\bar{\rho}$, $\sigma'=\sigma$. Once this fixed point has been located, the amplitude can be defined by the Mandelstam representation

$$F(s,t)=A(t,u)+\beta A(s,u)+\eta\beta\eta A(t,s)\,, \qquad (2.14)$$

where

$$A(t,u)=\frac{1}{\pi}\int_4^\infty \frac{ds'\rho(s')}{s'-s}+\frac{\beta}{\pi^2}\int_4^\infty dt'\int_{\sigma_0(t')}^\infty du'\frac{\rho(t',u')}{(t'-t)(u'-u)}\,, \qquad (2.15)$$

in which the single-spectral function is defined by

$$\rho(s)=\sigma(s)-\frac{1}{s-4}\int_{4-s}^0 dt\,[f(s,t)+\eta f(s,u)]\,, \qquad (2.16)$$

with

$$f(s,t)=\frac{1}{\pi^2}\int_{\sigma_0(s)}^\infty \frac{dt'\rho(s,t')}{t'-t}\,. \qquad (2.17)$$

The single-spectral function $\rho(s)$ ensures that the s-wave projection of Eq. (2.14) agrees with the N/D amplitude, and therefore satisfies unitarity (for the s wave). Provided that $D(s)$ has no zero on the physical sheet, the N/D amplitude and the s-wave projection must be equal, because their discontinuities on the right-hand and left-hand cuts are the same, and they both vanish asymptotically. As in Ref. 1, inelastic unitarity constraints in higher partial waves will be satisfied by imposing suitable restrictions on the input function $v(s,t)$.

The existence of a fixed point of the mappings (2.12) and (2.13) will be proved by an application of the contraction mapping theorem. The operators M and N will be defined on the direct product $\mathfrak{A}$ of two Banach spaces $\mathfrak{B}$ and $\mathfrak{C}$. The space $\mathfrak{B}$, which is to contain $\bar{\rho}(s,t)$, is defined to be the set of all real functions $F(s,t)$ with domain $4\leqslant s,t<\infty$, for which

$$\|F\|\equiv \sup_{4\leqslant s_1,s_2,t_1,t_2<\infty} |F(s_1,t_1)-F(s_2,t_2)|\ln^2 s\ln^2 t \Bigg/ \left[\left|\frac{(s_1-s_2)}{s_1s_2\bar{t}}\right|^\mu+\left|\frac{(t_1-t_2)}{t_1t_2\bar{s}}\right|^\mu\right] \qquad (2.18)$$

exists, where sup means least upper bound, $\bar{s}=\min(s_1,s_2)$, $\bar{t}=\min(t_1,t_2)$, and the Hölder index μ is a fixed number that satisfies $0<\mu<\frac{1}{2}$, and for which

$$\lim_{s\to\infty}F(s,t)=0=\lim_{t\to\infty}F(s,t)\,. \qquad (2.19)$$

The space $\mathfrak{C}$, which will contain $\sigma(s)$, is the set of all real, continuous functions $f(s)$ with domain $4\leqslant s<\infty$, for which

$$\|f\|\equiv \sup_{s\in C\Omega}|s^{2\mu}f(s)|+\epsilon^{1/2}\sup_{s\in\Omega}|f(s)| \qquad (2.20)$$

exists, where Ω is the interval $[s_1-\epsilon,s_1+\epsilon]$, with $s_1-\epsilon>4$, and $C\Omega$ is the complement of Ω with respect to $[4,\infty)$. Here ϵ is a small, positive number, the magnitude of which will be dictated by the technical requirements of the proof.

Linear combinations in the product space $\mathfrak{A}=\mathfrak{B}\times\mathfrak{C}$ are defined by

$$\lambda_1(F_1,f_1)+\lambda_2(F_2,f_2)=(\lambda_1F_1+\lambda_2F_2,\lambda_1f_1+\lambda_2f_2)\,, \qquad (2.21)$$

and, for the norm in the product space, we take

$$\|(F,f)\|\equiv\max\{\|F\|,\|f\|\}\,. \qquad (2.22)$$

The fixed point $(\bar{\rho},\sigma)$ will be sought in the subset

$$\mathfrak{A}_1=\mathfrak{B}_1\times\mathfrak{C}_1\,. \qquad (2.23)$$

Here $\mathfrak{B}_1$ is defined by the requirements

$$\|\bar{\rho}\|\leqslant b\,, \qquad (2.24)$$

and

$$\rho(s,t)=0\,, \qquad (2.25)$$

for

$$s\leqslant\min\left(\frac{4t}{t-16},\frac{16t}{t-4}\right)\,. \qquad (2.26)$$

The set $\mathfrak{C}_1$ is defined by

$$\|\sigma\|\leqslant c\,. \qquad (2.27)$$

Since $\mathfrak{C}_1$ is a closed subset of a Banach space, it is itself a complete metric space in the norm topology.

According to the contraction mapping principle, there is a unique fixed point, in $\mathfrak{A}_1$, of the mapping (2.12), (2.13), provided $\mathfrak{A}_1$ is mapped into itself, and the following Lipschitz condition holds:

$$\|(\bar{\rho}_1',\sigma_1')-(\bar{\rho}_2',\sigma_2')\|\leqslant\kappa\|(\bar{\rho}_1,\sigma_1)-(\bar{\rho}_2,\sigma_2)\|, \quad (2.28)$$

for any $(\bar{\rho}_1,\sigma_1)$ and $(\bar{\rho}_2,\sigma_2)$ belonging to $\mathfrak{A}_1$, where $\kappa<1$. In Sec. III, it is shown that, under suitable restrictions on b, c, c_1, ϵ, $\eta(s)$, and $v(s,t)$, the operator N takes $\mathfrak{A}_1\rightarrow\mathfrak{C}_1$ with

$$\|N(\bar{\rho}_1,\sigma_1)-N(\bar{\rho}_2,\sigma_2)\| \leqslant\kappa\max\{\|\bar{\rho}_1-\bar{\rho}_2\|,\|\sigma_1-\sigma_2\|\}, \quad (2.29a)$$

and in Sec. IV that M takes $\mathfrak{A}_1\rightarrow\mathfrak{B}_1$ with

$$\|M(\bar{\rho}_1,\sigma_1)-M(\bar{\rho}_2,\sigma_2)\| \leqslant\kappa\max\{\|\bar{\rho}_1-\bar{\rho}_2\|,\|\sigma_1-\sigma_2\|\}. \quad (2.29b)$$

These conditions imply (2.28), and so conclude the existence proof.

The constraints which are imposed upon $\eta(s)$ and $v(s,t)$, for the proof of Eqs. (2.29a) and (2.29b), will now be described. Let $\phi(s)=[1-\eta(s)]/2q(s)$. Unitarity requires $\phi=0$, $4\leqslant s\leqslant 16$. We assume that $\phi(s)$ is twice differentiable, $4\leqslant s<\infty$, and that

$$\phi(\infty)=\phi'(\infty)=\phi''(\infty)=0, \quad (2.30)$$

$$|\phi''(s)-\phi''(s')|\leqslant O(\|\phi\|)\left|\frac{s-s'}{ss'}\right|^{2\mu}\frac{1}{\bar{s}^2}, \quad (2.31)$$

where

$$\|\phi\|=\sup_{16\leqslant s\leqslant\infty}|s^{2\mu}\phi(s)|<\infty, \qquad \bar{s}=\min(s,s'). \quad (2.32)$$

Here and in the sequel, $y=O(x)$ means that $|y|\leqslant M|x|$, for some $M>0$, at all x. As in (2.31), we also write $y\leqslant O(x)z$, meaning $\sup|y/z|\leqslant M|x|$. The least possible value of M, although definite in (2.31), may be different at different places where the symbol $O(x)$ is employed. From (2.30) and (2.31) it follows immediately that

$$|\phi(s)-\phi(s')|\leqslant O(\|\phi\|)\left|\frac{s-s'}{ss'}\right|^{2\mu}, \quad (2.33)$$

$$|\phi'(s)-\phi'(s')|\leqslant O(\|\phi\|)\left|\frac{s-s'}{ss'}\right|^{2\mu}\frac{1}{\bar{s}}. \quad (2.34)$$

Since η is (sectionally) the boundary value of an analytic function, existence of ϕ'' between thresholds is assured. The lowest threshold, $s=16$, is the most doubtful regarding differentiability. Studies of threshold behavior of the s-wave 4π state[13] indicate, however, a behavior like $(s-16)^{7/2}$, which implies that ϕ'' exists and is Hölder-continuous.

The function $v(s,t)$ must belong to the space $\mathfrak{B}$, and must satisfy

$$v(s,t)=0 \quad (2.35)$$

for all s if $t\leqslant 16$ and for $s\leqslant\sigma_1(t)$ if $t\geqslant 16$, where $s=\sigma_1(t)$, the boundary of the support of $v(s,t)$, must satisfy $\sigma_1(t)\geqslant 16$ for all $t\geqslant 16$. In addition, the following positivity conditions will be imposed, which will be necessary for the proof of the inelastic unitarity bounds:

$$v(s,t)\geqslant 0 \quad (2.36)$$

for all s and t, and also

$$g(s,t)\geqslant 0 \quad\text{and}\quad \beta g(s,t)\geqslant 0 \quad (2.37)$$

for $4\leqslant s\leqslant 20$ and $16\leqslant t<\infty$, where

$$g(s,t)=P\int_{\sigma_1(t)}^{\infty}ds' \times\left[\frac{1}{s'-s}+\frac{\eta}{s'-u}-\frac{2}{t-4}\ln\left(1+\frac{t-4}{s'}\right)\right] \times[v(s',t)+\beta v(t,s')]. \quad (2.38)$$

It has been shown in Ref. 1 that one can find functions $v(s,t)$ that satisfy these positivity requirements. Moreover, one can work the proof with the weaker requirements of the Introduction of Ref. 2, according to which $v(s,t)$ may oscillate, but for simplicity this refinement will be omitted here.

III. ANALYSIS OF s-WAVE N/D EQUATION

In abbreviated notation the N/D equation (2.6) is written as

$$n=f+Kn, \quad (3.1)$$

where

$$f(s)=\eta^{-1}(s)\left[B(s)+c_1\frac{B(s)-B(s_1)}{s-s_1}\right], \quad (3.2)$$

$$Kx(s)=\int_4^{\infty}\frac{K(s,s')x(s')ds'}{s-s'}, \quad (3.3)$$

$$K(s,s')=\eta^{-1}(s)[B(s)-B(s')]q(s'). \quad (3.4)$$

We study (3.1) as an equation in the Banach space $\mathfrak{D}$ of all real, continuous functions $x(s)$ on $[4,\infty)$ with the norm

$$\|x\|=\sup_{4\leqslant s<\infty}|s^{2\mu}x(s)|. \quad (3.5)$$

The exponent μ is to be the same as in (2.18) and (2.20). We shall make sure that $f\in\mathfrak{D}$ and that K is a bounded, completely continuous operator on $\mathfrak{D}$. Then, with f and K fixed, the Fredholm theorems imply that (3.1) has a unique solution in $\mathfrak{D}$, provided that $\|K\|$ is less than 1. The norm of K is defined by

$$\|K\|=\sup_{x\in\mathfrak{D}}\frac{\|Kx\|}{\|x\|}. \quad (3.6)$$

[13] L. M. Delves, Nucl. Phys. 9, 391 (1958).

Our analysis of the nonlinear mapping, (2.12) and (2.13), will require the following behavior of n:

$$\|n\|=O(\psi), \tag{3.7}$$

$$|n(s)-n(s')|\leqslant O(\psi)\left|\frac{s-s'}{ss'}\right|^{2\mu}. \tag{3.8}$$

We use the convenient notation ψ for the quantity

$$\psi=\max(b,c,\|v\|,\|1-\eta\|), \tag{3.9}$$

where b and c are defined in (2.24) and (2.27). The scale of our scattering amplitudes is set by ψ, which will be small (in an appropriate sense).

Suppose that $\|K\|\leqslant k<1$, and let n be the unique solution of (3.1). Then

$$\|n\|\leqslant\|f\|+\|K\|\times\|n\| \tag{3.10}$$

and

$$\|n\|\leqslant\|f\|/(1-k). \tag{3.11}$$

To get such a bound k, we note the inequality

$$\|Kx\|=\sup_s\left|s^{2\mu}\int\frac{K(s,t)}{s-t}t^{-2\mu}[t^{2\mu}x(t)]dt\right|$$

$$\leqslant\|x\|\sup_s\left|\int\frac{K(s,t)}{s-t}\left(\frac{s}{t}\right)^{2\mu}dt\right|, \tag{3.12}$$

i.e.,

$$\|K\|\leqslant\sup_s\left|\int\frac{K(s,t)}{s-t}\left(\frac{s}{t}\right)^{2\mu}dt\right|. \tag{3.13}$$

Similarly, to derive the bound (3.8) we begin with the inequality

$$|n(s)-n(s')|\leqslant|f(s)-f(s')|+\|n\|\int\left|\frac{K(s,t)}{s-t}-\frac{K(s',t)}{s'-t}\right|\frac{dt}{t^{2\mu}}. \tag{3.14}$$

After these remarks, it is easy to state conditions on f and K which guarantee that the N/D equation has a unique solution in $\mathfrak{D}$ with the properties (3.7) and (3.8):

(a) $\|f\|=O(\psi)$,

(b) $|f(s)-f(s')|\leqslant O(\psi)\left|\frac{s-s'}{ss'}\right|^{2\mu}$,

(c) $|K(s,s')|\leqslant O(\psi)\left|\frac{s-s'}{ss'}\right|^{2\mu}$, (3.15)

(d) $\left|\frac{K(s,t)}{s-t}-\frac{K(s',t)}{s'-t}\right|\leqslant O(\psi)\frac{1}{t^\lambda}\left|\frac{s-s'}{ss'}\right|^{2\mu}$, $\lambda+2\mu>1$

(e) ψ is sufficiently small.

By definition, f is a real, continuous function on $[4,\infty)$, so (a) means that $f\in\mathfrak{D}$. Condition (c) and (3.13) show that K is a bounded operator on $\mathfrak{D}$. If ψ is suitably small, then $\|K\|<1$. Conditions (b) and (d) with (3.14) show that K maps any bounded set in $\mathfrak{D}$ into a uniformly bounded, equicontinuous set of functions. By reference to Ascoli's theorem it follows that K is completely continuous. Finally, (a)–(d) combined with (3.11) and (3.14) yield the desired properties (3.7) and (3.8) of the solution $n(s)$.

The properties (3.15a)–(3.15d) are easily derived from the results of Sec. IV. The details of the derivation are explained in Appendix A.

Our main task in this section is to analyze the s-wave absorptive part constructed from the unique solution of the N/D equation. This is expressed by the formula

$$\sigma(s)=\operatorname{Im}\frac{N}{D}=\frac{(1-\eta)(\operatorname{Re}D)^2+(1+\eta)(qn)^2}{2q[(\operatorname{Re}D)^2+(qn)^2]}, \tag{3.16}$$

$$D(s)=1+P(s)-\operatorname{Re}I(s)-iq(s)n(s),$$

where

$$\operatorname{Re}I(s)=\frac{P}{\pi}\int_4^\infty\frac{q(s')n(s')ds'}{s'-s},\quad P(s)=\frac{c_1}{s-s_1}. \tag{3.17}$$

Unitarity is not *explicit* in (3.16) when $\eta\neq1$, but one may nevertheless show[14] that N/D is indeed unitary, whenever n satisfies (3.1). Thus, we have the unitarity bound

$$\sigma(s)\leqslant(1+\eta)/2q\leqslant1/q. \tag{3.18}$$

The first step is to control the magnitude of the integral which appears in ReD. It is decomposed as follows:

$$\operatorname{Re}I(s)=\frac{P}{\pi}\int_4^\infty\frac{q(s')n(s')ds'}{s'-s}=\frac{1}{\pi}[I_1+I_2+I_3+I_4]$$

$$=\frac{1}{\pi}\int_{(1-\epsilon)s}^{(1+\epsilon)s}q(s')\left[\frac{n(s')-n(s)}{s'-s}\right]ds'$$

$$+n(s)\frac{P}{\pi}\int_{(1-\epsilon)s}^{(1+\epsilon)s}\frac{q(s')ds'}{s'-s}$$

$$+\frac{1}{\pi}\left[\int_4^{(1-\epsilon)s}+\int_{(1+\epsilon)s}^\infty\right]\frac{q(s')n(s')ds'}{s'-s}. \tag{3.19}$$

By (3.8) and the substitution $s'=sx$ one finds

$$|I_1|\leqslant O(\psi)\int_{1-\epsilon}^{1+\epsilon}\frac{dx}{x^{2\mu}|1-x|^{1-2\mu}}=O(\psi). \tag{3.20}$$

[14] R. L. Warnock, in *Lectures in Theoretical High Energy Physics*, edited by H. Aly (Wiley-Interscience, Inc., New York, 1968), Chap. 10.

Also, (3.7) shows that I_2, I_3, and I_4 are all $O(\|n\|)$. Hence,

$$\mathrm{Re}I(s)=O(\psi), \tag{3.21}$$

uniformly in s.

The following analysis proceeds by discussing separately a neighborhood Ω of the CDD pole. Define

$$\Omega=[s_1-\epsilon, s_1+\epsilon], \quad s_1-\epsilon>4$$
$$C\Omega=[4,\infty)-\Omega. \tag{3.22}$$

The pole term of (3.17) is bounded in $C\Omega$:

$$\sup_{C\Omega}|P(s)|=\left|\frac{c_1}{\epsilon}\right|. \tag{3.23}$$

We choose $|c_1/\epsilon|$ and ψ to be so small that

$$\left|\frac{c_1}{\epsilon}\right|+\sup_s|\mathrm{Re}I|<1. \tag{3.24}$$

This means that $\mathrm{Re}D(s)\neq 0$ when s is in $C\Omega$, and that $\sigma(s)$ may be given a small bound for s in that region. In Ω, on the other hand, $\mathrm{Re}D$ may vanish and we have only the large unitary bound (3.18).

In the region $C\Omega$ of good behavior, we have

$$\sup_{C\Omega}|s^{2\mu}\sigma|\leqslant\frac{2\|(1-\eta)q^{-1}\|+8^{-\mu}\|n\|^2}{(1-|c_1/\epsilon|-\sup|\mathrm{Re}I|)^2}. \tag{3.25}$$

In Ω, on the other hand,

$$\sup_\Omega|\sigma|\leqslant\frac{1}{q(s_1-\epsilon)}\equiv\frac{1}{q_1}. \tag{3.26}$$

It follows that

$$\|\sigma\|=\sup_{C\Omega}|s^{2\mu}\sigma|+\epsilon^{1/2}\sup_\Omega|\sigma|$$
$$=O[\max(\|1-\eta\|,\psi^2,\epsilon^{1/2})]. \tag{3.27}$$

We see that the set $\mathfrak{A}_1$ of Sec. II is mapped into $\mathfrak{C}_1$ by the N/D operator, provided the quantities ψ, ϵ, and $|c_1/\epsilon|$ are sufficiently small, i.e.,

$$\|\sigma'\|\leqslant c. \tag{3.28}$$

Next we must establish the contraction property (2.29a) of the N/D operator: For all $(\bar{\rho}_1,\sigma_1)$ and $(\bar{\rho}_2,\sigma_2)$ in $\mathfrak{A}_1$, we must have

$$\|N(\sigma_1,\bar{\rho}_1)-N(\sigma_2,\bar{\rho}_2)\|$$
$$\leqslant\kappa\max(\|\bar{\rho}_1-\bar{\rho}_2\|,\|\sigma_1-\sigma_2\|)\equiv\kappa d_{12}, \quad \kappa<1. \tag{3.29}$$

We write $\sigma=x/y$, $y=|D|^2$. Then

$$|\sigma_1-\sigma_2|\leqslant(1/y_1)(|x_1-x_2|+\sigma_2|y_1-y_2|), \tag{3.30}$$

$$|x_1-x_2|\leqslant\frac{1-\eta}{2q}|(\mathrm{Re}D_1)^2-(\mathrm{Re}D_2)^2|+\frac{1+\eta}{2}q|n_1-n_2|$$
$$\times|n_1+n_2|, \tag{3.31}$$

$$|y_1-y_2|\leqslant|(\mathrm{Re}D_1)^2-(\mathrm{Re}D_2)^2|+q^2|n_1-n_2|$$
$$\times|n_1+n_2|, \tag{3.32}$$

$$|(\mathrm{Re}D_1)^2-(\mathrm{Re}D_2)^2|\leqslant[2(1+|P|)+|\mathrm{Re}(I_1+I_2)|]$$
$$\times|\mathrm{Re}(I_1-I_2)|. \tag{3.33}$$

In Appendix B it is shown that

$$\|n_1-n_2\|=O(d_{12}), \tag{3.34}$$

$$|(n_1-n_2)(s)-(n_1-n_2)(s')|\leqslant O(d_{12})\left|\frac{s-s'}{ss'}\right|^{2\mu}. \tag{3.35}$$

When (3.35) is combined with the argument of Eq. (3.19) and following, we get

$$|\mathrm{Re}(I_1-I_2)|=O(d_{12}). \tag{3.36}$$

From Eqs. (3.30)–(3.36) one finds the required bound in $C\Omega$:

$$\sup_{C\Omega}|s^{2\mu}(\sigma_1-\sigma_2)|=O(\psi)d_{12}. \tag{3.37}$$

In Ω the situation is more delicate, since we must be able to bound $1/y_1$ in spite of a possible zero of $\mathrm{Re}D_1$, and because the pole term P occurs in the numerator via (3.33). We bound $1/y_1$ by forbidding $n(s)$ to vanish in Ω. Then

$$\frac{1}{y_1}=\frac{1}{|D_1|^2}\leqslant\frac{1}{(q_1n_0)^2},$$
$$|n(s)|\geqslant n_0>0, \quad s\in\Omega. \tag{3.38}$$

Presently we shall describe the method for preventing zeros of n in Ω, which gives an n_0 vanishing as ψ:

$$n_0=O(\psi). \tag{3.39}$$

To handle the pole term P in (3.33), we pick out a subset θ of Ω as follows:

$$\theta=\{\omega\,|\,|c_1/\epsilon|\leqslant|P|<3\}. \tag{3.40}$$

In θ we employ the bound

$$|P|/y_1\leqslant 3/(q_1n_0)^2, \quad s\in\theta. \tag{3.41}$$

In $\Omega-\theta$ we have

$$\frac{|P|}{y_1}\leqslant\frac{|P|^{-1}}{[(1-\mathrm{Re}I_1)/P-1]^2}<3, \quad s\in\Omega-\theta, \tag{3.42}$$

since $|1-\mathrm{Re}I_1|<2$, $|P|\geqslant 3$. Thus, $\sup_\Omega|\sigma_1-\sigma_2|=O(d_{12}/n_0^2)$, and

$$\|\sigma_1-\sigma_2\|=O(\psi+\epsilon^{1/2}n_0^{-2})d_{12}. \tag{3.43}$$

If we take $\epsilon=O(\psi^\lambda)$, $\lambda>4$, then for sufficiently small ψ we have the required contraction property

$$\|\sigma_1-\sigma_2\|\leqslant\kappa d_{12}, \quad \kappa<1. \tag{3.44}$$

To rule out zeros of n in Ω, we make the following observation: The contribution of $v(s,t)$ to the left-cut term can be made to dominate the N/D equation. This part of the left-cut term is denoted by $B_v(s)$:

$$B_v(s)=\frac{2\beta}{\pi}\int_{-\infty}^{0}\frac{ds'}{s'-s}\frac{1}{4-s'}\int_{4}^{4-s'}dt\,\mathrm{Re}g(s',t)\,,\quad(3.45)$$

where $g(s,t)$ was defined in Eq. (2.38). Since $B_v(s)$ is analytic and not identically zero in a neighborhood of the right cut, it is certainly possible to find some closed interval $[s_1-\epsilon, s_1+\epsilon]$, $s_1-\epsilon>4$, in which $B_v(s)$ has no zero. We identify this interval with Ω, putting the CDD pole at $s=s_1$. The difference between $\eta(s)n(s)$ and $B_v(s)$ can be made small enough to ensure that $n(s)$ and $B_v(s)$ have the same sign in Ω, and that $n(s)$ does not vanish in Ω. That is, we prove existence of a $\beta_0>0$ such that

$$\sup_{\Omega}|\eta(s)n(s)-B_v(s)|\leqslant\beta_0<\inf_{\Omega}|B_v(s)|\,.\quad(3.46)$$

Therefore, we have a lower bound n_0:

$$|n(s)|\geqslant\sup_{\Omega}(1/\eta)[\inf_{\Omega}|B_v(s)|-\beta_0]=n_0\,,\quad s\in\Omega.\quad(3.47)$$

To find β_0, one applies the N/D equation (3.1) to obtain

$$|\eta(s)n(s)-B_v(s)|$$
$$\leqslant\|C-B_v\|+|c_1|\left\|\frac{C(s)-C(s_1)}{s-s_1}\right\|+\|K\|\times\|n\|$$
$$=O(\|\bar\rho\|+\|\sigma\|)+|c_1|O(\psi)+O(\psi^2)\,.\quad(3.48)$$

Now $c_1=O(\psi^\lambda)$, $\lambda>4$, since $|c_1|<\epsilon=O(\psi^\lambda)$. If $\|\bar\rho\|+\|\sigma\|$ and ψ are sufficiently small, then the right-hand side of (3.48) may be made less than $\inf_\Omega|B_v|$. Although $B_v=O(\|v\|)=O(\psi)$, the terms of the right-hand side are of higher order in ψ, or else independent of v, and therefore can be relatively small. Since $\inf_\Omega|B_v|$ and the right-hand side of (3.48) are $O(\psi)$, we obtain the bound (3.39) on n_0.

The dominance of $v(s,t)$ over the elastic spectral function $\bar\rho(s,t)$ is also a feature of the second part of our proof described in Sec. IV.

To complete the discussion of the s-wave equation we must show that there are no ghosts, i.e., no zeros of the D function on the physical sheet. This is done by showing that outside a particular circle with center at s_1 the integral $-I(z)$ is too small to cancel the remainder $1+P(z)$ of $D(z)$. Inside this circle we use the positive (negative) definite property of $n(\mathrm{Re}z)$ to show that $\mathrm{Im}D(z)$ cannot vanish.

The uniform bound (3.21) is easily extended to the entire cut plane. Consequently, for all z we have

$$|D(z)-1|\leqslant\left|\frac{c_1}{z-s_1}\right|+O(\psi)\,.\quad(3.49)$$

For $|z-s_1|\geqslant 2c_1$, say, D cannot vanish if ψ is sufficiently small. To treat the interior of this circle, we suppose that $n(s)>0$ for $s\in\Sigma=[s_1-3c_1, s_1+3c_1]$ and take $c_1>0$. [The same argument will apply if $n(s)<0$, $c_1<0$.] We make $3c_1<\epsilon$, so that $\Sigma\subset\Omega$. In Ω the property $\mathrm{Im}D=-\rho n\neq0$ is already established. For $\mathrm{Im}z\neq0$, we employ the identity

$$\mathrm{Im}D(z)=-\mathrm{Im}z\left\{\frac{c_1}{|z-s_1|^2}+\frac{1}{\pi}\int_{s_1-3c_1}^{s_1+3c_1}\frac{q(s)n(s)ds}{|s-z|^2}\right.$$
$$+n(\mathrm{Re}z)\frac{1}{\pi}\left[\int_4^{s_1-3c_1}+\int_{s_1+3c_1}^{\infty}\right]\frac{ds}{|s-z|^2}$$
$$\left.+\frac{1}{\pi}\left[\int_4^{s_1-3c_1}+\int_{s_1+3c_1}^{\infty}\right]\frac{ds[n(s)-n(\mathrm{Re}z)]}{|s-z|^2}\right\}.\quad(3.50)$$

The first three terms in the bracket are positive. We show that the fourth term, although not necessarily positive, is less than $c_1/|z-s_1|^2$ in magnitude. In view of Eq. (3.8), the fourth term is majorized by the expression

$$\frac{1}{\pi}\left[\int_4^{s_1-3c_1}+\int_{s_1+3c_1}^{\infty}\right]ds\left|\frac{n(s)-n(\mathrm{Re}z)}{s-\mathrm{Re}z}\right|\frac{1}{|s-\mathrm{Re}z|}$$
$$\leqslant O(\psi)(\mathrm{Re}z)^{-2\mu}\left(\int_4^{s_1-3c_1}\frac{ds}{s^{2\mu}|s-s_1+2c_1|^{2-2\mu}}\right.$$
$$\left.+\int_{s_1+3c_1}^{\infty}\frac{ds}{s^{2\mu}|s-s_1-2c_1|^{2-2\mu}}\right).\quad(3.51)$$

We must know how this term behaves for small c_1. To ascertain the behavior one can change to a new integration variable u, where $s=s_1+c_1u$, in the integrals on the right-hand side of (3.51). The result is that (3.51) is bounded by $\zeta\psi c_1^{-1+2\mu}$, where ζ is a constant. The first term in (3.50), on the other hand, has a lower bound in the circle:

$$c_1/|z-s_1|^2\geqslant1/4c_1.\quad(3.52)$$

We merely have to choose ψ so small that $\zeta\psi c_1^{2\mu}<4$, to rule out zeros of $\mathrm{Im}D$ in the circle.

IV. ANALYSIS OF DOUBLE-SPECTRAL FUNCTION

The first problem in this section is to show that Eqs. (2.1)–(2.3), which were summarized by the formula

$$\bar\rho'=M(\bar\rho,\sigma)\quad(4.1)$$

of Eq. (2.12), map $\mathfrak{A}_1$ into $\mathfrak{B}_1$, for a suitably small value of the parameter ψ of Eq. (3.9).

Suppose one writes Eq. (2.3) as

$$d(s,t)=d_1(s,t)-d_2(t)+\sigma(t)\,,\quad(4.2)$$

where

$$d_1(s,t)=\frac{1}{\pi}\int_4^\infty ds'\left[\frac{1}{s'-s}+\frac{\eta}{s'-u}\right]\rho(t,s'),$$
$$d_2(t)=\frac{1+\eta}{\pi}\frac{1}{t-4}\int_4^\infty ds'\rho(t,s')\ln\left(1+\frac{t-4}{s'}\right). \tag{4.3}$$

As in Ref. 2, one can show that

$$|d_1(s,t)|\leqslant O(\psi)(st)^{-\mu}\ln^{-2}s\ln^{-2}t \tag{4.4}$$

and

$$|d_1(s_1,t)-d_1(s_2,t)|\leqslant O(\psi)\left|\frac{s_1-s_2}{s_1s_2t}\right|^{\mu}\ln^{-2}\bar{s}\ln^{-2}t. \tag{4.5}$$

Since $d_2(t)$ and $\sigma(t)$ do not depend on s, it follows that an equation similar to Eq. (4.5) holds also for $d(s,t)$. However, an equation like (4.4) does not hold for $d_2(t)$ and $\sigma(t)$. Instead, one has

$$|d_2(t)|\leqslant O(\psi)t^{-2\mu} \tag{4.6}$$

for $4\leqslant t<\infty$, and also

$$|\sigma(t)|\leqslant O(\psi)[1+\epsilon^{-1/2}\theta_\Omega(t)]t^{-2\mu}, \tag{4.7}$$

where $\theta_\Omega(t)=1$ for $s_1-\epsilon\leqslant t\leqslant s_1+\epsilon$ and $\theta_\Omega(t)=0$ otherwise. One can then show, following a slight generalization of Ref. 2, which is outlined in Appendix C, that Eqs. (4.4)–(4.7) imply

$$\|\bar{\rho}'\|=O(\psi^2). \tag{4.8}$$

Hence one can certainly choose ψ so small that the right-hand side of Eq. (4.8) is less than ψ, so that $\mathfrak{A}_1\to\mathfrak{B}_1$.

Next, in a closely similar way, as in Refs. 1 and 2, one can show that, given $(\bar{\rho}_1,\sigma_1)$ and $(\bar{\rho}_2,\sigma_2)$ belonging to $\mathfrak{A}_1$,

$$\|\bar{\rho}_1'-\bar{\rho}_2'\|\leqslant O(\psi)\max(\|\bar{\rho}_1-\bar{\rho}_2\|,\|\sigma_1-\sigma_2\|). \tag{4.9}$$

Hence, if ψ is small enough, one has proved the condition (2.29b), which completes the contraction-mapping proof.

It remains to be shown that the positivity constraints (2.36)–(2.38) suffice to demonstrate that the partial-wave projection of $F(s,t)$ [Eq. (2.14)],

$$F_l(s)=\frac{1}{s-4}\int_{4-s}^0 dt\,P_l\left(1+\frac{2t}{s-4}\right)F(s,t), \tag{4.10}$$

satisfies the inelastic unitarity condition

$$\mathrm{Im}F_l(s)\geqslant q(s)|F_l(s)|^2, \tag{4.11}$$

for $l=1, 2, 3, \ldots$, and $s\geqslant16$. For the s wave, (4.11) is guaranteed by the fact that $F_0(s)$ agrees with N/D, and that $0\leqslant\eta(s)\leqslant1$, of course.

The proof of (4.11) is closely modeled on Sec. 3 of the second paper in Ref. 1. One defines first the subset of $\mathfrak{B}_1\times\mathfrak{C}_1$ obtained by adding the conditions

$$\beta P\int_4^\infty ds'\left(\frac{1}{s'-s}+\frac{\eta}{s'-u}\right)\rho(t,s')\geqslant0 \tag{4.12}$$

for $4\leqslant s\leqslant20$, $20\leqslant t<\infty$ and

$$\int_4^\infty ds'\left(\frac{1}{s'-s}+\frac{\eta}{s'-u}\right)\rho(t,s')\geqslant0 \tag{4.13}$$

for $4\leqslant s\leqslant\min[4t/(t-16), 16t/(t-4)]$, $4\leqslant t\leqslant20$ to the defining equations (2.24)–(2.27). Let this subset be called $\bar{\mathfrak{B}}_1\times\mathfrak{C}_1$. Then one can show that M maps $\bar{\mathfrak{B}}_1\times\mathfrak{C}_1$ into $\bar{\mathfrak{B}}_1$, much as in Ref. 1. The only new points arise from the subtraction in Eq. (2.3). In Ref. 1, a key point was the positivity of the term $1/(s'-s)+\eta/(s'+s+t-4)$ for $4\leqslant s<s'$, $t\geqslant4$. For the isospin-0 and -2 states this is now replaced by

$$\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{2}{t-4}\ln\left(1+\frac{t-4}{s'}\right)$$
$$=\frac{2}{t-4}\left[\frac{1}{z'-z}+\frac{1}{z'+z}-2Q_0(z')\right], \tag{4.14}$$

where $z=2s/(t-4)$ and $z'=2s'/(t-4)$. In Appendix D, it is shown that the quantity in large square brackets [ ] is positive if $z'>z\geqslant1$. For the isospin-1 state, one has simply the term

$$\frac{1}{s'-s}-\frac{1}{s'-u}=\frac{2s+t-4}{(s'-s)(s'+s+t-4)}, \tag{4.15}$$

which is again positive. Moreover, the subtraction term $\sigma(t)$ in Eq. (2.3) is non-negative at the fixed point of $\mathfrak{A}_1\to\mathfrak{A}_1$, as can be seen from Eq. (2.4). This allows one to prove that $d(s,t)$ is non-negative throughout $4\leqslant s\leqslant\max[4t/(t-16), 16t/(t-4), 20]$, $4\leqslant t<\infty$, just as in Ref. 1, which means that, for $\|v\|$ sufficiently small, one can certainly arrange that $\bar{\rho}(t,s)+v(t,s)+\beta v(s,t)$ is non-negative everywhere, and this implies Eq. (4.11), as in Ref. 1.

Finally, we turn to the question of the l-plane analyticity of the partial-wave projection of our solution. The partial-wave projection of Eq. (2.14) is

$$F_l(s)=\frac{\delta_{l,0}}{\pi}\int\frac{ds'\rho(s')}{s'-s}+[1+\eta(-1)^l]h_l(s), \tag{4.16}$$

where

$$h_l(s)=\frac{\beta}{\pi}\int dt'\rho(t')Q_l\left(1+\frac{2t'}{s-4}\right)$$
$$+\frac{\eta\beta\eta}{\pi^2}\int\int\frac{dt'du'\rho(t',u')}{t'+u'+s-4}Q_l\left(1+\frac{2u'}{s-4}\right)$$
$$+\frac{1}{\pi^2}\int\int\frac{ds'du'\rho(s',u')}{s'-s}Q_l\left(1+\frac{2u'}{s-4}\right). \tag{4.17}$$

The signatured Froissart-Gribov amplitudes are

$$F_l^{\pm}(s)=(1\pm\eta)h_l(s)\,. \tag{4.18}$$

The (+) analytic interpolation agrees with (4.16) for even l and even I, and the (−) interpolation agrees with (4.16) for odd l and $I=1$, *except* for the s wave, since the first term in (4.16) is a Kronecker δ at $l=0$ (assuming that the single-spectral function does not vanish).

It will be shown that, in our case, the single-spectral function does not vanish everywhere, so that necessarily the continuation of the Froissart-Gribov amplitude to $l=0$ does not agree with the s wave. This is evident from (2.16), since $\sigma(s)$ attains the unitarity limit $\frac{1}{2}(1+\eta)$ at a point near the CDD pole where $\text{Re}D=0$. The other term in (2.16) is of order ψ, so that it cannot cancel $\sigma(s)$ at this point for sufficiently small ψ. The vanishing of $\text{Re}D$ near $s=s_1$ is certain when the CDD residue c_1 is sufficiently small, since the difference between $\text{Re}D$ and the pole term is uniformly close to unity.

V. APPROACH TO STRONG-COUPLING PROBLEM

In this section we outline a program for passing into the physically interesting region of large coupling constants. For the present we discuss only the equations without subtractions or CDD poles, purely for simplicity.

Our basic concern is with the Fréchet derivative[15] of the crossing-unitarity operator. Once we gain control of this linear operator, we then should be able to take a step from one point in the Banach space to another point nearby. We shall see that one might, in principle, generate a "global solution curve" corresponding to larger and larger values of the coupling.

Define the crossing-unitarity operator Φ from Eq. (2.2) as

$$\Phi(\bar\rho;s,t)=\bar\rho(s,t)-\gamma^{M,N}\int\int dt_1dt_2 \times K(s;t,t_1,t_2)d_M^*(s,t_1)d_N(s,t_2)\,, \tag{5.1}$$

where the isospin index I is suppressed, and

$$d(s,t)=\frac{1}{\pi}\int ds'\left(\frac{1}{s'-s}+\frac{\eta}{s'-u}\right) \times[\bar\rho(s',t)+\beta\bar\rho(t,s')+v(s',t)+\beta v(t,s')]\,. \tag{5.2}$$

The equation we wish to solve is

$$\Phi(\bar\rho)=0\,. \tag{5.3}$$

[15] For an introduction to the ideas of this section, including references to mathematical literature, see R. L. Warnock, in *Lectures in Theoretical Physics*, edited by K. T. Mahanthappa *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1969), Vol. 16.

The Fréchet derivative of Φ, evaluated at the point $\bar\rho$, operating on a member $h(s,t)$ of our Banach space $\mathfrak{B}$, is defined by

$$\Phi'(\bar\rho)h(s,t)=-2\,\text{Re}\left[\gamma^{M,N}\int\int dt_1dt_2 \times K(s;t,t_1,t_2)d_M^*(s,t_1)g_N(s,t_2)\right]+h(s,t)\,, \tag{5.4}$$

where

$$g(s,t)=\frac{1}{\pi}\int ds'\left(\frac{1}{s'-s}+\frac{\eta}{s'-u}\right)[h(s',t)+\beta h(t,s')]\,. \tag{5.5}$$

The second Fréchet derivative, which acts in $\mathfrak{B}\times\mathfrak{B}$, is given by

$$\Phi''h_1h_2(s,t)=-2\,\text{Re}\left[\gamma^{M,N}\int\int dt_1dt_2 \times K(s;t,t_1,t_2)g_{1,M}(s,t_1)g_{2,N}(s,t_2)\right]\,. \tag{5.6}$$

Here g_1 and g_2 are related to h_1 and h_2 in the same way that g is related to h above. The second derivative is a constant operator, independent of $\bar\rho$.

We wish to apply the implicit function theorem to show that a small change in the input function $v(s,t)$ will produce, in general, a small change in the solution $\bar\rho(s,t)$. This requires that the inverse Φ'^{-1} of the Fréchet derivative exist. Thus, we must show that the following equation has a unique solution in $\mathfrak{B}$ for every $\omega\in\mathfrak{B}$:

$$\Phi'(\bar\rho)h(s,t)=\omega(s,t)\,. \tag{5.7}$$

For a suitably restricted $\bar\rho$, this may be done by a further application of the contraction-mapping principle. We write $\Phi'=1-\hat K$, corresponding to Eq. (5.4). Then solving (5.7) is equivalent to finding a fixed point of the mapping

$$\hat K h+\omega\to h'\,. \tag{5.8}$$

First we note that for any fixed ω, (5.8) takes $\mathfrak{B}$ into itself. That is clear from the work of Sec. IV. Furthermore, if $h_1,h_2\in\mathfrak{B}$, then

$$h_1'-h_2'=\hat K(h_1-h_2) \tag{5.9}$$

and

$$\|h_1'-h_2'\|\leqslant\|\hat K\|\times\|h_1-h_2\|\,, \tag{5.10}$$

where

$$\|\hat K\|=\sup_{x\in\mathfrak{B}}\|\hat Kx\|/\|x\|\,. \tag{5.11}$$

Again, by Sec. IV we know that $\|\hat K\|=O(\psi)$, so that for sufficiently small ψ, (5.8) is a contraction mapping. It follows that for small ψ there is one and only one solution of (5.7) for each ω, i.e., Φ'^{-1} exists.

To apply the implicit function theorem we multiply the inelastic function $v(s,t)$ by a real parameter λ, and consider the corresponding equation

$$\Phi(\lambda,\bar\rho)=0\,. \tag{5.12}$$

The operator λ, $\bar\rho \rightarrow \Phi(\lambda,\bar\rho)$ is a mapping from $R\times\mathfrak{B}$ into $\mathfrak{B}$, where R is the real line. Suppose that we have a solution $\bar\rho_0$ for some λ_0:

$$\Phi(\lambda_0,\bar\rho_0)=0. \tag{5.13}$$

We note that $\Phi(\lambda,\bar\rho)$ possesses derivatives of all orders with respect to λ and $\bar\rho$. In fact, all derivatives beyond the second vanish. If, in addition, the derivative with respect to $\bar\rho$ at $(\lambda_0,\bar\rho_0)$ has an inverse, then the implicit function theorem[16] guarantees existence of a function $\bar\rho(\lambda)$ in some neighborhood G of λ_0 such that

$$\begin{aligned}&\text{(i)}\quad \Phi(\lambda,\bar\rho(\lambda))=0,\quad \lambda\in G\\ &\text{(ii)}\quad \bar\rho(\lambda_0)=\bar\rho_0,\\ &\text{(iii)}\quad \bar\rho(\lambda) \text{ has derivatives of all orders.}\end{aligned} \tag{5.14}$$

We have proved the existence of the inverse $\Phi_{\bar\rho}^{-1}(\lambda_0,\bar\rho_0)$ for small ψ, so in that case the conclusions (5.14) follow. We expect that the inverse will exist at most points $(\lambda_0,\bar\rho_0)$ even when ψ is not small, and, therefore, that there will be an infinitely differentiable solution curve $\bar\rho(\lambda)$ passing through almost every solution $\bar\rho_0$.

The problem of how to compute the inverse of the Fréchet derivative at large ψ is not yet solved; nor do we have a proof of its existence when ψ is large. To solve these problems, which do not seem insuperable, we must master the linear, multidimensional, singular, integral equation (5.7) in the region where its kernel is large. The equation is of an unfamiliar type, but we shall soon see that an understanding of its properties will be important for further progress.

In applying the implicit function theorem we may consider Fréchet derivatives with respect to $v(s,t)$ instead of the simple derivatives with respect to λ. Once more, $\Phi(v,\bar\rho)$ is infinitely differentiable with respect to both variables, and if $\Phi_{\bar\rho}^{-1}(v_0,\bar\rho_0)$ exists, we find that there is a solution curve $\bar\rho(v)$ having Fréchet derivatives with respect to v of all orders.

We see that, in general, a small change δv in v will produce a new solution $\bar\rho(v_0+\delta v)$ from a given solution $\bar\rho(v_0)$. If we knew how to compute this new solution, we would have a procedure for following our weak-coupling (small-v) solution into the strong-coupling (large-v) domain. The Newton-Kantorovich method,[17,15] which is a generalization to Banach space of the classical Newton procedure, provides a specific means of computing the new solution. For any given inelastic function we wish to solve

$$\Phi(\bar\rho)=0 \tag{5.15}$$

when an approximate solution $\bar\rho_0$ is known (here we suppress reference to λ or v). The approximate solution $\bar\rho_0$ may be, for example, an exact solution for a smaller value of v. The (modified) Newton-Kantorovich method is based on the iteration

$$\bar\rho_{n+1}=\bar\rho_n-\Phi'(\bar\rho_0)^{-1}\Phi(\bar\rho_n), \tag{5.16}$$

which just amounts to successive linearizations of (5.15). Suppose $\bar\rho_1$ is so close to $\bar\rho_0$ that

$$\|\bar\rho_1-\bar\rho_0\|\times\|\Phi'(\bar\rho_0)^{-1}\|\times\|\Phi''\|\leqslant\tfrac{1}{2}. \tag{5.17}$$

Then by Kantorovich's theorem[17] one knows that there is a unique solution of (5.15) in the closed ball

$$\|\bar\rho-\bar\rho_0\|\leqslant 2\|\bar\rho_1-\bar\rho_0\|. \tag{5.18}$$

Instead of taking $\bar\rho_0$ to be a solution for a smaller v, we may make a linear interpolation between one v and the next. For simplicity, suppose that v is altered only by varying its real multiplier λ. Then we may start the Newton-Kantorovich method with the linear expression

$$\bar\rho_0=\bar\rho(\lambda_0)+\left.\frac{d\bar\rho}{d\lambda}\right|_{\lambda_0}(\lambda-\lambda_0), \tag{5.19}$$

for λ near λ_0. By differentiating $\Phi(\lambda,\bar\rho(\lambda))=0$ we may calculate $d\bar\rho/d\lambda$ in terms of quantities that are already required for the Newton-Kantorovich procedure:

$$\left.\frac{d\bar\rho}{d\lambda}\right|_{\lambda_0}=-\Phi_{\bar\rho}^{-1}(\lambda_0,\bar\rho(\lambda_0))\Phi_\lambda(\lambda_0,\bar\rho(\lambda_0)). \tag{5.20}$$

In practice, it may occur that, as λ is increased step by step by successive applications of the Newton-Kantorovich method, one eventually gets to a point where Eq. (5.7) is ill conditioned, i.e., where $\Phi'(\bar\rho)^{-1}$ no longer exists. This does not necessarily mean that one cannot reach larger values of λ: One may be able to skirt the bad point by going out into the complex λ plane, and then returning to the real axis beyond the point of ill condition (which need not correspond to a singularity of the solution, considered as a function of λ). To show that this is a sensible procedure, we now demonstrate that $\bar\rho(s,t)$ is holomorphic in λ at fixed s,t for sufficiently small $|\lambda|$. Consequently, an analytic continuation exists which should allow one to attain large, real values of λ. First we must say what we mean by a solution of (5.3) at complex λ. In place of (5.2) we write

$$d(s,t;\lambda)=\frac{1}{\pi}\int ds'\left(\frac{1}{s'-s}+\frac{\eta}{s'-u}\right)\times[\bar\rho(s',t;\lambda)+\beta\bar\rho(t,s';\lambda)+\lambda v(s',t)+\lambda\beta v(t,s')]. \tag{5.21}$$

In Eq. (5.1) the factor $d_M{}^*(s,t_1)d_N(s,t_2)$ is replaced by

$$d_M{}^*(s,t_1;\lambda^*)d_N(s,t_2;\lambda), \tag{5.22}$$

while the first term $\bar\rho(s,t)$ is replaced by $\bar\rho(s,t;\lambda)$. Now if we begin our iterative solution of the resulting equation with a polynomial in λ—for instance, the trivial one $\bar\rho=0$—then every iterate $\bar\rho_n$ will be a polynomial

[16] J. Dieudonné, *Foundations of Modern Analysis* (Academic Press Inc., New York, 1960), Theorems (10.2.1) and (10.2.3).

[17] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces* (Pergamon Press Ltd., Oxford, 1964).

in λ. The estimates which establish the contraction property of the mapping go through as before, with $|\lambda|$ replacing λ in the estimates when λ is complex. Thus the sequence $\{\bar{\rho}_n(st;\lambda)\}$ of polynomials in λ is uniformly bounded in a circle of the λ plane $|\lambda|\leqslant\lambda_0$ at fixed s,t, where λ_0 is the same limit that one has at real λ for validity of the contraction-mapping proof. The uniform bound comes just from the fact that $\bar{\rho}_n$ belongs to the subspace $\mathfrak{B}_1$ [cf. (2.24)]. It now follows from Vitali's theorem that the limit of the sequence, $\bar{\rho}(s,t,\lambda)$, is analytic in $|\lambda|<\lambda_0$.

Although there may be singularities for larger values of λ, one should in general be able to go round them. In practice, then, upon hitting a bad point on the real λ axis, one could continue into the complex plane, by the Newton-Kantorovich method, and then try to get back to the real axis after a few complex steps. One of three things could happen:

(1) On reaching the real axis again, $\bar{\rho}$ becomes real again, so that one can then continue along the real axis as before. In such a case, the point of ill condition of Eq. (5.7) would correspond either to an isolated singularity of $\bar{\rho}$ or to no singularity at all.

(2) On returning to the real axis, $\bar{\rho}$ is complex. This could happen if there were a branch point on the real axis, and probably no further continuation would be possible, consistent with $\bar{\rho}$'s being real.

(3) A natural boundary frustrates the attempt to regain the real axis.

In either of cases (2) or (3), one would not be able to continue analytically to larger, real values of v. As a matter of fact, one expects that a barrier, either of type (2) or (3), will eventually be encountered, since Martin and Lukaszuk[18] have shown that analyticity, crossing, and unitarity imply an absolute bound on the modulus of the scattering amplitude. Below this barrier, the Newton-Kantorovich method should suffice for a computer construction of a solution.

As an alternative to detours into the complex λ plane, one may sometimes get past a singularity of the Fréchet derivative by means of a procedure due to Anselone and Moore.[19] This "change of parameter" method is discussed in Ref. 15.

Finally, we note that the Fréchet derivative (5.4) is not expected to have an inverse when evaluated at a CDD-type solution $\bar{\rho}$ of the sort discussed in Secs. II–IV. One cannot, therefore, follow a CDD solution to large couplings using the equations employed in this section. Instead, we must compute Fréchet derivatives of the coupled mappings M and N defined in Sec. II. A Newton-Kantorovich iteration based on those derivatives can be used to follow a CDD solution, with either fixed or variable CDD parameters.

[18] A. Martin and L. Lukaszuk, Nuovo Cimento **52A**, 122 (1967).
[19] P. M. Anselone and R. H. Moore, J. Math. Anal. Appl. **13**, 476 (1966).

At large values of $v(s,t)$ there may be problems in enforcing the unitarity constraints in the inelastic region, as well as the possible entry of ghosts into the N/D sector. Hopefully, one can use the freedom in choosing CDD parameters and the form of $v(s,t)$ and $\eta(s)$ to keep these difficulties in abeyance.

ACKNOWLEDGMENT

One of us (D.A.) would like to thank Dr. K. C. Wali for his hospitality at the Argonne National Laboratory, where this work was performed.

APPENDIX A

In this Appendix we wish to verify properties (3.15a)–(3.15d) of the kernel K and the inhomogeneous term f of the N/D integral equation. First, one shows that the discontinuity Δ over the s-wave left cut is bounded as follows:

$$\Delta(s)\leqslant O(\psi)(-s)^{-2\mu},\quad s\to-\infty\,. \tag{A1}$$

This is a straightforward deduction from Eq. (2.10), if the following bounds from Secs. III and IV are invoked:

$$\begin{aligned}\sigma(t)&\leqslant O(\psi)t^{-2\mu},\\ \rho(s,t)&\leqslant O(\psi)(st)^{-2\mu},\\ |\rho(s,t)-\rho(s',t)|&\leqslant O(\psi)\left|\frac{s-s'}{ss't}\right|^{2\mu}.\end{aligned} \tag{A2}$$

The s' integral is majorized easily; the principal-value integral with denominator $s'-u$, $0\leqslant u<\infty$, is treated by the technique of Eq. (3.19), while a bound on the logarithm term may be read off by means of the change of variable $s'\to x$, $s'=(t-4)x$. The three terms of the s' integral are seen to be of orders $(-st)^{-\mu}$, $(ut)^{-\mu}$, and $t^{-2\mu}$, respectively, and when these bounds are integrated over t, the result (A1) is apparent.

It is now immediate to establish (3.15a)–(3.15d) for the left-cut parts of f and K; the latter are denoted by f_L and K_L. We note two simple lemmas, (A3) and (A4). Without loss of generality we take $s>s'$ to prove that

$$\begin{aligned}&|B_L(s)-B_L(s')|\\ &=|s-s'|\left|\frac{1}{\pi}\int_{-\infty}^{0}\frac{\Delta(u)du}{(u-s)(u-s')}\right|\\ &\leqslant O(\psi)\left|\frac{s-s'}{s}\right|\left|\frac{1}{s'^{2\mu}}\right|\left|\int_{-\infty}^{0}\frac{dx}{x^{2\mu}(x-1)}\right|\\ &\leqslant O(\psi)\left|\frac{s-s'}{ss'}\right|^{2\mu},\end{aligned} \tag{A3}$$

$$\left|\frac{B_L(s)-B_L(t)}{s-t}-\frac{B_L(s')-B_L(t)}{s'-t}\right|$$
$$=|s-s'|\frac{1}{\pi}\left|\int_{-\infty}^{0}\frac{\Delta(u)du}{(u-s)(u-s')(u-t)}\right|$$
$$\leqslant O(\psi)\left|\frac{s-s'}{s}\right|\frac{1}{ts'^{2\mu}}\left|\int_{-\infty}^{0}\frac{dx}{x^{2\mu}(x-1)}\right|$$
$$\leqslant O(\psi)\frac{1}{t}\left|\frac{s-s'}{ss'}\right|^{2\mu}. \quad \text{(A4)}$$

With these results and our assumptions about η [cf. (2.30) and (2.31)], we have

$$\|f_L\|=\sup_{4\leqslant s<\infty}|s^{2\mu}f_L(s)|$$
$$=\sup\left|\frac{s^{2\mu}}{\eta(s)}\left[B_L(s)+c_1\frac{B_L(s)-B_L(s_1)}{s-s_1}\right]\right|$$
$$=O(\psi), \quad \text{(A5)}$$
$$|f_L(s)-f_L(s')|\leqslant\frac{1}{\eta(s)}\left|B_L(s)-B_L(s')+c_1\left[\frac{B_L(s)-B_L(s_1)}{s-s_1}-\frac{B_L(s')-B_L(s_1)}{s'-s_1}\right]+f_L(s')[\eta(s')-\eta(s)]\right|\leqslant O(\psi)\left|\frac{s-s'}{ss'}\right|^{2\mu}. \quad \text{(A6)}$$

It follows directly from (A3), (A4), and (2.33) that

$$|K_L(s,s')|=\frac{1}{\pi}\frac{q(s')}{\eta(s)}|B_L(s)-B_L(s')|\leqslant O(\psi)\left|\frac{s-s'}{ss'}\right|^{2\mu}, \quad \text{(A7)}$$
$$\left|\frac{K_L(s,t)}{s-t}-\frac{K_L(s',t)}{s'-t}\right|$$
$$\leqslant\frac{1}{\pi}\frac{q(s')}{\eta(s)}\left|\frac{B_L(s)-B_L(t)}{s-t}-\frac{B_L(s')-B_L(t)}{s'-t}+\frac{B_L(s')-B_L(t)}{s'-t}\left(\frac{\eta(s')-\eta(s)}{\eta(s')}\right)\right|$$
$$\leqslant O(\psi)\frac{1}{t}\left|\frac{s-s'}{ss'}\right|^{2\mu}. \quad \text{(A8)}$$

To complete the proof of the bounds (3.15), we shall prove that the right-cut terms, f_R and K_R, obey conditions similar to (A5)–(A8). The only difference will be that t^{-1} in (A8) is replaced by $t^{-\lambda}$, $\lambda+2\mu>1$. First we note that the Hölder continuity of η, Eq. (2.33), and the end-point conditions $\eta(16)=\eta(\infty)=1$ imply a similar Hölder continuity of the right-cut integral B_R:

$$|B_R(s)-B_R(s')|\leqslant O(\|\phi\|)\left|\frac{s-s'}{ss'}\right|^{2\mu},$$
$$\phi(s)=\frac{1-\eta(s)}{2\pi q(s)}. \quad \text{(A9)}$$

This follows from a well-known theorem on Hölder continuity of Cauchy integrals with density functions vanishing at the end points (make the change of variable $s=1/t$, and apply the theorem of Sec. 19 in Ref. 4). We now study $B_R'(s)$ with the help of the identity

$$\phi(s)-\phi(s')=(s-s')\int_0^1\phi'[xs+(1-x)s']dx. \quad \text{(A10)}$$

It is useful to compute $B_R'(s)$ from the formula

$$B_R(s)=\int_{(1-\epsilon)s}^{(1+\epsilon)s}\frac{\phi(t)-\phi(s)}{t-s}dt+\left[\int_{16}^{(1-\epsilon)s}+\int_{(1+\epsilon)s}^{\infty}\right]\frac{\phi(t)dt}{t-s}, \quad 0<\epsilon<1. \quad \text{(A11)}$$

The result is

$$B_R'(s)=\frac{2\phi(s)}{\epsilon s}+\int_{(1-\epsilon)s}^{(1+\epsilon)s}\left[\frac{\phi(t)-\phi(s)}{t-s}-\frac{d\phi}{ds}\right]\frac{dt}{t-s}. \quad \text{(A12)}$$

After two applications of (A10) the integral in (A12) becomes

$$\int_{(1-\epsilon)s}^{(1+\epsilon)s}dt\int_0^1dy\int_0^1xdx\,\phi''[xyt+(1-xy)s]. \quad \text{(A13)}$$

We can now bound $B_R'(s)-B_R'(s')$ by making use of Eq. (2.31). Supposing that $s<s'$, we have

$$|B_R'(s)-B_R'(s')|\leqslant\frac{2}{\epsilon}\frac{1}{s}|\phi(s)-\phi(s')|+\frac{2}{\epsilon}\left|\phi(s')\left(\frac{s'-s}{ss'}\right)\right|$$
$$+O(\|\phi\|)\int_{(1-\epsilon)s}^{(1+\epsilon)s}dt\int_0^1dy\int_0^1dx\frac{x(1-xy)^2|s-s'|^{2\mu}}{[xyt+(1-xy)s]^{2\mu}[xyt+(1-xy)s']^{2\mu}[xyt+(1-xy)s]^2}. \quad \text{(A14)}$$

The integral in (A14) is majorized by discarding xyt in the first two factors of the denominator. In the third factor, replace t by its smallest value $(1-\epsilon)s$. Then the integral is smaller than

$$\frac{1}{s^2}\left|\frac{s-s'}{ss'}\right|^{2\mu}\int_{(1-\epsilon)s}^{(1+\epsilon)s}dt\int_0^1 dy\int_0^1 dx\frac{x}{(1-xy)^{2\mu}(1-\epsilon xy)^2}<\left|\frac{s-s'}{ss'}\right|^{2\mu}\frac{1}{s}\frac{2\epsilon}{(1-\epsilon)^2}\int_0^1 dy\int_0^1 dx\frac{x}{(1-xy)^{2\mu}}$$

$$=O\left(\frac{1}{s}\left|\frac{s-s'}{ss'}\right|^{2\mu}\right). \quad \text{(A15)}$$

The first two terms in (A14) are readily seen to have similar bounds, and we conclude that

$$|B_R'(s)-B_R'(s')|\leqslant O(\|\phi\|)\left|\frac{s-s'}{ss'}\right|^{2\mu}\frac{1}{\bar{s}}. \quad \text{(A16)}$$

Now (A16) and one more application of (A10) give us the last inequality required:

$$\left|\frac{B_R(s)-B_R(t)}{s-t}-\frac{B_R(s')-B_R(t)}{s'-t}\right|=\int_0^1 dx|B_R'[xs+(1-x)t]-B_R'[xs'+(1-x)t]|$$

$$\leqslant O(\|\phi\|)\int_0^1 dx\frac{x^{2\mu}|s-s'|^{2\mu}}{[xs+(1-x)t]^{2\mu}[xs'+(1-x)t]^{2\mu}[xs+(1-x)t]^{\lambda}4^{1-\lambda}}$$

$$\leqslant O(\|\phi\|)\frac{1}{t^\lambda}\left|\frac{s-s'}{ss'}\right|^{2\mu}\int_0^1 dx\, x^{-2\mu}(1-x)^{-\lambda}\leqslant O(\|\phi\|)\frac{1}{t^\lambda}\left|\frac{s-s'}{ss'}\right|^{2\mu}. \quad \text{(A17)}$$

This is enough to prove Eqs. (3.15a)–(3.15d).

It should be noted that (A9) is easily proved by means of (A10) if Eq. (2.34) is assumed. The more difficult proof quoted from Ref. 4 does not require ϕ' to exist. Correspondingly, it is probable that (A16) can be deduced from appropriate Hölder continuity of ϕ', without existence of ϕ''.

APPENDIX B

Here we show how to obtain Eqs. (3.34) and (3.35) from the integral equation (3.1). Let n_1 and n_2 be the solutions of (3.1) corresponding to $(\bar{\rho}_1,\sigma_1)$ and $(\bar{\rho}_2,\sigma_2)$, respectively. The integral equation gives

$$(n_1-n_2)(s)=(f_1-f_2)(s)+\frac{1}{\pi\eta(s)}\int_4^\infty\frac{(B_1-B_2)(s)-(B_1-B_2)(s')}{s-s'}q(s')n_1(s')ds'+\frac{1}{\pi\eta(s)}\int_4^\infty\frac{B_2(s)-B_2(s')}{s-s'}q(s')(n_1-n_2)(s')ds'. \quad \text{(B1)}$$

We write this in a shorter notation, and then take norms

$$n_1-n_2=f_1-f_2+K_{12}n_1+K_2(n_1-n_2), \quad \text{(B2)}$$

$$\|n_1-n_2\|\leqslant\|f_1-f_2\|+\|K_{12}\|\times\|n_1\|+\|K_2\|\times\|n_1-n_2\|. \quad \text{(B3)}$$

Since we have already required $\|K_2\|<1$, this yields

$$\|n_1-n_2\|\leqslant\frac{\|f_1-f_2\|+\|K_{12}\|\times\|n_1\|}{1-\|K_2\|}. \quad \text{(B4)}$$

In f_1-f_2 and B_1-B_2, the contributions of $v(s,t)$ and $B_R(s)$ do not appear. In other respects the analysis of f_1-f_2 and B_1-B_2 is the same as that of f and B. Therefore, Appendix A implies that

$$\|f_1-f_2\|=O(d_{12}),\quad \|K_{12}\|=O(d_{12}), \quad \text{(B5)}$$

$$d_{12}=\max[\|\bar{\rho}_1-\bar{\rho}_2\|,\|\sigma_1-\sigma_2\|]. \quad \text{(B6)}$$

Since $\|K_2\|$ and $\|n_1\|$ are $O(\psi)$, Eq. (B4) yields (3.34):

$$\|n_1-n_2\|=O(d_{12}).$$

In a similar way we can use (B2) to treat the Hölder continuity of $(n_1-n_2)(s)$. One employs (3.15d), but with B_1-B_2 replacing B. The result is (3.35):

$$|(n_1-n_2)(s)-(n_1-n_2)(s')|\leqslant O(d_{12})\left|\frac{s-s'}{ss'}\right|^{2\mu}.$$

APPENDIX C

In this Appendix, it will be shown that Eqs. (4.4)–(4.7) imply Eq. (4.8). One certainly has

$$|d(s,t)|\leqslant O(\psi)\{(st)^{-\mu}\ln^{-2}s\ln^{-2}t+[1+\epsilon^{-1/2}\theta_\Omega(t)]t^{-2\mu}\}. \quad \text{(C1)}$$

The double integral (2.2) contains the quadratic form $d^*(s,t_1)d(s,t_2)$, which can be split into three terms corresponding to the two terms of Eq. (C1) and the cross term. The first gives an integral that has already been treated in Refs. 1 and 2. The second term can be shown to give an integral that is bounded by

$$O(\psi)(st)^{-2\mu}\ln t, \tag{C2}$$

which is not greater than a term of the required form

$$O(\psi)(st)^{-\mu}\ln^{-2}s\,\ln^{-2}t. \tag{C3}$$

Note the importance of the extra inverse powers of t, which allow both the $\ln^{-2}s$ and $\ln^{-2}t$ factors to be fabricated. The factor $1+\epsilon^{-1/2}\theta_\Omega(t)$ disappears, since the double integral produces a factor $\epsilon^{3/2}$ from the integration of $\theta_\Omega(t_1)\theta_\Omega(t_2)$, which more than cancels the coefficient ϵ^{-1}. Note that $1+\epsilon^{-1}\theta_\Omega(t)$ could not have been handled: The reason is essentially because of the inverse square-root kernel in the t_2 integration. The cross term from (C1) is certainly majorized by

$$O(\psi)s^{-\mu}\ln^{-2}s\,(t_1t_2)^{-\mu}\ln^{-2}t_1\ln^{-2}t_2\left[1+\epsilon^{-1/2}\theta_\Omega(t_2)\right] \tag{C4}$$

and this immediately yields a bound of the form (C3).

In a similar way, by considering the Hölder-continuity condition (4.5) and the bound (C1), one can show that

$$|\bar\rho'(s_1,t_1)-\bar\rho'(s_2,t_2)|$$
$$\leqslant O(\psi^2)\left[\left|\frac{s_1-s_2}{s_1s_2\bar t}\right|^\mu+\left|\frac{t_1-t_2}{t_1t_2\bar s}\right|^\mu\right]\ln^{-2}\bar s\,\ln^{-2}\bar t. \tag{C5}$$

APPENDIX D

In this Appendix, it will be shown that, if $1\leqslant z<z'$, then

$$\frac{1}{z'-z}+\frac{1}{z'+z}-2Q_0(z')\geqslant 0. \tag{D1}$$

The Darboux-Christoffel identity is

$$\frac{1}{z'-z}-\sum_{l=0}^{L}(2l+1)P_l(z)Q_l(z')$$
$$=\frac{L+1}{z'-z}\left[P_{L+1}(z)Q_L(z')-P_L(z)Q_{L+1}(z')\right]. \tag{D2}$$

For $L=1$, this gives

$$\frac{1}{z'-z}-P_0(z)Q_0(z')-3P_1(z)Q_1(z')$$
$$=\frac{2}{z'-z}\left[P_2(z)Q_1(z')-P_1(z)Q_2(z')\right]. \tag{D3}$$

Hence, on replacing z by $-z$, one finds

$$\frac{1}{z'+z}-P_0(z)Q_0(z')+3P_1(z)Q_1(z')$$
$$=\frac{2}{z'+z}\left[P_2(z)Q_1(z')+P_1(z)Q_2(z')\right]. \tag{D4}$$

Add (D3) and (D4):

$$\frac{1}{z'-z}+\frac{1}{z'+z}-2Q_0(z')$$
$$=\frac{2}{z'-z}\{\left[P_2(z)-P_1(z)\right]Q_1(z)'+P_1(z)\left[Q_1(z')-Q_2(z')\right]\}$$
$$+\frac{2}{z'+z}\left[P_2(z)Q_1(z')+P_1(z)Q_2(z')\right]. \tag{D5}$$

This proves (D1), since

$$P_2(z)\geqslant P_1(z)\geqslant 1,$$

and

$$Q_1(z')\geqslant Q_2(z')\geqslant 0,\quad z,z'\geqslant 1. \tag{D6}$$