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Origin and Propagation of Cosmic-Ray Electrons*

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We consider the modulating of the energy spectrum of cosmic-ray electrons as they propagate in interstellar space. The effects of spatial propagation, energy loss, boundary conditions, and the spatial distribution and energy spectrum of the electrons injected into interstellar space by the sources are all taken into account. Two models for spatial propagation are considered: isotropic diffusion and convection diffusion. In the latter, the particle motion parallel to the disk plane is characterized by diffusion, while particles are transported perpendicular to the disk plane by convection, caused by the outward expansion of the field lines. Examples are worked out for several source distributions. It is shown that the high-energy electron distribution (in both energy and space) is determined by the source distribution, rather than the properties of the confinement region (i.e., the propagation model). In order to illustrate how one can use the observable quantities (the electron spectrum and positron fraction at Earth, and the background Galactic radio and γ radiation) to determine the properties of the injection spectrum, we have considered the construction of a model of primary cosmic-ray electron sources in some detail. It is found that a combination of a source with a disk distribution and a source concentrated at the Galactic center may be necessary to explain the observed disk γ rays.

I. INTRODUCTION

EVER since the discovery of cosmic rays, their origin and propagation in space have remained interesting problems. Until the last few years the study had been restricted to the nuclear component. In the last decade discovery of the electron component and its relationship with the background electromagnetic radiation have broadened the scope of the subject. The high-energy cosmic-ray electrons are of particular interest in the study of cosmic rays for two reasons. First of all, electrons lose energy much faster than nuclei, so modulation in interstellar space introduces a significant difference between the injection energy spectrum of electrons and the spectrum observed at Earth. Secondly, in losing energy, electrons produce electromagnetic radiation which indicates their distribution throughout the Galaxy. Therefore, a combined study of electrons and the related observable quantities would provide valuable information about the origin and propagation of cosmic rays. We shall summarize briefly the present-day knowledge in this subject.

That electrons must be present in cosmic radiation was first demonstrated by Hayakawa.[1] Direct detection of cosmic-ray electrons was first made by Earl and by Meyer and Vogt.[2] Since then, a number of experiments[3–16] have yielded considerable data on the intensity, energy spectrum, and charge composition of cosmic-ray electrons. The most recent data are summarized in Fig. 6 and Table I. The energy spectrum at Earth from 10 to 300 BeV is well represented by a power law with spectral index 2.6±0.1, with an intensity about 1% of the nuclear cosmic rays. Below a few BeV the observed spectrum flattens considerably; but because of effects of solar modulation, the observed intensities must be regarded as an upper limit to the interstellar spectrum near, but outside of, the solar system. The positron fraction is about 30% at a few hundred MeV and appears to decrease rapidly above 1 BeV. The low

TABLE I. Cosmic-ray electron-positron fraction.

Energy range (BeV)	Hartmann[a]	Beuermann *et al.*[b]
0.01–0.02		0.35±0.25
0.02–0.03		0.35±0.20
0.03–0.06		0.30±0.30
0.1–0.2		0.30±0.20
0.2–0.5	0.35±0.10	
0.5–1.0	0.10±0.09	
1–2	0.05±0.03	
2–5	0.07±0.05	
5–10	<0.20	

[a] Reference 9.
[b] Reference 8.

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[1] S. Hayakawa, Prog. Theoret. Phys. (Kyoto) **8**, 517 (1952).
[2] J. A. Earl, Phys. Rev. Letters **6**, 125 (1961); P. Meyer and R. Vogt, *ibid.* **6**, 193 (1961).
[3] K. C. Anand, R. R. Daniel, and S. A. Stephens, Phys. Rev. Letters **20**, 764 (1968).
[4] R. R. Daniel and S. A. Stephens, Proc. Indian Acad. Sci. **65**, 319 (1967).
[5] J. A. M. Bleeker, J. J. Burger, A. J. M. Deerenberg, A. Scheepmaker, B. N. Swanenburg, and T. Tanaka, Can. J. Phys. **46**, S522 (1968).
[6] J. L'Heureux, Astrophys. J. **148**, 399 (1968).
[7] W. R. Webber and C. Chotkowski, J. Geophys. Res. **72**, 2783 (1967).
[8] K. P. Beuermann, C. J. Rice, E. C. Stone, and R. E. Vogt, Phys. Rev. Letters **22**, 412 (1969).
[9] R. C. Hartman, Astrophys. J. **150**, 371 (1967).
[10] C. H. Costain, Monthly Notices Roy. Astron. Soc. **120**, 248 (1960).
[11] A. J. Turtle, J. F. Pugh, S. Kenderdine, and I. I. K. Pauliny-Toth, Monthly Notices Roy. Astron. Soc. **124**, 297 (1962).
[12] R. Parthasarthy and G. M. Lerfald, Monthly Notices Roy. Astron. Soc. **129**, 395 (1965).
[13] B. H. Andrew, Monthly Notices Roy. Astron. Soc. **132**, 79 (1966).
[14] C. R. Purton, Monthly Notices Roy. Astron. Soc. **133**, 463 (1966).
[15] A. H. Bridle, Monthly Notices Roy. Astron. Soc. **136**, 219 (1967).
[16] G. W. Clark, G. P. Garmire, and W. L. Kraushaar, Astrophys. J. **153**, L203 (1968).

positron fraction is convincing evidence that in addition to secondary electrons (those produced by collisions of cosmic-ray nuclei with ambient matter) there is a sizable flux of primary negatrons which must be directly accelerated in the cosmic-ray sources.

In addition, measurements of the background radiation have covered a wide frequency range, from radio waves to high-energy γ photons. The radiation data are summarized in Figs. 1 and 5. The Galactic radio background is generally attributed to synchrotron radiation of electrons with energies from a few hundred MeV to a few BeV. The flatness of the radio spectrum at low frequencies indicates that the low-energy electron spectrum does in fact deviate from the power law observed above a few BeV.[17,18]

In keeping with the rapid observational progress, many theoretical papers concerning the production and propagation of cosmic-ray electrons in space have appeared in the literature. For complete treatments see Ramaty and Lingenfelter,[19] Felten and Morrison,[20] and Gould and Burbridge.[21] The theoretical model which the work of these authors is based upon may be described as follows: The majority of the cosmic-ray electrons observed at Earth were produced inside the Galaxy and were assumed to fill uniformly the Galactic "sphere" (consisting of the disk and halo). Since the cosmic rays are known to pass through approximately 3 g/cm² of material (as is observed for the nuclear component) before leaking into intergalactic space, the average confinement time in the galactic sphere was taken to be $T_L \approx 10^8$ yr, during which the electrons lose energy through interaction with the ambient matter, magnetic fields, and photons. Because the electrons were assumed *a priori* to be distributed uniformly throughout the confinement region, the effects of spatial propagation were neglected and the equilibrium spectrum was calculated from the transfer equation in energy space only,

$$\frac{\partial}{\partial E}\left(\frac{dE}{dt}N\right)+\frac{N}{T_L}=Q, \tag{1.1}$$

where $N(E)$ and $Q(E)$ are the equilibrium electron density and injection rate, respectively, and for electrons above a few tens of MeV, the rate of energy change is given by $dE/dt \approx -bE^2$ and is due to synchrotron and inverse Compton radiation. For a power-law injection spectrum $Q(E)=KE^{-\alpha}$, the result is that for $E<E_c \approx 1/bT_L$ (E_c is the critical energy at which the radiative lifetime equals the confinement lifetime), the production spectral index is preserved, but for $E>E_c$ the equilibrium spectrum becomes one power steeper than the production spectrum. Thus it was expected that a "knee"

[17] S. D. Verma, Astrophys. J. **152**, 537 (1968).
[18] K. C. Anand, R. R. Daniel, and S. A. Stephens, Nature **217**, 25 (1968).
[19] R. Ramaty and R. E. Lingenfelter, J. Geophys. Res. **71**, 3687 (1966).
[20] J. E. Felten and P. Morrison, Astrophys. J. **146**, 686 (1966).
[21] R. J. Gould and G. R. Burbidge, Ann. Astrophys. **28**, 171 (1965).

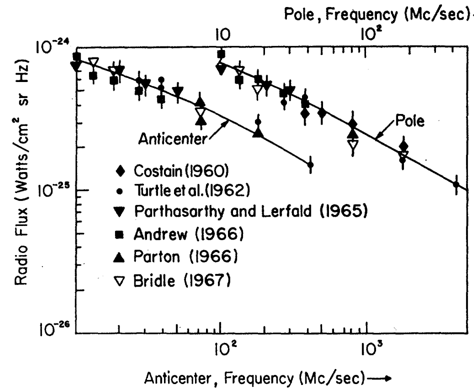

FIG. 1. Background Galactic radio spectrum in the anticenter and pole directions. The data are taken from Refs. 10–15. The curves are free-hand fits to the data.

would be found in the electron spectrum at $E_c \approx 10$ BeV. In order to explain the absence of this knee, models were proposed in which the disk and halo were treated as two separate confinement regions[22] or in which confinement was in the disk alone.[23]

That this confinement-leakage approach is inadequate for cosmic-ray electron propagation in the Galaxy had been pointed out by Shen[24] and by Jokipii and Meyer.[25] Use of T_L implies that a particle has constant probability of escape from the Galaxy. But the electrons, unlike the nuclei, are subject to deceleration. Thus, if the cosmic rays are produced in a region whose spatial extent is smaller than the confinement volume, the high-energy electrons would have little chance to reach the boundary before losing most of their energy. For them the concept of leakage is misleading, and Eq. (1.1) leads to incorrect results. For a proper treatment of electron propagation, as pointed out in Refs. 24 and 25, one must include the spatial diffusion, the boundary condition, and the distribution of sources. It shall become evident that above the critical energy E_c, the equilibrium electron distribution in space depends not so much on the confinement region as on the spatial distribution of the electron sources.

With the above conditions in mind, we develop in Sec. II the mathematical formalism needed to calculate the equilibrium electron distribution for an arbitrary time-dependent source in two propagation models. One is the usual isotropic-diffusion model and the other we call the convection-diffusion model, in which the cosmic-ray motion parallel to the disk plane is a diffusion process controlled by the random-walk collisions of the particles with small-scale magnetic irregularities along the field lines, while the motion perpendicular to the disk plane is a convection process. The particles are carried away from the disk into the halo by expansion of

[22] R. Ramaty and R. E. Lingenfelter, Phys. Rev. Letters **17**, 1230 (1966).
[23] R. F. O'Connell, Phys. Rev. Letters **17**, 1232 (1966).
[24] C. S. Shen, Phys. Rev. Letters **19**, 399 (1967).
[25] J. R. Jokipii and P. Meyer, Phys. Rev. Letters **20**, 752 (1968).

the Galactic magnetic field. In Sec. III, solutions are obtained in the isotropic-diffusion model for several plausible source distributions by neglecting the boundary conditions. In Sec. IV, the diffusion equation is solved with a general boundary condition and the solution for a simple-source distribution is discussed to illustrate the effect of the boundary condition on the equilibrium spectrum. In Sec. V we consider the convection-diffusion approach and the discrepancies between the different propagation models are estimated. In the last section the theoretical results of the earlier sections are compared with the pertinent experimental data. It appears that a two-component–source model, in which the production rate per unit volume in the Galactic core is much higher than in the disk, is most natural in explaining the recent observational results.

II. GENERAL TRANSFER EQUATION

The basic transport equation which governs the propagation of cosmic-ray electrons in the Galaxy is given by[26]

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial E}\left\{\frac{dE}{dt}N-\frac{1}{2}\frac{\partial}{\partial E}\left[\frac{d}{dt}(\langle\Delta E^2\rangle)N\right]\right\}+\nabla\cdot\mathbf{J}=Q\,, \quad (2.1)$$

where $N(E,\mathbf{r},t)$ is the number density, $\mathbf{J}(E,\mathbf{r},t)$ is the "net flow" of particles in coordinate space, and $Q(E,\mathbf{r},t)$ is the production rate. Equation (2.1) is a direct consequence of Liouville's theorem. The second and third terms represent, respectively, the transfer of particles in energy space and in coordinate space.

For electrons in interstellar space, statistical energy fluctuations are small compared to energy losses, so the term

$$-\frac{1}{2}\frac{\partial}{\partial E}\left[\frac{d}{dt}(\langle\Delta E^2\rangle)N\right]$$

is unimportant. For dE/dt, the energy-loss mechanisms include bremsstrahlung, ionization, and Compton and synchrotron radiation. Bremsstrahlung is a "catastrophic" rather than a "continuous" process, for the particle loses so much of its energy during the encounter that it will no longer stay in the energy range under consideration. Therefore, to include bremsstrahlung, a loss term N/T_c, with T_c the appropriate lifetime, should be added to Eq. (2.1). It is known, however, that the average amount of material transversed by Galactic cosmic rays is $\sim$3–5 g/cm^2, while the mean free path for bremsstrahlung is 62 g/cm^2. The fraction of electrons lost in interstellar space due to bremsstrahlung is, therefore, around 5% at all energies.

The rate of ionization loss is given by

$$dE/dx=1.5\times10^5[3\ln(E/mc^2)+18.8]\text{ eV }(\text{g/cm}^2)^{-1}\,, \quad (2.2)$$

so the total ionization energy loss of a relativistic cosmic-ray electron in the Galaxy is only a few times 10 MeV. Therefore, in studying the energy losses of Galactic electrons in the energy range $E\gg10$ MeV, one need consider only synchrotron and Compton processes. These losses are given by

$$dE/dt=-bE^2\,, \quad (2.3)$$

where

$$b\cong10^{-25}(w_{\text{ph}}+w_H)\text{ eV}^{-1}\text{ sec}^{-1}\,, \quad (2.4)$$

and w_{ph} and w_H are, respectively, the energy densities of the ambient photons and magnetic fields in eV/cm^3. [Equation (2.4) is an adequate approximation for astrophysical application as long as the average energy of the ambient photon $\langle\epsilon\rangle\ll(mc^2)^2/E$, where E is the energy of the electron.] At present, the *known* significant contributions to w_{ph} are the universal blackbody photons w_{bb} and the stellar photons w_{st}. At 2.7°K $w_{\text{bb}}\approx0.25$ eV/cm^3. The values of w_{st} and w_H, although not accurately known, are comparable to w_{bb}. Using the often-quoted values of $w_{\text{st}}=0.2$ eV/cm^3, $w_H=0.6$ eV/cm^3 ($H=5\times10^6$ G) in the disk, and $w_{\text{st}}=0.4$ eV/cm^3, $w_H=0.1$ eV/cm^3 ($H=2\times10^{-6}$ G) in the halo, we have $\langle w_{\text{ph}}+w_H\rangle\approx0.8$ eV/cm^3 and $b=8\times10^{-26}$ eV^{-1} sec^{-1}.

It is also possible that a strong infrared-radiation field exists in our Galaxy. A preliminary report by Shivanandan, Houck, and Harwit[27] indicates an energy density $w_{\text{IR}}=13$ eV/cm^3 in the wavelength range $0.4<\lambda<1.3$ mm. This alone would increase b to 1.4×10^{-24} eV^{-1} sec^{-1}. Indirect measurements on the microwave intensity from analysis of intermolecular lines suggest that the background radiation cannot be 8°K blackbody radiation,[28] but the possibility that it is concentrated into one or more lines cannot be ruled out. Since the total photon intensity in the infrared range is still an open question, in the computation carried out below, b will be considered as a free parameter. Only in Sec. VI, when we compare the theoretical model with the observational results, is a value of b substituted to illustrate the effect of the radiation field on cosmic-ray electrons.

The third term on the left-hand side of Eq. (2.1) describes the motion of electrons in coordinate space. In a frame of reference where the scattering centers (i.e., the magnetic irregularities superposed on the field lines) possess a systematic motion, such as the inflation of the disk magnetic field into the halo,[29] the net flow $\mathbf{J}$ is given by

$$\mathbf{J}=-\mathbf{D}\cdot\nabla N+\mathbf{v}_dN\,, \quad (2.5)$$

where $\mathbf{D}$ is the diffusion tensor and $\mathbf{v}_d$ is the velocity of the scattering centers. (The drift of the magnetic field may also cause an energy change of order v_d/v, where $v\approx c$ is the particle velocity. This effect can be ignored

[26] V. L. Ginzburg and S. I. Syrovatskii, *The Origin of Cosmic Rays* (The Macmillan Co., New York, 1964).

[27] K. Shivanandan, J. R. Houck, and M. O. Harwit, Phys. Rev. Letters **21**, 1460 (1968).

[28] V. J. Bortolot, Jr., J. F. Clauser, and P. Thaddens, Phys. Rev. Letters **22**, 307 (1969).

[29] E. N. Parker, Astrophys. J. **142**, 584 (1965).

for cosmic-ray propagation in the Galaxy since $v_d \approx 100$ km/sec$\ll c$.)

As was discussed in the Introduction, most previous treatments of electron propagation have neglected the spatial dependence of N and Q and have replaced $\nabla \cdot \mathbf{J}$ by N/T_L. It has been pointed out[24,25] that such a treatment is not only physically unjustifiable, but also leads to incorrect results for high-energy electrons. The most reasonable description of the spatial motion of cosmic rays in the Galaxy, i.e., the exact expression for the net flow $\mathbf{J}$, is still an open question. The simplest model, which is often used in the study of the propagation of cosmic-ray nuclei, is the isotropic-diffusion approximation, in which

$$\mathbf{J} = -D\nabla N, \tag{2.6}$$

where D is a constant. In this case, Eq. (2.1) reduces to

$$\frac{\partial N}{\partial t} - \frac{\partial}{\partial E}(bE^2N) - D\nabla^2 N = Q. \tag{2.7}$$

The solutions of Eq. (2.7) will be discussed with and without boundary conditions in Secs. IV and III, respectively.

The isotropic-diffusion approach, although simplifying considerably the computation, is not wholly convincing. Recent observations have shown that the large-scale disk magnetic field is parallel to the Galactic plane, while the field lines generally follow the direction of the spiral arms. Since the Larmor radius ρ of a cosmic-ray particle with energy less than 10^6 BeV is small compared to the scale l of the magnetic field inhomogeneities, it is expected to move along the lines of force with only a slow drift across the lines. Each collision with a magnetic "bump" would cause a particle's guiding center to shift perpendicular to the field lines by a distance $\approx \rho$. Hence across the field line the step length is $\approx \rho$, while along the line it is $\approx l$, and the diffusion coefficients perpendicular and parallel to the field are in the ratio $D_\perp/D_{\parallel} \approx (\rho/l)^2 \approx 10^{-16} [E\text{ (BeV)}]^2$. The applications of a strict anisotropic-diffusion model, in which the particles are approximately confined to move along orderly, stationary magnetic field lines, to the propagation of Galactic cosmic rays leads to observational contradictions. One is the observed high isotropy (better than 0.1% at 10^{14} eV)[30] of cosmic-ray nuclei in the neighborhood of Earth; particles confined to streaming along the field lines would exhibit a much larger anisotropy. Another problem arises from the observed nonthermal radiation of the Galactic halo. In this anisotropic-diffusion model the time for a charged particle to leak out the open end of the spiral arm (of length $L \approx 2\times 10^{23}$ cm) would be $T_{\parallel} \approx L^2/2D_{\parallel}$, while the time to diffuse across the Galactic plane (half-width $d \approx 4\times 10^{20}$ cm) is $T_\perp \approx d^2/2D_\perp$. In the adiabatic limit, $T_{\parallel}/T_\perp \approx (\rho L/ld)^2 \approx 2\times 10^{-10} [E\text{ (BeV)}]^2$, so that particles produced in the Galactic plane would leak out the open ends of the spiral arms and would have no chance to diffuse into the halo. It would then be necessary to postulate a "halo" source of cosmic-ray electrons to explain the nearly isotropic halo radio emission; as is evident from the structure of the Galaxy, this is rather unlikely.

[30] K. Greisen, Ann. Rev. Nucl. Sci. **10**, 63 (1960).

For a simple yet realistic description of cosmic-ray motion in the Galaxy, one can perhaps write

$$\nabla \cdot \mathbf{J} = -D_{\parallel}\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right) - D_\perp \frac{\partial^2 N}{\partial z^2} + \frac{\partial}{\partial z}[v_d(z)N]. \tag{2.8}$$

In Eq. (2.8) the first term on the right-hand side describes diffusion parallel to the disk plane due to the random-walk collisions of the particles with small-scale field irregularities. (We are neglecting the details of the spiral-arm structure.) The second term describes diffusion perpendicular to the disk plane, which is caused primarily by the random walk of the field lines[31]; $D_\perp$ in the halo could be considerably larger than in the disk. The third term represents the convection of cosmic rays being carried along with the field lines into the halo with a drift velocity $v_d(z)$.

The adiabatic approximation discussed above corresponds to $v_d(z) = 0$ and $D_\perp \ll D_{\parallel}$. In order to achieve near-isotropy in the cosmic-ray intensity and to allow cosmic rays to enter the halo in sufficient numbers, it is necessary that the combined magnitude of the second and third terms on the right-hand side of Eq. (2.8) be comparable to the first term. In the isotropic-diffusion approximation, convection is neglected, and $D_\perp = D_{\parallel} = D$ throughout the Galaxy; in Sec. V we shall examine the opposite extreme, letting $D_\perp = 0$ and assuming the perpendicular motion to be governed by convection. In this case Eq. (2.1) becomes

$$\frac{\partial N}{\partial t} - \frac{\partial}{\partial E}(bE^2N) + \frac{\partial}{\partial z}[v(z)N] - D_{\parallel}\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right) = Q. \tag{2.9}$$

A main distinction between Eqs. (2.7) and (2.9), representing the isotropic-diffusion and the convection-diffusion models, respectively, is that in the latter case the motion of the electrons across the line of force is "regular." The particles are flowing away from the disk into the halo and will not return. Therefore, although Eq. (2.9) is obtained by the convection approach, its solution closely represents that found in any "closed-disk model," in which the diffusion coefficient in the halo is much larger than in the disk. In these models once the particle leaves the disk, whether because of diffusion or convection, it has little chance to return. Thus, the average age of the particles observed at Earth can be inferred from the 3 g/cm² of material traversed to be $\sim 10^6$ yr (or somewhat larger if the "confinement disk" is thicker than the matter disk). By the same rea-

[31] J. R. Jokipii and E. N. Parker, Phys. Rev. Letters **21**, 44 (1968).

soning, the average age of the particles observed in the disk in the isotropic-diffusion model would be 10^8 yr, since there is free exchange between the disk and halo.

III. ISOTROPIC DIFFUSION IN INFINITE SPACE

The basic assumption of the isotropic-diffusion model is that charged particles are scattered randomly by magnetic irregularities, which are in turn wandering randomly in space; hence the diffusion is isotropic and **D** reduces to a scalar of magnitude $D=\frac{1}{3}lv$, where l is the effective mean free path between scatterings and $v\approx c$ is the particle velocity. For particles with $E<10^6$ BeV, the Larmor radius is small compared to the scale of the magnetic inhomogeneities in interstellar space, and l is given by the average distance between magnetic "clouds." From direct observation of the interstellar magnetic field, l is estimated to be about 10^{19} cm.[26] Therefore, $D\approx 10^{29}$ cm^2 sec^{-1} inside the Galaxy, including the halo and disk. The time for a particle to diffuse a distance equal to the Galactic radius $R\approx 12$ kpc $=3.7\times 10^{22}$ cm is given by

$$T_d\approx R^2/2D\approx 2\times 10^8 \text{ yr}. \tag{3.1}$$

(For a more detailed discussion of the isotropic diffusion model, see Secs. 10 and 14 of Ginzburg and Syrovatskii.[26])

The solution of Eq. (2.7) in infinite space can be found by the Green's-function method[26] and is given by

$$N(E,\mathbf{r},t)=\iiint_{-\infty}^{\infty} d\mathbf{r}'\int_E^{\infty} dE'\int_{-\infty}^{t} dt' \times Q(E',\mathbf{r}',t')G_\infty(E,\mathbf{r},t;E',\mathbf{r}',t'), \tag{3.2}$$

where

$$G_\infty(E,\mathbf{r},t;E',\mathbf{r}',t')=\frac{1}{bE^2\{4\pi D[\tau(E)-\tau(E')]\}^{3/2}} \times\exp\left\{-\frac{(\mathbf{r}-\mathbf{r}')^2}{4D[\tau(E)-\tau(E')]}\right\}\delta(t-t'-\tau(E)+\tau(E')) \tag{3.3}$$

and

$$\tau(E)=\int_E^{\infty}\frac{dE}{bE^2}=\frac{1}{bE}. \tag{3.4}$$

When studying isotropic diffusion in infinite space, it is mathematically convenient (and, as we discuss below, physically reasonable) to consider as a general form for the source distribution

$$Q(E',\mathbf{r}',t')=K(t')E'^{-\alpha} \times\exp\left[-\frac{(x'-x_0)^2}{d_1^2}-\frac{(y'-y_0)^2}{d_2^2}-\frac{(z'-z_0)^2}{d_3^2}\right]. \tag{3.5}$$

The source in Eq. (3.5) has a Gaussian spatial distribution centered at the point (x_0,y_0,z_0), with $K(t')$ and α being the injection intensity and the slope, respectively. For this source the integrations over coordinate and energy variables can be performed in closed form to obtain

$$N(E,\mathbf{r},t)=E^{-\alpha}\int_{t-1/bE}^{t} dt'K(t')[1-bE(t-t')]^{\alpha-2}f(\mathbf{r},t-t'), \tag{3.6}$$

where

$$f(\mathbf{r},\xi)=\exp\left[-\left(\frac{(x-x_0)^2}{\xi+T_1}+\frac{(y-y_0)^2}{\xi+T_2}+\frac{(z-z_0)^2}{\xi+T_3}\right)\Big/4D\right]\Big/\left(\frac{1+\xi}{T_1}\right)^{1/2}\left(\frac{1+\xi}{T_2}\right)^{1/2}\left(\frac{1+\xi}{T_3}\right)^{1/2} \tag{3.7}$$

and

$$T_i=d_i^2/4D. \tag{3.8}$$

Equation (3.6) gives the contribution of a particular source to the cosmic-ray electron density at time t and position $\mathbf{r}$. The integral over t' can be computed for any choice of $K(t')$.

Before proceeding to discuss the result given by Eq. (3.6) for plausible Galactic sources, we shall first illustrate the physical effects of the competition between energy loss and spatial diffusion by considering the special case $K(t')=K_\delta\delta(t_0-t')$ and the limit $d_1=d_2=d_3\to 0$, i.e.,

$$Q_\delta(E',\mathbf{r}',t')=K_\delta E^{-\alpha}\delta(t_0-t')\delta(\mathbf{r}_0-\mathbf{r}'). \tag{3.9}$$

This source distribution represents contributions from a single event, such as a supernova explosion at the position $\mathbf{r}_0$ at time t_0. From Eq. (3.2) one easily obtains the result

$$N_\delta(E,\mathbf{r},t)=K_\delta g(\mathbf{r},t)f(E,t), \tag{3.10}$$

where

$$g(\mathbf{r},t)=[4\pi D(t-t_0)]^{-3/2} \times\exp[-(\mathbf{r}-\mathbf{r}_0)^2/4D(t-t_0)] \tag{3.11}$$

and

$$f(E,t)=E^{-\alpha}[1-bE(t-t_0)]^{\alpha-2},\quad E<1/b(t-t_0)$$
$$=0,\quad E>1/b(t-t_0). \tag{3.12}$$

Note that the factor $g(\mathbf{r},t)$ in (3.10) is small unless $t-t_0\approx(\mathbf{r}-\mathbf{r})^2/4D$, the approximate time for a particle to diffuse from the source to the observer. Also, $f(E,t)$ exhibits a cutoff for $t-t_0\gtrsim 1/bE$, the radiative lifetime of an electron with energy E. The injection slope α is retained for $E\ll 1/b(t-t_0)$, and the spectrum drops sharply as $E\to 1/b(t-t_0)$ for $\alpha>2$. For $1<\alpha<2$ the choice of a δ function for $K(t')$ causes a singularity in the differential spectrum as $E\to 1/b(t-t_0)$, but the integral spectrum

$$f(>E,t)=\frac{E^{-(\alpha-1)}}{\alpha-1}[1-bE(t-t_0)]^{\alpha-1}\left(E<\frac{1}{b(t-t_0)}\right) \quad (3.13)$$

remains finite.

Equation (3.10) is a good approximation for a source whose duration of production is much less than $t-t_0$, the time since the source became active. It is interesting to use Eq. (3.10) to estimate the impact of a nearby supernova explosion on the cosmic-ray intensity (both nuclei and electrons) at Earth. For nuclei we may take $b=0$ and disregard the modulation of the interstellar medium, since for close sources the propagation time is so short that neither Fermi acceleration nor ionization loss can have much effect. Consider, for example, the Crab Nebula at a distance of approximately 1200 pc with a total cosmic-ray energy of about 10^{49} erg. The bulk of the charged cosmic-ray particles will first reach the vicinity of earth $\sim 10^6$ yr after the explosion, but with an energy density $\sim 10^{-16}$ erg/cm^3, only $\sim 10^{-4}$ of the average background cosmic-ray intensity. This slight increase will last a few million years. A supernova occuring sufficiently close to earth could, however, have a more significant effect; a type-II supernova releasing $\sim 10^{50}$ erg in cosmic-ray particles exploding at a distance of 150 pc would approximately double the cosmic-ray intensity at Earth during the period of its maximum influence (approximately 10^4 yr). Unfortunately, such an increase, because of its short duration, is not likely to be detectable by the study of radioactive nuclei produced by cosmic rays in meteorites.

For electrons, energy loss in interstellar space cannot be neglected. Since an electron requires an average time $T_d(L)\approx L^2/2D$ to diffuse a distance L, we expect to find few electrons with radiative lifetime $T_R(E)=1/bE<T_d(L)$ at a distance L from any source. Conversely, electrons with energy E observed at Earth have been produced within a distance $L\approx(2D/bE)^{1/2}$ and a time $T\approx 1/bE$. For $E\approx 300$ BeV, the highest-energy electrons yet observed at Earth, the appropriate values are

$$L\approx 1\text{ kpc}, \; T\approx 1\times 10^6\text{ yr} \quad \text{(without the infrared)}$$

and

$$L\approx 300\text{ pc}, \; T\approx 7\times 10^4\text{ yr} \quad \text{(with the infrared)}.$$

These results indicate how recently and how close the high-energy electrons were produced.

Let us now specify the forms of the source distributions for which we shall calculate $N(E,\mathbf{r},t)$. Galactic cosmic-ray electrons may be divided into two components, those which were accelerated directly inside the cosmic-ray sources (the primary component) and those which were produced by nuclear collisions in space (the secondary component). The production spectrum and the source distribution of secondary electrons are well known; to a good approximation they can be expressed by

$$Q_s(E,\mathbf{r})=K_sE^{-2.67}\exp\left(-\frac{x^2+y^2+(z/p)^2}{R^2}\right) \quad (3.14)$$

for $E\gtrsim 1$ BeV. For smaller values of E the spectrum deviates from a power law.[19,32] The sources of primary electrons can be grouped, somewhat hypothetically, into a disk component

$$Q_d(E',\mathbf{r}')=K_dE'^{-\alpha}\exp\left(-\frac{x'^2+y'^2+(z'/p)^2}{R^2}\right), \quad (3.15)$$

a time-independent core component

$$Q_c(E',\mathbf{r}')=K_cE'^{-\alpha}\exp\left(-\frac{x'^2+y'^2+z'^2}{p^2R^2}\right), \quad (3.16)$$

and a time-dependent core component

$$Q_t(E',\mathbf{r}',t')=\sum_i K_i\delta(t'-t_i)E'^{-\alpha}\exp\left(-\frac{x'^2+y'^2+z'^2}{p^2R^2}\right), \quad (3.17)$$

where $p\approx 10^{-2}$ characterizes both the width of the disk and the radius of the core, the t_i are the times at which core explosions occurred, and the K's and α's give, respectively, the intensities and spectral indices of the various components.

The disk component represents the contribution from the so-called active stars such as supernovae, novae, flare stars, etc., which are often mentioned as likely sources of cosmic rays. All of these objects are concentrated near the Galactic plane, and there is no evidence that their frequency and character have changed significantly over the past few hundred million years (the time for cosmic rays to diffuse out of the Galaxy). Hence the disk component is assumed to be time-independent. The Galactic core, of which we know very little, has often been suggested as a powerful source of cosmic rays. Two core sources, one steady (Q_c) and one time-dependents (Q_t), are therefore considered. The former represents "quiet-time" production and takes into account the possibility that a large amount of cosmic rays is generated continuously in the Galactic center. The latter is associated with the hypothesis of Burbidge and Hoyle[33] that gigantic explosions, on the scale of those in Seyfert galaxies, have occurred in the Galactic core, and they contribute significantly to the cosmic-ray production rate.

The three source components are chosen for detailed discussion not because we believe that they represent exactly the distribution of actual cosmic-ray sources, but rather to illustrate the dependence of the resultant electron spectrum on the location of the sources. Any

[32] R. Ramaty and R. E. Lingenfelter, Phys. Rev. Letters **20**, 120 (1968).
[33] G. R. Burbidge and F. Hoyle, Astrophys. J. **138**, 57 (1963).

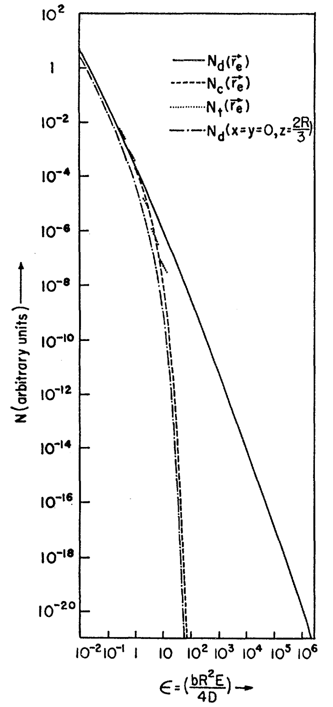

FIG. 2. Electron spectra at Earth for the disk, time-independent core, and time-dependent core sources, and in the halo for the disk source, in the isotropic-diffusion model. We have chosen $\alpha=2$, and equal time-averaged total production rates for all three sources.

reasonable source-distribution model for the Galactic primary electrons can always be, on a gross scale, represented by superposition of the three components Q_d, Q_c, and Q_t. Comparison of the theoretical resultant spectrum with the observational results will then indicate the relative importance of contributions from each source component to the present-day cosmic-ray flux. The resultant electron density $N(E,\mathbf{r},t)$ for each of the source components can be obtained readily by substituting the appropriate values of $K(t')$, α, d_1, d_2, d_3, and $\mathbf{r}_0=0$ into Eqs. (3.6)–(3.8).

Before displaying numerical solutions to $N(E,\mathbf{r},t)$ for the above sources, we shall first discuss physically the solution for steady-state $[K(t')=\text{const}]$ sources of the form (3.5). To discern the general features of the result we may, with sufficient accuracy, approximate the solution by

$$N(E,\mathbf{r})\propto E^{-(\alpha+\delta)}|\mathbf{r}-\mathbf{r}_0|^{-\gamma}\exp\left[-\frac{(x-x_0)^2}{d_1^2}\frac{E}{E_1+E}-\frac{(y-y_0)^2}{d_2^2}\frac{E}{E_2+E}-\frac{(z-z_0)^2}{d_3^2}\frac{E}{E_3+E}\right], \tag{3.18}$$

where $E_i=2D/bd_i^2$, $\delta=0$ for $E\ll\min E_i$, and $\delta=1$ for $E\gg\max E_i$. The index $\gamma=0$ in or near the source region and increases to $\gamma=1$ for $|\mathbf{r}-\mathbf{r}_0|\gg\max d_i$. It is evident that the exponential term in Eq. (3.18) is nearly unity within the source, where the equilibrium density $N(E,\mathbf{r})$ retains the injection spectral index α for $E<\min(E_i)$, but gradually steepens until the index becomes $\alpha+1$ for $E>\max(E_i)$. If $d_1\approx d_2\approx d_3$, a spherical source region, there is one rather well-defined break; if the source is sufficiently far from spherical, there is a larger transition region over which the spectrum within the source steepens. For the case of an extremely flattened oblate spheroid, such as the Galactic disk, $d_1\approx d_2\ll d_3$, there is a well-defined intermediate-energy range $2D/bd_1^2\ll E\ll 2D/bd_3^2$, in which the spectral index is $\alpha+\frac{1}{2}$. Outside the source region the spectrum is a power law with index α for $E<2D/bd_s^2$, where d_s is the distance to the source, and for $E>2D/bd_s^2$ the spectrum drops off exponentially with increasing energy. Note that even at low energy the particle density falls off as r^{-1} for a compact source, where r is the distance from the source. This is as expected since at low energy for a steady-state source, Eq. (2.7) reduces to Poisson's equation.

In order to illustrate the effects of interstellar modulation, we now discuss the properties of $N(E,\mathbf{r})$ for each of the source distributions discussed above. The electron spectrum at Earth, for which the experimental data have been extended to the few-hundred-BeV range, can be readily obtained for each source by choosing $\mathbf{r}=\mathbf{r}_e=(\frac{2}{3}R,0,0)$. To demonstrate the general shape of $N(E,\mathbf{r}_e)$ for each source component we have plotted in Fig. 2 the quantities $N_d(E,\mathbf{r}_e)$, $N_c(E,\mathbf{r}_e)$, and $N_t(E,\mathbf{r}_e)$ for a common production spectral index $\alpha=2$. Since the disk distribution is not spherically symmetric, we have also plotted $N_d(E,x=y=0,\ z=\frac{2}{3}R)$ in order to illustrate the electron spectrum in the halo for the disk source. The times t_i in Eq. (3.17) have been chosen such that $t-t_1=10^7$ yr, $t-t_2=2\times10^7$ yr, etc. Accordingly, there is a "step" in the spectrum at $E_i=R^2/4D(t-t_i)$. The horizontal axis is plotted in terms of the dimensionless energy $\epsilon=bR^2E/4D$. Without the infrared radiation discussed above,

$$1<4D/bR^2<10\text{ BeV}, \tag{3.19}$$

while with the infrared ($w_{IR}=13$ eV/cm^3),

$$50<4D/bR^2<500\text{ MeV}. \tag{3.20}$$

The normalizations in Fig. 2 are chosen such that the production rates (or in the case of Q_t the time-averaged rate) integrated over the source volume are the same for all these sources. Thus, for a core source to contribute as many low-energy electrons at Earth as a disk source, it would be necessary to have $K_c\approx p^{-2}K_d$, or the production rate per unit volume in the core would need to be 10^4 times larger than in the disk.

The differences between N_d and N_c illustrate the effects of interstellar modulation. A particle travels a distance L by random walk (diffusion) through the irregular Galactic magnetic field in a time of the order of $L^2/2D$; the particle also loses most of its original energy E_0 after a time $1/bE_0$. Hence, the majority of electrons with energy E observed at any point in space must have been produced within a local sphere of radius $(2D/bE)^{1/2}$, and within a time of the order of $1/bE$. Since Earth is about 8 kpc from the Galactic core, few electrons with $E>30$ BeV (or 1 BeV if the infrared radiation fills the Galaxy) can reach Earth from a core source. On the other hand, the disk-component electrons are produced throughout the Galactic plane, so the shrinking of $(2D/bE)^{1/2}$ does not exclude high-energy electrons produced in the disk from reaching us (although they are still excluded from the halo). Consequently, the resultant disk-equilibrium spectrum from a disk source steepens very gradually at high energies. For our choice of the times of the core "explosions" there are no electrons left from the source Q_t for $E>1/b(t-t_1)\approx 40$ BeV (or 2 BeV with the infrared radiation).

It is also of interest to study the spatial dependence of $N(E,\mathbf{r})$ for the various sources, since the background radio and γ radiation observed at Earth depend on the electron distribution throughout the Galaxy. In Fig. 3 we have plotted (in terms of the dimensionless distance $\xi=|\mathbf{r}|/R$) the quantities $N_d(\epsilon,\ x=\xi R,\ y=z=0)$, $N_d(\epsilon,\ x=y=0;\ z=\xi R)$, $N_c(\epsilon,\ |\mathbf{r}|=\xi R)$, and $N_t(\epsilon,\ |\mathbf{r}|=\xi R)$ for the values $\epsilon=10^{-2}$, 1, 10^2, and 10^3. At $\epsilon=10^{-2}$ the effects of spatial diffusion in the limit of low energy loss (this limit is applicable to nuclei also) are evident; at $\epsilon=1$ a particle can still diffuse a distance approximately equal to R before losing most of its energy; at $\epsilon=10^3$ energy loss has become dominant, limiting electrons to regions close to the source. Radiation in the radio-frequency range is produced with electrons having $\epsilon\lesssim 1$. γ rays are produced by much higher-energy electrons, $\epsilon\approx 10^2$ without the infrared and $\epsilon\approx 10^3$ if the infrared fills the Galaxy.

IV. ISOTROPIC DIFFUSION WITH BOUNDARY CONDITION

In the previous section we studied in detail the solution of the isotropic-diffusion equation in infinite space. In reality the diffusion coefficient is much larger outside the Galaxy than inside. (It is also evident that D should be an increasing function of distance from the Galactic disk; our present knowledge of cosmic conditions, however, is too limited for an elaborate treatment of the positional dependence of D, so in this section we consider $D=$const throughout the Galaxy. The case $D_{\text{halo}}\gg D_{\text{disk}}$ is discussed in Sec. V, where we consider the convection-diffusion model.) Therefore, once a particle escapes across the boundary into metagalactic space, it has little chance of returning to the Galaxy. The metagalaxy is, in fact, a sink for cosmic-ray electrons. Hence a more realistic approach within the isotropic-diffusion model is to impose at the Galactic boundary the condition

$$\left.\frac{\partial N}{\partial r}\right|_{r=R}=-\frac{\beta}{R}N\bigg|_{r=R}, \qquad (4.1)$$

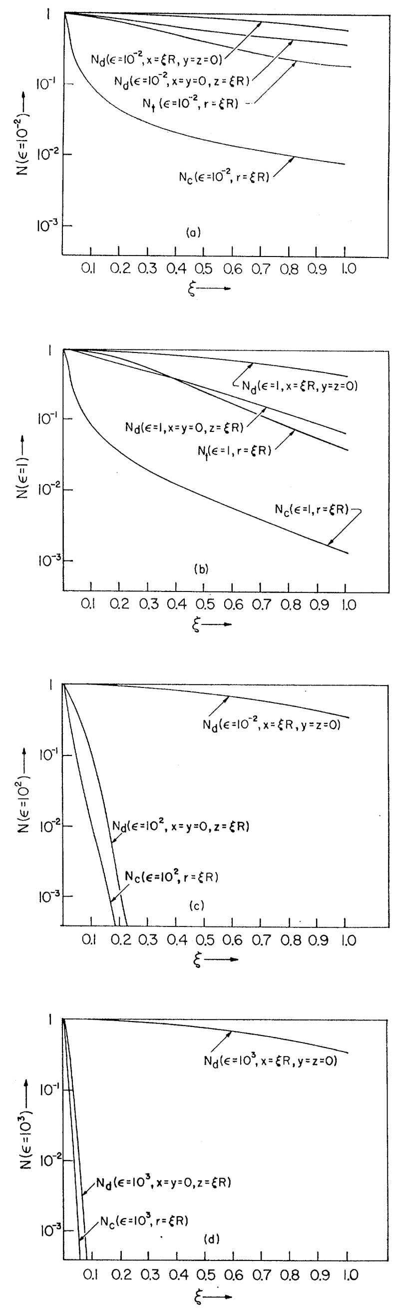

FIG. 3. Spatial dependence of N for the disk, time-dependent core, and time-independent core sources in the isotropic-diffusion model, for $\epsilon=10^{-2}$, 1, 10^2, and 10^3.

where β is a parameter characterizing the ease with which particles leak out of the Galaxy; $\beta=0$ and $\beta=\infty$ correspond, respectively, to total reflection and free departure at the Galactic boundary. The general solution of Eq. (2.7) with boundary condition (4.1) can be expressed easily in an integral form. Numerical evaluation for an arbitrary source involves, however, considerable labor. To illustrate the proper treatment for

TABLE II. Properties of $N(E,r)$ in a bounded region.

β	r_0	$N(E,r)$ for small E	T_L
∞	0	$(E^{-2}/4\pi D)(1/r-1/R)$ $(E\ll\pi^2D/bR^2)$	$R^2/6D$
∞	R	$(E^{-2}/8\pi D)(1/R-r^2/R^3)$ $(E\ll\pi^2D/bR^2)$	$R^2/15D$
1	0	$(E^{-2}/4\pi D)(1/r)$ $(E\ll\pi^2D/4bR^2)$	$R^2/2D$
1	R	$(E^{-2}/8\pi D)(3/R-r^2/R^3)$ $(E\ll\pi^2D/4bR^2)$	$2R^2/5D$
$\rightarrow 0$	0	$(3/4\pi bR^3)E^{-3}$	$R^2/3D\beta$
$\rightarrow 0$	R	$(3/4\pi bR^3)E^{-3}$	$R^2/3D\beta$

the "leakage" of particles and to estimate the effects of the boundary condition on the equilibrium cosmic-ray concentration $N(E,\mathbf{r})$, we shall consider the spherically symmetric source

$$Q(E,r)=(3/4\pi r_0^3)/E^{-2} \quad (0\leq r\leq r_0<R). \tag{4.2}$$

Then from Appendix A, Eqs. (A16)–(A19), we have

$$N(E,r)=\frac{3E^{-2}}{2\pi Dr}\sum_{n=1}^{\infty}\frac{[\lambda_n^2+(\beta-1)^2][1-e^{-\lambda_n^2/4\epsilon}]}{[\lambda_n^2+\beta(\beta-1)]\lambda_n^4}\times\zeta_n(r_0)\sin(\lambda_n r/R), \tag{4.3}$$

where

$$\zeta_n(r_0)=(R/r_0)^3[\sin(\lambda_n r_0/R)-(\lambda_n r_0/R)\cos(\lambda_n r_0/R)], \tag{4.4}$$

$$\epsilon=R^2bE/4D, \tag{4.5}$$

and λ_n satisfies the transcendental equation

$$\lambda_n\cot\lambda_n=1-\beta. \tag{4.6}$$

The leakage lifetime of the particles in a region characterized by the boundary condition (4.1) may be defined as

$$T_L=\int d\mathbf{r}\int dE\,N(E,\mathbf{r})\Big/\int d\mathbf{r}\int dE\,Q(E,\mathbf{r}). \tag{4.7}$$

The integration over energy is to ensure that no particle be lost due to deceleration.

For a source distribution of the form (4.2),

$$T_L=T_d\sum_{n=1}^{\infty}\frac{12[\lambda_n^2+(\beta-1)^2]\beta\sin\lambda_n}{\lambda_n^6[\lambda_n^2+\beta(\beta-1)]}\zeta_n(r_0), \tag{4.8}$$

where $T_d=R^2/2D$ characterizes the time of a particle to travel across the confinement region through diffusion. The values of T_L for $\beta\ll1$, $\beta=1$, and $\beta=\infty$ are evaluated for a point source ($r_0=0$) and a uniform source ($r_0=R$) and listed in Table II. As expected, for $\beta\geq1$ (weak confinement), T_L is approximately given by the diffusion time $T_d=R^2/2D$, while for $\beta\ll1$, T_L varies as T_d/β. The average "age" of electrons with energy E contained within the confinement region is given approximately by

$$T(E)=\int d\mathbf{r}\,N(E,\mathbf{r})\Big/\int d\mathbf{r}\,Q(E,\mathbf{r})=T_d\sum_{n=1}^{\infty}\frac{12[\lambda_n^2+(\beta-1)^2]\beta\sin\lambda_n\zeta_n(r_0)[1-e^{-\lambda_n^2/4\epsilon}]}{\lambda_n^6[\lambda_n^2+\beta(\beta-1)]} \tag{4.9}$$

for the source (4.2). It is evident that for $\epsilon\ll1$, $T(E)\rightarrow T_L$. For $\epsilon\gg1$, because of the fast convergence of the series, $1-e^{-\lambda_n^2/4\epsilon}$ can be replaced by $\lambda_n^2/4\epsilon$ and

$$T(E)\underset{\epsilon\gg1}{\approx}\frac{1}{bE}\sum_{n=1}^{\infty}\frac{6[\lambda_n^2+(\beta-1)^2]\beta\sin\lambda_n}{[\lambda_n^2+\beta(\beta-1)]\lambda_n^4}. \tag{4.10}$$

In Appendix A it is shown that the summation is equal to unity for all values of β and r_0, so the age of a high-energy particle is, as expected, approximately its radiative lifetime $(bE)^{-1}$, independent of the boundary condition and the source distribution.

In order to estimate the effects of the boundary condition on the equilibrium intensity we have evaluated $N(E,r)$ explicitly for the cases $\beta=0$, 1, and ∞. It is obvious that for $\epsilon\gg1$ the Galactic boundary plays little role because few electrons of that energy ever reach there. In this energy range, the equilibrium distribution calculated with no boundary condition is a good representation of $N(E,r)$ regardless of the value of β. The situation is a little different in the energy range $\epsilon\lesssim1$, for which $N(E,r)$ is listed in Table II for the three values of β. It is interesting to note that in the low-energy limit $N(E,r)\propto E^{-2}$ for $\beta\gtrsim1$, but rises more sharply as E^{-3} for $\beta\ll1$. The case $\beta=1$ corresponds closely to the situation of isotropic diffusion in infinite space; they are identical if the source is at the origin. This can be shown by substituting $Q(E',\mathbf{r}')=\delta(\mathbf{r}')E^{-2}$ into Eq. (3.2). We have

$$N_\infty(E,r)=\frac{E^{-2}}{4\pi^{3/2}Dr}\int_{r^2bE/4D}^{\infty}dx\,x^{-1/2}e^{-x}\approx E^{-2}/4\pi Dr\quad(\text{for }E\ll4D/r^2b). \tag{4.11}$$

This is precisely the result obtained with the boundary condition $\beta=1$ and $r_0=0$.

The results obtained above are, of course, expected. They illustrate the inadequacy of replacing the boundary condition (4.1) and the diffusion term $-D\nabla^2N$ in the transfer equation by a uniform-leakage term N/T_L. The electrons, unlike the nuclei, are subject to deceleration; thus for a compact source the high-energy ones have little chance to fill the Galaxy and cannot escape, since such escape can in reality occur only at the boundary. On the other hand, even for low-energy electrons (and hence nuclei) the equilibrium distribution may be far from uniform in space.

By inspecting Table II we can draw several conclusions about the effects of the boundary on the equilibrium intensity $N(E,\mathbf{r})$. In the case of weak confinement ($\beta\gg1$), even though the boundary condition forces N to equal zero at $r=R$, it does not greatly affect the energy spectrum of electrons at positions not near the boundary. Comparison with the results obtained from diffusion in infinite space reveals that while the age of a low-energy particle is smaller by a factor of $\sim$3–5 the high-energy flux is unaffected and the low-energy flux still has a spectral index equal to the production spectral index. But the "break" now occurs at a higher energy, $E\approx\pi^2D/R^2b(\epsilon\approx\frac{1}{4}\lambda_1^2)$; consequently, the low-energy intensity is somewhat reduced (because electrons are not allowed to return after they have leaked out of the Galaxy). A strongly reflecting boundary ($\beta\ll1$), on the other hand, produces significant changes in the equilibrium energy spectrum; it is steeper at all energy, by one power, than the production spectrum for both a point source and a uniform source. In general, if the observation is made inside an extended source region, such as from Earth for a disk source, one would see a straight power-law spectrum with no breaks in the case of strong reflection.

With regard to cosmic-ray propagation in the Galaxy, the matter traversed is related to T_L by

$$x\approx\rho cT_L,$$

where ρ is the mean density. From our knowledge of ρ and x, T_L can hardly be much larger than 10^8 yr (it could be smaller). Therefore, unless we are willing to increase the value of D to much greater than 10^{29} cm^2/sec, β cannot be much smaller than 1. Cosmic-ray particles are likely to depart from the Galaxy with little or no reflection from the boundary. In this case, corrections introduced by the existence of a metagalactic "sink" can, for all practical purposes, be neglected.

V. CONVECTION-DIFFUSION MODEL

In Secs. III and IV we presented a detailed analysis of the propagation of cosmic rays in the isotropic-diffusion approximation. In this section we discuss briefly the propagation of cosmic-ray electrons in the convection-diffusion approximation

$$-\frac{\partial}{\partial E}(bE^2N)+\frac{\partial}{\partial z}[v_d(z)N]-D_{\shortparallel}\left(\frac{\partial^2N}{\partial x^2}+\frac{\partial^2N}{\partial y^2}\right)=Q. \quad (5.1)$$

In this model the electrons drift away from the disk along with the outwardly expanding field lines. The solution of Eq. (5.1) is given in Appendix B for a disk source and the drift velocity

$$v_d(z)=v_0z/pR \quad (|z|\leq pR) \quad (5.2)$$

$$=v_0 \quad (|z|>pR). \quad (5.3)$$

where $v_0\approx10^2$ km/sec characterizes the expansion velocity. In this model the characteristic time for a disk particle to move into halo through convection is $T_c\simeq pR/v_0$,while the time for a particle to diffuse a distance of R in the disk is $\sim T_{\shortparallel}=R^2/2D_{\shortparallel}$. For $D_{\shortparallel}\approx10^{29}$ cm^2/sec, we have $T_c\cong10^{-2}T_{\shortparallel}=10^6$ yr; hence most electrons drift into the halo instead of diffusing out the open ends of the spiral arms. Because $T_c\ll T_{\shortparallel}$, we may approximately express the solution for $\alpha=2$ as

$$N(E,\mathbf{r})\approx KT_cR^{-(x^2+y^2)/R^2}E^{-2}$$

$$\times[1-e^{-1/bET_c}] \quad (|z|\leq pR)$$

$$\approx KT_c\frac{e^{-(x^2+y^2)/R^2(1+z/v_0T_{\shortparallel})}}{1+z/v_0T_{\shortparallel}}E^{-2} \quad (5.4)$$

$$\times[1-e^{-(1/bET_c-z/pR+1)}] \quad (pR\leq|z|\leq pR+v_0/bE) \quad (5.5)$$

$$=0 \quad (|z|\geq pR+v_0/bE). \quad (5.6)$$

The convection-diffusion approach, of course, does not strictly correspond to reality; perpendicular diffusion and disorder in the field lines are likely to play an important role in bringing cosmic rays into the halo. In addition, the particular form of $v_d(z)$ is chosen more for mathematical convenience than for physical reasons. Nevertheless, $N(E,\mathbf{r})$ obtained in Eqs. (5.4)–(5.6) should, as pointed out in Sec. II, indicate the general features expected from any "closed-disk model," i.e., one in which particles are confined in the disk for an average lifetime T_c, but do not reenter the disk after leaking into the halo.

For comparison, we have plotted in Fig. 4 the equilibrium spectra at Earth and at $z=\frac{2}{3}R$ for a disk source in the convection-diffusion model and in the isotropic-diffusion model. We have assumed $\alpha=2$ and the same total production rate for the two models. It is interesting to note that in the convection-diffusion approach the spectral index at Earth exhibits one break of one power at $\epsilon\approx10^2$ ($E\approx1/bT_c$), while in the isotropic-diffusion model there are two half-power breaks at $\epsilon\approx1$ and $\epsilon\approx10^4$. In the halo the equilibrium spectrum in the convection model has an even sharper drop, but at a higher energy ($\epsilon\approx10^2pR/z$) than in the isotropic-diffusion model.

VI. COMPARISON OF THEORY AND EXPERIMENT

In Secs. II–V we have used the general transfer equation for cosmic-ray electrons to discuss various aspects of electron propagation in the Galaxy. It has been shown that the spatial distribution of high-energy cos-

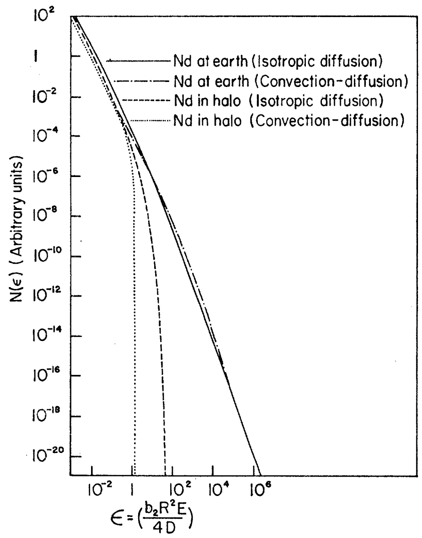

FIG. 4. Electron spectra at Earth and in the halo for a disk source in the isotropic-diffusion and convection-diffusion models. We have chosen $\alpha=2$ and equal production rates in both models.

mic-ray electrons is insensitive both to the boundary condition at the edge of the Galaxy and to the exact model of electron propagation; the dominating factor is the spatial distribution of the sources of the electrons. With these results in mind we proceed to discuss the related observable quantities, such as the electron spectrum and positron fraction at Earth and the background electromagnetic radiations. By studying these experimental quantities we shall be able to draw some interesting conclusions about when and where the bulk of the Galactic cosmic-ray electrons, and perhaps also the nuclei, were produced.

The most crucial data for determining the electron source distribution are, of course, the background cosmic γ rays produced by high-energy electrons, since these electrons will not have time to propagate very far from the sources. As has been discussed elsewhere by Shen[34] and by Cowsik and Pal,[35] a possible source of the background γ ray flux above 100 MeV detected in the disk plane of the Galaxy[16,36,37] is the inverse Compton scattering of cosmic-ray electrons of energy $\gtrsim 100$ BeV with the aforementioned infrared radiation. This hypothesis is attractive in that it brings these two rather unexpected discoveries into good numerical agreement.

It had been shown[38] that the γ-ray photons produced in interstellar space by other mechanisms, such as decay of neutral pions, are not sufficient to give the observed intensity. The only alternative, other than Compton scattering, is to assume the diffuse flux to be the superposition of discrete but unresolved γ sources. But, according to the calculation of Ögelman,[39] in order to give the observed intensity, an average flux of 5×10^{-5} cm^{-2} sec^{-1} per source would be required. Recent observation by Frye and Wang[37] found no source above their threshold of 10^{-5} cm^{-2} sec^{-1} either on or off the Galactic plane.[40] We shall then, mainly to demonstrate the proper way of constructing cosmic-ray source models, assume that an infrared flux of 13 eV/cm^3 exists in our Galaxy. For the purpose of calculation, we have also assumed that the photons are of average energy $\epsilon_0\approx 2\times10^{-3}$ eV and distributed uniformly in space. The results obtained, however, are not strongly dependent on these two assumptions as long as $\epsilon_0 \ll (mc^2)^2/E_\gamma$ and the infrared does exist throughout the disk plane.

The differential γ-ray intensity toward a certain direction produced by Compton scattering for gamma-ray energies $E_\gamma \ll (mc^2)^2/\epsilon_0$ is related to the cosmic-ray electron intensity by (see Ref. 41 and references cited therein)

$$I_c(E_\gamma)\approx E_\gamma^{-1/2}A_1\int d\mathbf{l}\, w_{\rm ph}(\mathbf{r})I_e(A_2E_\gamma^{1/2},\mathbf{r})\,, \quad (6.1)$$

where the integration is along the direction of observation,

$$A_1=\tfrac{1}{4}\sqrt{3}\sigma_T mc^2\epsilon_0^{-3/2}\,, \quad (6.2)$$

$$A_2=(3/4\epsilon_0)^{1/2}mc^2\,, \quad (6.3)$$

σ_T is the Thomson cross section, $w_{\rm ph}(\mathbf{r})$ is the photon energy density, and ϵ_0 is the mean photon energy of the ambient radiation. For the infrared, $w_{\rm ph}\approx 13$ eV/cm^3 and $\epsilon_0\approx 2\times10^{-3}$ eV,

$$A_1 w_{\rm ph}(\mathbf{r})\approx 2.2\times10^{-14}\ ({\rm eV})^{1/2}/{\rm cm}\,, \quad (6.4)$$

$$A_2\approx 1.0\times10^{7}\ ({\rm eV})^{1/2}. \quad (6.5)$$

The directional flux calculated from (6.1) can be compared directly with experiment only if the angular acceptance of the detector used is smaller than the width of the γ-ray-producing region. For results obtained with a detector of poor resolution, integration over the acceptance angle is necessary.

As has been shown in Ref. 34, a disk-source distribution of the form (3.15),

$$Q(E,\mathbf{r})=K_dE^{-\alpha}\exp\left[-\frac{x^2+y^2+(z/p)^2}{R^2}\right], \quad (6.6)$$

with $\alpha=2$ and $K_d\approx10^{-25}$ electrons BeV (cm^3 sec)$^{-1}$, predicts a γ-ray disk of line intensity 2×10^{-4} photons (cm^2

[34] C. S. Shen, Phys. Rev. Letters **22**, 568 (1969).
[35] R. Cowsik and Y. Pal, Phys. Rev. Letters **22**, 550 (1969).
[36] R. K. Sood, Nature **222**, 650 (1969).
[37] G. M. Frye and C. P. Wang, Report, 1969 (unpublished).
[38] G. G. Fazio, Ann. Rev. Astronomy Astrophys. **5**, 481 (1967).
[39] H. Ögelman, Nature **221**, 754 (1969).
[40] *Note added in manuscript.* In a later report, Frye *et al.* [in Proceedings of the Eleventh International Conference on Cosmic Rays (unpublished)] also fail to find the strong γ flux from the Galactic center detected by Clark *et al.* (Ref. 16). Apparently there is some discrepancy between Frye *et al.* and Clark *et al.* in the calibrating of their apparatus. Therefore, the negative result of Frye *et al.* in detecting point γ-ray sources probably cannot be considered as a decisive evidence against Ögelman's suggestion.
[41] C. S. Shen and G. Berkey, Astrophys. J. **151**, 895 (1968).

sec rad)$^{-1}$. The equilibrium spectrum $N_e(E,\frac{2}{3}R)$ calculated from (6.6) also agrees well with the electron intensity at Earth for $E\geq 1$ BeV, the energy at which solar modulation is insignificant. There is one discrepancy, though: The OSO-III data[16] show a maximum by a factor of more than 2 in the direction of the Galactic center, while the intensity calculated from (6.6) indicates only a 50% increase. The observed maximum exceeds that which would result from the Compton scattering of the disk cosmic-ray sources [Eq. (3.15)] by

$$I_\gamma(\text{excess})\approx 2\times 10^{-4}\ \text{photons/cm}^2\ \text{sec}.$$

The direction and the angular extension of this excess flux strongly indicate that the source is in the Galactic center. If we make this assumption, we can estimate the total emission rate of the >100-MeV photons from there:

$$Q_\gamma(\text{excess})\approx 4\pi r^2 I_\gamma(\text{excess}) \approx 1.5\times 10^{42}\ \text{photons/sec}. \quad (6.7)$$

If the origin of at least a substantial part of this excess flux is due to Compton scattering, then either the radiation density or the cosmic-ray density (or both) must be higher at the Galactic center than at Earth. If we assume the far-infrared intensity at the core is similar to that at Earth, the required excess number of electrons in the Galactic center at $E\geq 100$ BeV would be

$$N(\text{excess})\approx \frac{Q_\gamma(\text{excess})}{cE_\gamma^{1/2}A_1 w_{\text{ph}}} \times 10^{11}\ \text{eV}\approx 2\times 10^{52}\ \text{electrons}. \quad (6.8)$$

It is interesting to compare this with the total number of electrons at $E>100$ BeV from the source equation (6.6) obtained by integrating $N(E,\mathbf{r})$ over all space:

$$N(\text{disk})=2\times 10^{52}\ \text{electrons}.$$

Since at this energy the electron lifetime is the radiative lifetime, we are led to the interesting conclusion that the total production rate of the excess core electrons is comparable to that of the disk electrons at $E\geq 100$ BeV. The production rate per unit volume in the Galactic center, is of course, much higher than that in the disk. The ratio will be of the order 10^2, depending on the size of the core production volume.

On the other hand, the infrared intensity may be larger in the center of the Galaxy than elsewhere in the disk. Hoffmann and Frederick[42] have reported an excess flux of 6×10^{-2} erg/(cm^2 sec sr) from 80 to 120 μ centered on the Galactic core with an extension more than 6.5° along the disk plane but less than 2° perpendicular to it. This by itself is insufficient to produce the observed excess γ-ray flux by Compton scattering if the electron intensity in the Galactic center is the same as at Earth.

[42] W. F. Hoffmann and C. L. Frederick, Astrophys. J. Letters **155**, L9 (1969).

To illustrate this point we shall derive, from Eq. (6.1), a formula which gives the relation among the intensity of a compact infrared source, the ambient cosmic-ray electron intensity, and the resultant Compton flux. The radiation density $w_{\text{ph}}(\mathbf{r})$ due to a compact source with emission power $P_{\text{IR}}(\mathbf{r})$ per unit volume is

$$w_{\text{ph}}(\mathbf{r})=\int_{\text{emisson region}} \frac{P_{\text{IR}}(\mathbf{r}')}{4\pi c|\mathbf{r}-\mathbf{r}'|^2}d\mathbf{r}'. \quad (6.9)$$

Because the infrared flux outside the emission region is not locally isotropic, the resultant γ-ray photons will not, strictly speaking, be produced isotropically. The deviation from isotropy is, however, insignificant as long as the electrons which scatter the infrared photons are isotropic and ultrarelativistic. Therefore, for cases of interest we can calculate the excess γ-ray flux from Compton scattering of the excess infrared photons by substituting Eq. (6.9) into (6.1). Exact evaluation requires a knowledge of the size, geometry, and local emission rate of the infrared source. For estimation purposes one may replace $w_{\text{ph}}(\mathbf{r})$ by

$$\begin{aligned} w_{\text{ph}}(\mathbf{r})&=Q_{\text{IR}}/4\pi c r_s^2 \quad \text{for } |\mathbf{r}|\leq r_s \\ &=Q_{\text{IR}}/4\pi c r^2 \quad \text{for } |\mathbf{r}|>r_s, \end{aligned} \quad (6.10)$$

where r_s is the extension of the infrared source and Q_{IR} is its total emission rate. The approximation is especially apt for a spherical source of uniform emissivity. Clearly Q_{IR} is related to the excess infrared intensity observed at Earth, I_{IR} (in eV/cm^2 sec), by

$$Q_{\text{IR}}=4\pi R_e^2/I_{\text{IR}}, \quad (6.11)$$

where $R_e\approx 8$ kpc is the distance from the source to Earth. Combining Eqs. (6.1), (6.10), and (6.11) gives the excess γ-ray intensity at Earth

$$I(E_\gamma)=\frac{A_1}{c}I_{\text{IR}}E_\gamma^{-1/2}\int d\mathbf{l}\, I_e(A_2E_\gamma^{1/2},\mathbf{r})F(r), \quad (6.12)$$

where $F(\mathbf{r})=(R_e/r_s)^2$ inside the source and $F(\mathbf{r})=(R_e/r)^2$ outside the source. From Earth the infrared source subtends an angle $\theta_s\approx r_s/R_e\approx 0.05$ rad. For a γ-ray detector with resolution θ_d such that $\theta_s<\theta_d\ll 1$, the total excess counting rate above an energy E_γ obtained by integrating Eq. (6.11) over the resolution cone of the detector is

$$J(>E_\gamma)\approx 2\pi^2 R_e\theta_d(A_1/c)I_{\text{IR}}E_\gamma^{+1/2}\langle I_e(A_2E_\gamma^{1/2})\rangle, \quad (6.13)$$

where $I_e(A_2E_\gamma^{1/2})$ is a weighted average of the electron intensity, weighted heavily for the electron density near $|\mathbf{r}|\lesssim r_0$, because the infrared photon density is largest there. On the other hand, the excess counting rate for a detector of angular resolution $\theta_d\lesssim\theta_s$ is

$$J(>E_\gamma)\approx 4\pi R_e\frac{\theta_d^2}{\theta_s}\frac{A_1}{c}I_{\text{IR}}E_\gamma^{+1/2}\langle I_e(A_2E_\gamma^{1/2})\rangle. \quad (6.14)$$

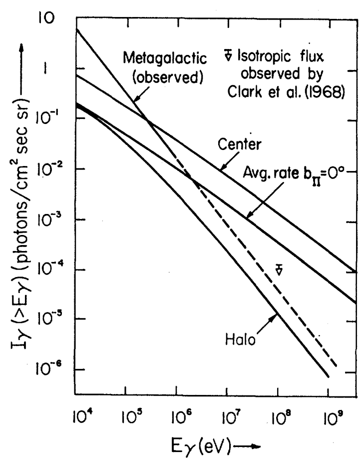

FIG. 5. Integral spectrum of Compton photons from scattering of infrared radiation by electrons from a disk and a time-independent core source. The dashed line represents the expected contribution of the metagalactic component.

The flux density from the infrared source discovered by Hoffmann and Frederick[42] is $1.4\pm0.6\times10^{8}$ eV/cm² sec, so the total excess counts above 100 MeV that should be observed with the detector in the OSO-III experiment ($\theta_d\approx\frac{1}{4}$) is approximately

$$J(>100\text{ MeV})\lesssim10^{-6}\text{ photons/cm}^2\text{ sec}. \quad (6.15)$$

This result, which is based on the assumption that the electron intensity in the Galactic center is similar to that at Earth, is two orders of magnitude lower than the excess measured by the OSO-III detector.[16] This conclusion, however, must be viewed as tentative since the observed infrared flux covers only a 40-μ wavelength range and the source apparently extends beyond the scanning range of 6.5° along the Galactic plane. It is likely that the densities of both far-infrared photons and high-energy electrons are higher in the Galactic center than that at Earth.

In consistency with discussions presented above, let us consider a two-component–source model for primary cosmic-ray electrons: a disk component

$$Q_d=10^{-25}E^{-2}\exp\left(-\frac{x^2+y^2+10^4z^2}{R^2}\right)\quad\text{electrons/(BeV cm}^3\text{ sec)} \quad (6.16)$$

and a core component

$$Q_c=10^{-23}E^{-2}\exp\left[-\frac{10^2(x^2+y^2)+10^4z^2}{R^2}\right]\quad\text{electrons/(BeV cm}^3\text{ sec)}. \quad (6.17)$$

(The extension of the "core component" in the disk plane to $0.1R$ is necessary because the excess core γ-ray photons apparently have an extension $\approx15°$ along the Galactic plane.[16] Thus the source must extend ≈1.5 kpc.) Together with the collision-produced secondary electrons Q_s, these constitute the Galactic cosmic-ray electrons. The equilibrium density $N=N_d+N_c+N_s$ and the observable quantities related to it can be readily calculated by the solutions obtained in the previous sections. We shall first consider the case of isotropic diffusion with a uniform infrared background of 13 eV/cm³. The integral spectrum of the Compton photons produced by scattering of $N(E,\mathbf{r})$ with the infrared is shown in Fig. 5. As expected, the flux above 100 MeV agrees well with the OSO-III results.[16] The equilibrium electron spectra at Earth, $N(E,\frac{2}{3}R)$, from these three source components are shown in Fig. 6. As can be seen from Fig. 2, a core source and a disk source with the same production spectral indices and total production rates produce comparable amounts of electrons at Earth for $E\lesssim300$ MeV (the increase in size of the core source does not affect this conclusion); above 300 MeV the core electron flux is insignificant at Earth. The secondary intensity is derived from the production spectrum of Ramaty and Lingenfelter,[19,31] but owing to the modulation of the infrared and spatial diffusion, our equi-

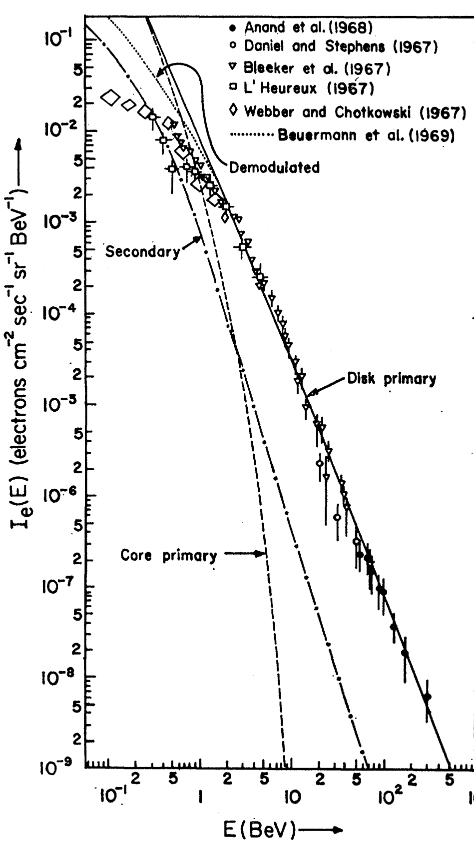

FIG. 6. Contributions to the electron spectrum at Earth from secondary, disk primary, and core primary electrons in the isotropic-diffusion model. The primary sources have a production spectral index $\alpha=2$ and equal total production rates.

librium spectrum is somewhat steeper than they obtain. The flux at a few hundred MeV is, however, equal to theirs

If one calculates the positron fraction in this model, one finds that $f_+ \approx 0.06$ around 200 MeV, which is a factor of 5 lower than the value $f_+ \approx 0.3$ found by Hartman[9] and by Beuermann *et al.*[8] (The value of f_+ would be too small even without the infrared and without the core electron component.) This discrepancy can be eased somewhat if one considers the possibility that positrons and negatrons may be modulated differently by the deceleration of the solar wind,[8] although the energy loss of particles in this energy range should be small.[43] Another possibility is that the normalization of the secondary component be raised, but this could hardly account for so large a discrepancy in the positron fraction.

A more natural explanation is that the equilibrium electron spectral index α_e is smaller than 2 at energies less than 1 BeV. This conclusion, as pointed out by previous authors,[17,18] is consistent with the radio data. The synchrotron radiation in the frequency range where it is separable from the blackbody background is produced by low-energy ($E<10$ BeV) electrons, which are not so sensitive to interstellar modulation as the γ-ray-producing electrons. The radio spectrum depends more on the selection of a Galactic-magnetic-field model than the selection of cosmic-ray source model. We shall not, therefore, present here details of our calculation of the halo radio spectrum, which, in any case, differs little from the previous calculations (see, e.g., Refs. 17 and 18). There are, however, two qualitative conclusions worthy of mention. First, the apparent depletion of energetic electrons in the halo due to the existence of a strong infrared field does not contradict the observed radio data. Even in the presence of the infrared, the half-width $(2D/bE)^{1/2}$ of the spatial distribution of the radio electrons measured from the Galactic plane is $\gtrsim 1$ kpc, which is wider than the width of the radio disk. This point, in fact, strengthens the idea that the sharp drop of the intensity of the nonthermal radio emission from disk to halo is largely due to field intensity variation instead of electron intensity variation. On the other hand, it is difficult, if not impossible, to achieve a radio spectrum with spectral index flatter than $\alpha_r = \frac{1}{2}(\alpha_e - 1) \approx 0.5$ in the isotropic-diffusion model unless the injection spectral index α of the cosmic-ray electrons in the relevant energy range is less than 2.

All these facts indicate that it is more natural to assume that the equilibrium electron spectrum in the range 200 MeV$<E<$2 BeV has $\alpha_e<2$, rather than $\alpha_e \gtrsim 2$ as assumed in the above model. There are two ways to get around this difficulty. The first, as mentioned above, is to abandon the idea that the injection spectrum of the disk primary electrons has a constant spectral index in the energy range considered. In the isotropic-diffusion

[43] J. R. Jokipii and E. N. Parker, Planetery Space Sci. **15**, 1375 (1967).

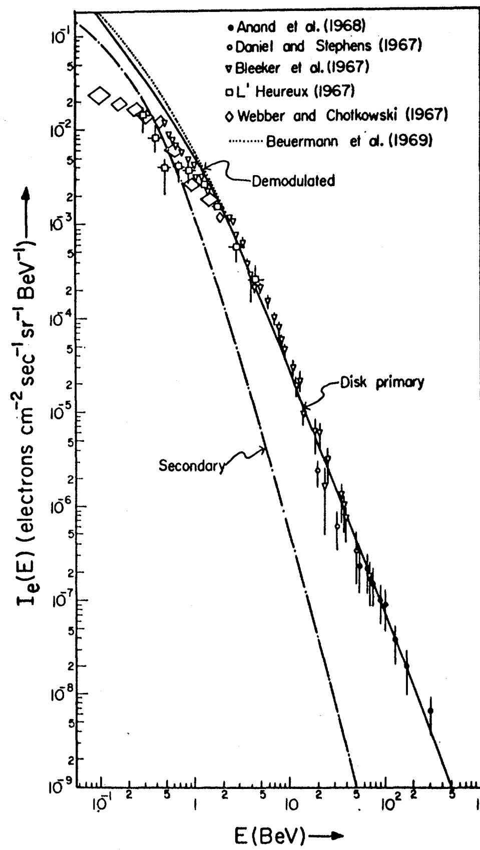

FIG. 7. Contributions to the electron spectrum at Earth from secondary and disk primary electrons in the convection-diffusion model. The disk primary source has a production spectral index $\alpha=1.6$. We have taken $T_c \approx 10^7$ yr.

model, the injection index must be ≈ 2 for $E \gtrsim 1$ BeV because the equilibrium index is ≈ 2.5. It is perhaps difficult to understand why the electron source spectrum should flatten at lower energy, since the nuclear injection spectrum is apparently a power law over a large energy range (assuming interstellar acceleration is insignificant for nuclei). But if the injection index is ≈ 2 for electrons, the acceleration mechanism for the electron and nuclear species is likely to be different anyway.

A second way to get a flatter low-energy electron spectrum is to abandon the isotropic-diffusion picture in favor of the convection-diffusion model discussed in Sec. V. In this case one finds the disk source given by Eq. (3.15), and

$$\alpha \approx 1.5\,, \quad K \approx 5 \times 10^{-26} \text{ electrons BeV}^{1/2}/(\text{cm}^3 \text{ sec}) \qquad (6.18)$$

gives rise to the required electron spectrum at Earth for $E \gtrsim 2$ BeV provided

$$T_c = d/v_0 \gtrsim 10^7 \text{ yr}\,, \qquad (6.19)$$

where d is the thickness of the "confinement disk." As pointed out in the discussion in Sec. V, this model in

TABLE III. Some parameters in the isotropic-diffusion and convection-diffusion models.[a]

Model	f_+(200 MeV)	f_+(2 BeV)	Power of Q_d (erg/sec)	Power of Q_c (erg/sec)	Characteristic lifetime (yr)
Isotropic-diffusion	0.06	0.05	10^{39}–10^{40}	10^{39}–10^{40}	$R^2/4D\approx 10^8$
Convection-diffusion	0.32	0.05	$>5\times 10^{39}$	$>5\times 10^{39}$	$T_c\approx 10^7$

[a] Choice of $\alpha=2$ introduces a logarithmic divergence for the injection power Q. Observations, however, hint that the injection spectrum is flatter than 2 at $E<$a few hundred MeV and is steeper than 2 at $E>200$ BeV. We have accordingly chosen $E_{\min}=100$ MeV and $E_{\max}=1000$ BeV as the cutoff energies in the estimation of Q_c and Q_d

fact describes the case in which a particle is confined in the disk for the average time T_c, and is unlikely to return after leaving the disk. For T_c to be as large as 10^7 yr, the cosmic rays would have to spend this time in regions of average density 0.2 atoms per cm^3 in order for the nuclei to have traversed only 3 g/cm^2 of material. The possibility that cosmic rays may spend a significant fraction of their "disk lifetime" in low-density regions near, but outside of, the matter disk has been discussed recently by Jokipii and Parker.[44] To be consistent with the accepted value of $n\approx 1$ atom of hydrogen/cm^3 in the disk, we should chose $d\gtrsim 5\ pR$, where pR is approximately the half-width of the matter disk.

In Fig. 7 we plot the secondary and primary disk contributions to the electron spectrum at Earth, assuming

$$1/bT_c=2\text{ BeV}. \tag{6.20}$$

The core source will no longer contribute to the electron intensity at Earth in this propagation model. Note that the equilibrium secondary spectrum is somewhat different from that in Fig. 6 because of the different propagation model chosen. The γ-ray spectrum is not significantly altered from the previous (isotropic-diffusion) model, since the high-energy electron fluxes must be the same in both cases.

Some interesting parameters relating to the two models are displayed in Table III.

VII. CONCLUSION

We have given, we believe for the first time, a complete and consistent treatment of the propagation of high-energy cosmic-ray electrons in the Galaxy. In Sec. VI we proposed a model in which the primary cosmic-ray electron sources have two components, one distributed throughout the disk plane and another concentrated near the Galactic center. The total production rates of the two sources, integrated over their respective volumes, are comparable. Both sources may contribute to the low-energy electron intensity at Earth (depending on the propagation model), but the disk source is solely responsible for the observed electron flux at higher energies. This two-component model of cosmic-ray electrons seems capable of explaining many puzzling facts relevant to cosmic-ray electrons, especially if the 100-MeV γ-ray photons are indeed produced by cosmic-ray electrons. We must emphasize, however, that the data are still insufficient to prove the existence of the two components. Clearly, better measurements of the intensity and angular distributions of both the infrared background and the high-energy γ-ray flux are necessary.

[44] J. R. Jokipii and E. N. Parker, Astrophys. J. **155**, 799 (1969).

APPENDIX A: SOLUTION OF ISOTROPIC-DIFFUSION EQUATION WITH BOUNDARY CONDITION

In this appendix we solve the time-independent isotropic-diffusion equation

$$-\frac{\partial}{\partial E}(bE^2N)-D\nabla^2N=Q \tag{A1}$$

subject to the boundary condition

$$\left.\frac{\partial N}{\partial r}\right|_{r=R}=-\frac{\beta}{R}N\bigg|_{r=R}. \tag{A2}$$

(For a discussion of the various terms see Sec. II.) In order to reduce the problem to a soluble one we make the change of variable from E to τ defined by

$$\tau=\int_E^\infty\frac{dE'}{bE'^2}. \tag{A3}$$

Let us also define

$$\mathfrak{N}(\tau,\mathbf{r})=b[E(\tau)]^2N(E(\tau),\mathbf{r}), \tag{A4}$$

$$\mathcal{Q}(\tau,\mathbf{r})=b[E(\tau)]^2Q(E(\tau),\mathbf{r}), \tag{A5}$$

where $E(\tau)$ is found by inverting $\tau=\tau(E)$. Then $\mathfrak{N}$ and $\mathcal{Q}$ satisfy

$$\frac{\partial\mathfrak{N}}{\partial\tau}-D\nabla^2\mathfrak{N}=\mathcal{Q}, \tag{A6}$$

$$\left.\frac{\partial\mathfrak{N}}{\partial r}\right|_{r=R}=-\frac{\beta}{R}\mathfrak{N}\bigg|_{r=R}. \tag{A7}$$

Let $n(x,\mathbf{r};\tau')$ be the solution of

$$\frac{\partial n}{\partial x}-D\nabla^2 n=0 \quad (x>0)\,, \tag{A8}$$

$$\left.\frac{\partial n}{\partial r}\right|_{r=R}=-\frac{\beta}{R}n\bigg|_{r=R}\,, \tag{A9}$$

$$n(0,\mathbf{r};\tau')=\mathcal{Q}(\tau',\mathbf{r}')\,. \tag{A10}$$

Then

$$\mathfrak{N}(\tau,\mathbf{r})=\int_0^\tau d\tau' n(\tau-\tau',\mathbf{r};\tau') \tag{A11}$$

is the solution of Eqs. (A6) and (A7), and

$$N(E,\mathbf{r})=\frac{1}{bE^2}\mathfrak{N}(\tau(E),\mathbf{r}) \tag{A12}$$

is the solution of Eqs. (A1) and (A2).

The general solution of Eqs. (A8)–(A10) is

$$n(x,\mathbf{r};\tau')=\frac{2}{R^3}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}a_{lm}(\tau')\times j_l(\lambda_{ln}r/R)Y_{lm}(\theta,\varphi)e^{-\lambda_{ln}^2Dx/R^2}\,, \tag{A13}$$

where $j_l(x)$ is the spherical Bessel function of order l, the Y_{lm}'s are the spherical harmonics, the λ_{ln} are the solutions of

$$j_l'(\lambda_{ln})=-\beta j_l(\lambda_{ln}) \quad (0<\lambda_{l1}<\lambda_{l2}<\cdots)\,, \tag{A14}$$

and

$$a_{lmn}(\tau')=\frac{\lambda_{ln}^2}{[\lambda_{ln}^2+\beta(\beta-1)-l(l+1)]|j_l(\lambda_{ln})|^2}\times\iiint_{|\mathbf{r}'|<R}d\mathbf{r}'j_l(\lambda_{ln}r'/R)Y_{lm}^*(\theta,\varphi)\mathcal{Q}(\tau',\mathbf{r}')\,. \tag{A15}$$

For a spherically symmetric source distribution, the solution is greatly simplified since a_{lmn} is zero unless $l=m=0$, $j_0(x)=\sin(x)/x$, and the $\lambda_{0n}=\lambda_n$ are given by

$$\lambda_n\cot\lambda_n=1-\beta\,. \tag{A16}$$

In particular, for

$$\begin{aligned}Q(E',r')&=(3/4\pi r_0^3)E'^{-2} \quad (0\le r'<R)\\ &=0 \quad (r'>r)\,,\end{aligned} \tag{A17}$$

we find

$$n(x,\mathbf{r};\tau')=\frac{3b}{2\pi R^2 r}\sum_{n=1}^{\infty}\frac{\lambda_n^2+(\beta-1)^2}{[\lambda_n^2+\beta(\beta-1)]\lambda_n^2}\times\zeta_n(r_0)\sin(\lambda_n r/R)e^{-\lambda_n^2Dx/R}\,, \tag{A18}$$

and using Eqs. (A11) and (A12), one obtains the result

$$N(E,r)=\frac{3E^{-2}}{2\pi Dr}\sum_{n=1}^{\infty}\frac{[\lambda_n^2+(\beta-1)^2][1-e^{-\lambda_n^2/4\epsilon}]}{[\lambda_n^2+\beta(\beta-1)]\lambda_n^4}\times\zeta_n(r_0)\sin(\lambda_n r/R)\,, \tag{A19}$$

where

$$\epsilon=R^2bE/4D \tag{A20}$$

and

$$\zeta_n(r_0)=(R/r_0)^3[\sin(\lambda_n r_0/R)-(\lambda_n r_0/R)\cos(\lambda_n r_0/R)]\,. \tag{A21}$$

For a point source ($r_0\to 0$),

$$\zeta_n(0)=\lim_{r_0\to 0}\zeta_n(r_0)=\tfrac{1}{3}\lambda_n^3\,, \tag{A22}$$

while for a uniform source ($r_0=R$),

$$\begin{aligned}\zeta_n(R)&=\beta\sin\lambda_n \quad (\beta<\infty)\\ &=(-1)^{n+1}\lambda_n \quad (\beta=\infty)\,.\end{aligned} \tag{A23}$$

The solutions of (A16) are characterized by

$$\begin{aligned}&\lambda_n=n\pi\,, && \beta=\infty\\ &(n-\tfrac{1}{2})\pi<\lambda_n<n\pi\,, && 1<\beta<\infty\\ &\lambda_n=(n-\tfrac{1}{2})\pi\,, && \beta=1\\ &(n-1)\pi<\lambda_n<(n-\tfrac{1}{2})\pi\,, && 0<\beta<1\\ &\lambda_1\to(3\beta)^{1/2},\quad \lambda_n\lesssim(n-\tfrac{1}{2})\pi\,(n\ge 2)\,, && \beta\to 0\,.\end{aligned} \tag{A24}$$

Equations (A19)–(A23), together with the Fourier-series representations

$$1=2\sum_{n=1}^{\infty}\frac{\sin[(n-\frac{1}{2})\pi r/R]}{(n-\frac{1}{2})\pi} \quad (0<r<R)\,,$$

$$3R^2r-r^3=12R^3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin[(n-\frac{1}{2})\pi r/R]}{(n-\frac{1}{2})^4\pi^4} \quad (0\le r<R)\,,$$

$$1-r/R=2\sum_{n=1}^{\infty}\frac{\sin(n\pi r/R)}{n\pi} \quad (0<r\le R)\,,$$

$$rR^2-r^3=12R^3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi r/R)}{n^3\pi^3} \quad (0\le r\le R)\,,$$

can be used to derive the low-energy spatial dependence of $N(E,r)$ shown in Table II.

From the definition of $\mathcal{Q}(\tau,\mathbf{r})$, Eq. (A5), we obtain for the source (A17)

$$\begin{aligned}\mathcal{Q}(\tau',r')&=3b/4\pi r_0^3 \quad (0\le r'\le r_0<R)\\ &=0 \quad (r'>r_0)\,.\end{aligned} \tag{A25}$$

By integrating $n(x=0,\,\mathbf{r};\,\tau')$ from Eq. (A18) over all

$|\mathbf{r}|<R$, we obtain

$$\iiint_{|\mathbf{r}|<R} d\mathbf{r}\, n(0,r;\tau')=6b\sum_{n=1}^{\infty}\frac{[\lambda_n^2+(\beta-1)^2]\beta\sin\lambda_n}{[\lambda_n^2+\beta(\beta-1)]\lambda_n^4}\zeta_n(r_0)$$

$$=\iiint_{|\mathbf{r}|<R} d\mathbf{r}\,\mathcal{Q}(\tau',\mathbf{r})=b\,,$$

which yields the interesting result

$$6\sum_{n=1}^{\infty}\frac{[\lambda_n^2+(\beta-1)^2]\beta\sin\lambda_n}{[\lambda_n^2+\beta(\beta-1)]\lambda_n^4}\zeta_n(r_0)=1\,. \qquad \text{(A26)}$$

APPENDIX B: CONVECTION-DIFFUSION EQUATION FOR DISK SOURCE

Consider the equation

$$-\frac{\partial}{\partial E}(bE^2N)+\frac{\partial}{\partial z}[v_d(z)N]-D_{\rm II}\left(\frac{\partial^2N}{\partial x^2}+\frac{\partial^2N}{\partial y^2}\right)=Q. \qquad \text{(B1)}$$

If we choose $Q(E,x,y,-z)=Q(E,x,y,z)$ and $v_d(-z)=-v_d(z)$, as is reasonable from the structure of the Galaxy, then clearly $N(E,x,y,-z)=N(E,x,y,z)$, and we need solve Eq. (B1) only for $z>0$. It is easily shown that if $v_d(z)\propto z^\delta$ for small z, then $\delta<1$ and $\delta>1$ imply

$$\lim_{z\to 0} N(E,x,y,z)=0 \text{ and } \infty\,,$$

respectively, for any source distribution Q which is approximately independent of z for small z; $\delta=1$, however, implies $N(E,x,y,z)$ approximately independent of z for small z, as is expected on physical grounds.

It is mathematically convenient to choose

$$Q(E',\mathbf{r}')=KE'^{-\alpha}\exp\left(-\frac{x'^2+y'^2}{R^2}\right) \quad (z\le pR)$$

$$=0 \qquad (z>pR) \qquad \text{(B2)}$$

and

$$v_d(z)=v_0z/pR \quad (0<z\le pR)$$
$$=v_0 \qquad (z>pR)\,, \qquad \text{(B3)}$$

where $r_0\approx 10^2$ km/sec is a characteristic expansion velocity. Equation (B1) can be solved by making the change of variable from z to

$$\sigma=\int_{pR}^{z}\frac{dz'}{v_d(z')},$$

in which case σ and z are related by

$$\sigma(z)=(pR/v_0)\ln(z/pR)\,,\quad 0<z\le pR$$
$$=(z-pR)/v_0\,,\qquad z\ge pR \qquad \text{(B4)}$$

and

$$z(\sigma)=pRe^{\sigma v_0/pR}\,,\quad \sigma\le 0$$
$$=pR+v_0\sigma\,,\quad \sigma\ge 0. \qquad \text{(B5)}$$

Then the quantities

$$\mathfrak{N}(E,x,y,\sigma)=v_d(z(\sigma))N(E,x,y,z(\sigma))\,, \qquad \text{(B6)}$$

$$\mathcal{Q}(E,x,y,\sigma)=v_d(z(\sigma))Q(E,x,y,z(\sigma)) \qquad \text{(B7)}$$

satisfy

$$\frac{\partial\mathfrak{N}}{\partial\sigma}-\frac{\partial}{\partial E}(bE^2\mathfrak{N})-D_{\rm II}\left(\frac{\partial^2\mathfrak{N}}{\partial x^2}+\frac{\partial^2\mathfrak{N}}{\partial y^2}\right)=\mathcal{Q}, \qquad \text{(B8)}$$

which is the two-dimensional diffusion equation equivalent to Eq. (2.7) if we associate σ with t. It is now an easy task to find $\mathfrak{N}$, and inverting Eq. (B6), we have

$$N(E,\mathbf{r})=[1/v_d(z)]\mathfrak{N}(E,x,y,z(\sigma)). \qquad \text{(B9)}$$

Hence, for the source, Eq. (B2),

$$N(E,\mathbf{r})=KE^{-\alpha}\int_0^{1/bE}d\xi\frac{(1-bE\xi)^{\alpha-2}}{1+\xi/T_{\rm II}}$$
$$\times\exp\left[-\frac{\xi}{T_c}-\frac{(x^2+y^2)/R^2}{1+\xi/T_{\rm II}}\right] \text{ for } |z|\le pR \qquad \text{(B10)}$$

$$=Ke^{z/pR-1}E^{-\alpha}\int_{(z-pR)/v_0}^{1/bE}d\xi\frac{(1-bE\xi)^{\alpha-2}}{1+\xi/T_{\rm II}}$$
$$\times\exp\left[-\frac{\xi}{T_c}-\frac{(x^2+y^2)/R^2}{1+\xi/T_{\rm II}}\right]$$
$$\text{for } pR\le z\le pR+v_0/bE \qquad \text{(B11)}$$

$$=0 \qquad \text{for } z\ge pR+v_0/bE\,, \qquad \text{(B12)}$$

where $T_c=pR/v_0$ and $T_{\rm II}=R^2/4D_{\rm II}$ are the characteristic times for a particle to leave the disk through convection and diffusion, respectively.