Solution of a Multiatom Radiation Model Using the Bargmann Realization of the Radiation Field*

WILLIAM R. MALLORY
Physics Department, Syracuse University, Syracuse, New York 13210
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A model for interacting single-mode electromagnetic radiation and matter consisting of N two-level particles is solved exactly using a representation in which the creation and destruction operators for the field act by multiplication and differentiation, respectively. The model includes off-resonant interactions. Closed-form expressions can be given for some special cases. The solutions are transformed to the more familiar Fock and Schrödinger representations. Numerical results are given to indicate the behavior of the solutions. An equation for a generating function is derived which would give solutions of the (algebraic) matrix equation to which the problem has been reduced.

I. INTRODUCTION: MODEL AND REPRESENTATIONS

THE purpose of this paper is to give the exact solution for a quantum model of interacting radiation and matter. As a physical model we take the following: Consider a collection of particles—N in number—interacting with an electromagnetic (EM) radiation field consisting of a single mode in some region of space. The presence of the single mode can be considered as due to a resonant structure of some sort. The particles interact with each other only through the radiation field. The energy separation of the levels need not correspond exactly to the quantum energy of the radiation field. We may write the Hamiltonian as $\mathcal{H}=\mathcal{H}_f+\mathcal{H}_i$, where $\mathcal{H}_f=\omega a^\dagger a$ is the "free" radiation field Hamiltonian (we use $\hbar=c=1$) and $\mathcal{H}_i$ represents the interaction with the particles. As a specific example, consider that the interaction is through a magnetic dipole. Assume that the jth particle has a magnetic moment $\boldsymbol{\mu}_j$ and a gyromagnetic ratio γ such that $\boldsymbol{\mu}_j=-\gamma\mathbf{S}_j$, where $\mathbf{S}_j$ is the particle angular momentum. Then $\mathcal{H}_i=-\sum_j \boldsymbol{\mu}_j\cdot\mathbf{H}(\mathbf{x}_j)$, where the summation is over all particles and $\mathbf{x}_j$ is the position of the jth particle. $\mathbf{H}$ is the (time-dependent) magnetic field at the particle. If we now assume that $\mathbf{H}$ is essentially constant over the whole collection (which implies the collection occupies a small portion of the resonant cavity), then $\mathcal{H}_i=-\mathbf{H}\cdot\sum_j \boldsymbol{\mu}_j=\gamma\mathbf{H}\cdot\mathbf{S}$, where $\mathbf{S}\equiv\sum_j \mathbf{S}_j$ is the sum of the individual particle angular momenta. We assume herein that the particles all have an angular momentum amplitude of $\frac{1}{2}$.

For the field, assume

$$\mathbf{H}=H_0\hat{\epsilon}_3+H_1(\hat{\epsilon}_1\cos\omega t+\hat{\epsilon}_2\sin\omega t),$$

where the $\hat{\epsilon}_i$ are orthogonal unit vectors, i.e., a static field H_0 along the 3 axis and a circularly polarized transverse field. The transverse field can be written (so as to display the positive- and negative-frequency parts for second-quantization purposes) as

$$\mathbf{H}_T=(H_1/\sqrt{2})(e^{-i\omega t}\hat{\epsilon}_+ + e^{i\omega t}\hat{\epsilon}_-),$$

where $\hat{\epsilon}_\pm\equiv(\hat{\epsilon}_1\pm i\hat{\epsilon}_2)/\sqrt{2}$. The $\mathbf{x}$ dependence (in the entire cavity—not the sample) is contained in H_1. When we quantize the field, it becomes

$$\mathbf{H}_T=(K'/\sqrt{2})(a\hat{\epsilon}_+ + a^\dagger\hat{\epsilon}_-),$$

where K' is a constant. If we define in the usual way

$$S_\pm=\tfrac{1}{2}(S_1\pm iS_2),$$

then $\mathbf{S}$ can be written as $S_3\hat{\epsilon}_3+\sqrt{2}(\hat{\epsilon}_-S_+ + \hat{\epsilon}_+S_-)$, whence

$$\mathcal{H}_i=\gamma H_0S_3+\gamma K'(aS_+ + a^\dagger S_-).$$

(Note that $\hat{\epsilon}_\pm\cdot\hat{\epsilon}_\pm=0$, $\hat{\epsilon}_\pm\cdot\hat{\epsilon}_\mp=1$.) Define $\omega_0=+\gamma H_0$, $K=+\gamma K'$, and our Hamiltonian becomes

$$\mathcal{H}=\omega a^\dagger a+\omega_0S_3+K(aS_+ + a^\dagger S_-). \tag{1.1}$$

In addition to the circularly polarized case above, the model may be taken as a "rotating-wave" approximation when a linearly polarized radiation field is used.[1] The approximation is quite good near resonance.[2]

The terms in the Hamiltonian (1.1) can be interpreted in the following way: $\omega a^\dagger a$ is the energy of the free radiation field; ω_0S_3 is a Zeeman splitting into $2s+1$ levels caused by the static field H_0; the operator $a^\dagger S_-$ creates a photon and lowers the material excitation by one level (or, alternatively, one less particle is excited); the operator aS_+ destroys a photon and raises the material excitation. The last two terms include the action of the material on the field and the reaction of the field back on the material system.

To obtain this model Hamiltonian, it is not necessary that the interaction be via magnetic dipole; electric dipole interactions are handled analogously.[2] The essential points of the model are that the particles have effectively two energy levels (generated above by the H_0 term) and that the field operators appear linearly in the interaction Hamiltonian.

We shall be concerned with finding the eigenvalues and eigenfunctions of the Hamiltonian (1.1).

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[1] See, e.g., W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Co., New York, 1964), pp. 212 and 213.

[2] E. T. Jaynes and F. W. Cummings [Proc. IEEE **51**, 89 (1963)] discuss certain neglected terms in this approximation.

The above model and equivalent ones have been considered previously for various purposes. The single-particle model has been used most,[3] and the exact solution was obtained by Jaynes and Cummings.[2] The more general model was used by Dicke[4] and others.[5] Recently Tavis and Cummings[6] have found the exact solution for the resonant case $\omega=\omega_0$.

The model is important because it is one of the simplest for radiation-matter interaction and is hence more tractable than more complicated (albeit more realistic) models, which, for example, include the effect of spatial distribution of particles and perhaps randomizing effects of thermal reservoirs. The solutions found herein can be used as a starting point for more realistic models. The solutions may also by themselves reveal facts not obtainable from the usual perturbation treatment of radiation-matter interaction, especially in the presence of high-amplitude fields as in lasers.

The unique contributions of this paper are the treatment of the nonresonant model and the use of a realization of the field on the Bargmann Hilbert space of entire functions.[7,8] In this representation the field creation and annihilation operators $a^\dagger$ and a are represented by z^* and $\partial/\partial z^*$, respectively, where z is the complex eigenvalue of the annihilation operator a. Because of its close connection with the coherent-state description of the field, the representation is appropriate for "classical fields" where large uncertainties in photon number exist. Schweber used this representation in a similar case but found it necessary to make certain approximations for his model.[9] The nonresonant case can also be treated by an extension of the method of Tavis and Cummings.[10]

References 7–9 discuss extensively the continuous representation used herein for the fields. Briefly, it is as follows.

The coherent state $|z\rangle$,[11] labeled by the complex number z, satisfies $a|z\rangle=z|z\rangle$, $\langle z|a^\dagger=\langle z|z^*$, where a and $a^\dagger$ are the (boson) field annihilation and creation operators, respectively. In terms of the Fock states $|n\rangle$ which satisfy $a^\dagger a|n\rangle=n|n\rangle$,

$$\langle n|z\rangle=e^{-|z|^2/2}z^n/\sqrt{n!}. \tag{1.2}$$

We have the resolution of the identity

$$\pi^{-1}\int |z\times z|\,d^2z=1, \tag{1.3}$$

where $d^2z=d(\mathrm{Im}z)d(\mathrm{Re}z)$, and the integral extends over the entire complex plane. The states $|z\rangle$ are not orthogonal for different z. For an arbitrary state $|\psi\rangle$ we have, using the solution of the identity, Eq. (1.3),

$$|\psi\rangle=\pi^{-1}\int\langle z|\psi\rangle|z\rangle d^2z.$$

We define $\psi(z^*)$ by

$$\langle z|\psi\rangle\equiv e^{-|z|^2/2}\psi(z^*), \tag{1.4}$$

and therefore

$$|\psi\rangle=\pi^{-1}\int\psi(z^*)e^{-|z|^2/2}|z\rangle d^2z.$$

$\psi(z^*)$ must be an entire function[12] of z^*. It serves as a representation of $|\psi\rangle$ in the coherent-state expansion.

To find the operator representation, consider an analytic function of the creation and annihilation operators $Q(a^\dagger,a)$. Let

$$|\psi'\rangle=Q(a^\dagger,a)|\psi\rangle.$$

Now by Eq. (1.4),

$$|\psi'\rangle=\pi^{-1}\int\psi'(z^*)e^{-|z|^2/2}|z\rangle d^2z. \tag{1.5}$$

We seek an operator function $f(z,z^*)$ such that

$$\psi'(z^*)=f(z,z^*)\psi(z^*). \tag{1.6}$$

Now using Eq. (1.3), we obtain

$$|\psi'\rangle=\pi^{-1}\int\langle z|\psi'\rangle|z\rangle d^2z$$

or

$$|\psi'\rangle=\pi^{-1}\int\langle z|Q(a^\dagger,a)|\psi\rangle|z\rangle d^2z.$$

Use (1.3) again to expand $|\psi\rangle$:

$$|\psi'\rangle=\pi^{-2}\int\langle z|Q(a^\dagger,a)|z'\rangle\langle z'|\psi\rangle|z\rangle d^2zd^2z'. \tag{1.7}$$

Since Q is analytic, we can expand in a power series in a and $a^\dagger$. It is easily shown using the properties of the coherent state that[13] $\langle z|a^\dagger|z'\rangle=z^*\langle z|z'\rangle$ and $\langle z|a|z'\rangle=(\frac{1}{2}z+\partial/\partial z^*)\langle z|z'\rangle$. If this is combined with the series expansion for Q, we obtain

$$\langle z|Q(a^\dagger,a)|z'\rangle=Q(z^*,\tfrac{1}{2}z+\partial/\partial z^*)\langle z|z'\rangle.$$

Substitute this in Eq. (1.7) and use Eq. (1.3) to remove the z' dependence:

$$|\psi'\rangle=\pi^{-1}\int Q(z^*,\tfrac{1}{2}z+\partial/\partial z^*)\langle z|\psi\rangle|z\rangle d^2z.$$

[3] N. Chandra and H. Prakash, Phys. Rev. Letters **22**, 1068 (1969); J. A. Fleck, Jr., Phys. Rev. **149**, 309 (1966).
[4] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[5] A. D. Gazazyan, Zh. Eksperim. i Teor. Fiz. **51**, 1863 (1966) [English transl.: Soviet Phys.—JETP **24**, 1254 (1967)]; J. H. Shirley, Am. J. Phys. **36**, 949 (1968), and references therein.
[6] M. Tavis and F. W. Cummings, Phys. Rev. **170**, 379 (1968). Also see references of similar models therein.
[7] V. Bargmann, Rev. Mod. Phys. **34**, 829 (1962).
[8] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, Inc., New York, 1968), p. 127.
[9] S. Schweber, Ann. Phys. (N. Y.) **41**, 205 (1967).
[10] Reference 1, Eq. (5.178).
[11] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[12] Reference 8, p. 114.
[13] Reference 8, p. 127, Eq. (7-93).

Use the definition (1.4) to obtain

$$|\psi'\rangle=\pi^{-1}\int Q(z^*,\tfrac{1}{2}z+\partial/\partial z^*)e^{-|z|^2/2}\psi(z^*)\,|z\rangle d^2z.$$

By expanding $Q(z^*,\frac{1}{2}z+\partial/\partial z^*)$ it can be shown that

$$Q(z^*,\tfrac{1}{2}z+\partial/\partial z^*)e^{-|z|^2/2}\psi(z^*)=e^{-|z|^2/2}Q(z^*,\partial/\partial z^*)\psi(z^*).$$

Hence,

$$|\psi'\rangle=\pi^{-1}\int e^{-|z|^2/2}[Q(z^*,\partial/\partial z^*)\psi(z^*)]\,|z\rangle d^2z.$$

By comparison with Eqs. (1.5) and (1.6) we see that the desired operator representation is $Q(z^*,\partial/\partial z^*)$, which is obtained from the abstract operator $Q(a^\dagger,a)$ by replacing $a^\dagger$ by z^* and a by $\partial/\partial z^*$. For example, $\exp(-a^\dagger a)$ is represented here by $\exp[-z^*(\partial/\partial z^*)]$, which is identically equal to

$$\sum_{j=0}^{\infty}(-z^*\partial/\partial z^*)^n/n!.$$

Another relation we shall need is that between the continuous complex representation used herein and the Schrödinger wave function[14]:

$$\phi(y)=\tfrac{1}{2}(h)^{-3/4}e^{y^2/2\hbar}\int_{-\infty}^{\infty}\langle p,2y\,|\psi\rangle dp \qquad (1.8)$$

(we put in $\hbar$ explicitly only when dealing with the Schrödinger representation), where $\phi(y)$ is the Schrödinger wave function, and $|p,2y\rangle$ is the coherent state with real and imaginary parts explicitly indicated, i.e., $z=(2y+ip)/(2\hbar)^{1/2}$. According to Eq. (1.4),

$$\langle p,2y\,|\psi\rangle=\exp[-\tfrac{1}{4}(4y^2+p^2)/\hbar]\times[\psi(2y-ip)(2\hbar)^{-1/2}]. \qquad (1.9)$$

The Hamiltonian $\mathcal{H}$ of Eq. (1.1) commutes with the square of the total angular momentum $S^2=S_3^2+\frac{1}{2}(S_+S_-+S_-S_+)$. Therefore $\mathcal{H}$ does not connect states of different s, and we need consider only representations in which one value of s appears. The complete solution for the N atoms will then be just an arbitrary combination of solutions for $s=\frac{1}{2}N$, $\frac{1}{2}N-1$, ..., $s_{\min}$, where $s_{\min}$ is zero or 1 for N even or odd, respectively. Because of the form of the Hamiltonian we will also take as a basis for the matter state the state labeled by s_3, the eigenvalue of S_3. There are of course $2s+1$ such states for a given value of s.

One should keep in mind that values of s and s_3 do not completely specify the atomic states because of the equivalent ways in which particles may be combined to form a given value of s and s_3. The state $s=\frac{1}{2}N$, $s_3=\frac{1}{2}N$ is nondegenerate. The state with $s_3=\frac{1}{2}N-1$ may be chosen by picking $N-1$ particles with their individual $s_3=+\frac{1}{2}$ and one with $s_3=-\frac{1}{2}$. This may obviously be done in N ways, so there must be N states with $s_3=\frac{1}{2}N-1$. One of these belongs to the state $s=\frac{1}{2}N$; the rest must belong to states with the only other conceivable value of s, $s=\frac{1}{2}N-1$. Hence there must be $N-1$ such states. By reasoning along this line, we find[15] that the degeneracy of the state (s,s_3) is

$$\frac{N!(2s+1)}{(\frac{1}{2}N+s+1)!(\frac{1}{2}N-s)!}.$$

This degeneracy parameter does not enter into the calculations of this paper but would be important, e.g., in forming distributions of the solutions calculated herein in thermal equilibrium.

The representation space for the combined matter and radiation is just the direct product of the continuous representation for the radiation and the $(2s+1)$-dimensional matter representation.

In Sec. II we reduce the solution for arbitrary s to a matrix eigenvalue problem Subsequent sections consider the transformation of the solutions to more common representations, treat the matrix equation analytically for some special cases, and display and discuss solutions of the matrix problem. Also, a partial differential equation is derived for a generating function for the matrix eigenvalues and eigenvectors.

II. REDUCTION TO ALGEBRAIC FORM

Consider the Hamiltonian (1.1). We use the following notation: For the states let

$$\psi(z^*)=\sum_{j=1}^{2s+1}\psi^{(j)}(z^*)\epsilon_j,$$

where the z^* dependence represents the field as discussed in I, and ϵ_j is a $(2s+1)$-dimensional column vector with zeros everywhere except for a 1 in the jth row; it represents the matter state having total angular momentum s and $s_3=s-j+1$. With this particle state representation, the required particle operators are represented[16] by square matrices of dimension $2s+1$ having elements

$$(S_3)_{mn}=\delta_{mn}s_3=\delta_{mn}(s+1-m),$$
$$(S_+)_{mn}=\delta_{(m+1)n}[m(2s+1-m)]^{1/2},$$
$$(S_-)_{mn}=(S_+)_{nm},$$

where δ_{mn} is the Kronecker δ and s_3 is given by $s+1-m$. We have ignored the subscript s to simplify notation since it will be the same throughout. Combining the above with the representation for the field operators $a^\dagger\rightarrow z^*$, $a\rightarrow\partial/\partial z^*$, and with $|\psi\rangle\rightarrow\psi(z^*)$ in the

[14] Reference 8, p. 122, Eq. (7-76).

[15] Reference 4, Eq. (20).

[16] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., New York, 1955), 2nd ed., pp. 143–146.

Hamiltonian (1.1), the eigenvalue equation $\mathcal{H}|\psi\rangle = E|\psi\rangle$ becomes (with $x\equiv z^*$, $\dot{\psi}^{(j)}\equiv d\psi^{(j)}/dx$)

$$\sum_{j=1}^{2s+1}\{\omega x\dot{\psi}^{(j)}\epsilon_j+[\omega_0(s+1-j)-E]\psi^{(j)}\epsilon_j +Kxf_j\epsilon_{j+1}\psi^{(j)}+Kf_{j-1}\dot{\psi}^{(j)}\epsilon_{j-1}\}=0,$$

where $f_j\equiv[j(2s+1-j)]^{1/2}$. This is a vector differential equation and is equivalent to the $2s+1$ coupled scalar equations

$$\omega x\dot{\psi}^{(j)}+[\omega_0(s+1-j)-E]\psi^{(j)}+Kf_{j-1}x\psi^{(j-1)} +Kf_j\dot{\psi}^{(j+1)}=0,\quad j=1, 2, \ldots, (2s+1). \quad (2.1)$$

The first and last equations have the simpler forms, respectively,

$$\omega x\dot{\psi}^{(1)}+(\omega_0 s-E)\psi^{(1)}+K\sqrt{2}s^{1/2}\psi^{(2)}=0,$$
$$\omega x\dot{\psi}^{(2s+1)}-(\omega_0 s+E)\psi^{(2s+1)}+K\sqrt{2}s^{1/2}x\psi^{(2s)}=0,$$

since $f_0=f_{2s+1}=0$.

Since, as mentioned in Sec. I, we are only interested in analytic solutions for the $\psi^{(j)}$, we can look for a solution which is a power series in x. However, examination of the different terms of Eqs. (2.1) indicates that the x dependence is a simple power of x with $\psi^{(j+1)}$ having one higher power than $\psi^{(j)}$. For such a dependence the power of x in $x\dot{\psi}^{(j)}$, $\psi^{(j)}$, $x\psi^{(j-1)}$, and $\dot{\psi}^{(j+1)}$ is all the same. Therefore, we insert the trial solution

$$\psi^{(j)}=U^{(j)}x^{\gamma+j-1}, \quad (2.2)$$

where $U^{(j)}$ is independent of x and γ is an arbitrary constant. After dividing the resulting expression for the jth equation by $x^{\gamma+j-1}$ we obtain

$$\omega(\gamma+j-1)U^{(j)}+[\omega_0(s+1-j)-E]U^{(j)}+Kf_{j-1}U^{(j-1)} +Kf_j(\gamma+j)U^{(j+1)}=0.$$

The functions $\psi^{(j)}(x)$ must be entire functions[12] of x ($\equiv z^*$). They can only be so if γ is an integer n such that $n+j-1\geq 0$ for all j for which $U^{(j)}\neq 0$. If γ is not an integer, $\psi^{(j)}(x)$ will have a branch point at the origin; if $n+j-1$ is a negative integer and $U^{(j)}\neq 0$, $\psi^{(j)}(x)$ will have a pole there. Within this restriction n is an arbitrary integer. Thus there are solutions for $n\geq 0$, with all $U^{(j)}$ in general $\neq 0$; $n=-1$, $U^{(1)}=0$, $U^{(2)}$, $U^{(3)}$, ..., $\neq 0$ in general; and $n=-r$, $U^{(1)}=U^{(2)}=\cdots=U^{(r)}=0$, $U^{(r+1)}$, $U^{(r+2)}$, ..., $U^{(2s+1)}\neq 0$ in general ($r=1, 2, \ldots, 2s$). Then the solution to (2.1) is an arbitrary combination of

$$\psi_n(z^*)=\sum_{j=1}^{2s+1}U_n{}^{(j)}(z^*)^{n+j-1}\epsilon_j,$$

for $n=-2s, -2s+1, \ldots, -1, 0, 1, 2, \ldots$, where the $U_n{}^{(j)}$ for fixed n are to be determined from the set of equations

$$[\omega(n+j-1)+\omega_0(s+1-j)-E_n]U_n{}^{(j)} +Kf_{j-1}U_n{}^{(j-1)}+K(n+j)f_jU_n{}^{(j+1)}=0, \quad (2.3)$$

where $U_n{}^{(j)}\equiv 0$ for $j<1-n$ if $n<0$. Equivalently, Eq. (2.3) can be written as a matrix eigenvalue equation

$$[\mathbf{C}_n-E_n\mathbf{I}]\mathbf{U}=0, \quad (2.4)$$

where $\mathbf{I}$ is the unit matrix of proper order and we have labeled the eigenvalue E to distinguish between those for different n. $\mathbf{C}_n$ has the tridiagonal form

$$\mathbf{C}_n=\begin{bmatrix} g_n & K(n+1)f_1 & & 0 & 0 & \cdots & 0 \\ Kf_1 & g_n+\Delta\omega & & K(n+2)f_2 & 0 & \cdots & 0 \\ 0 & Kf_2 & & g_n+2\Delta\omega & K(n+3)f_3 & & \vdots \\ 0 & 0 & & Kf_3 & \ddots & \ddots & \\ \vdots & \vdots & & & \ddots & & K(n+2s)f_{2s} \\ 0 & 0 & \cdots & & Kf_{2s} & & g_n+2s\Delta\omega \end{bmatrix}, \quad (2.5)$$

where $g_n\equiv n\omega+s\omega_0=(n+s)\omega-s\Delta\omega$, $\Delta\omega\equiv\omega-\omega_0$. Note that $f_{2s-k}=f_{k+1}$. It is to be understood in (2.5) that if $n<0$, the first $|n|$ rows and columns are to be replaced by zeros everywhere before proceeding with the eigenvalue problem

$$\det(\mathbf{C}_n-E_n\mathbf{I})=0, \quad (2.6)$$

since these elements are the coefficients of those $U^{(j)}$ in (2.4), which must be zero.

Then for $n\geq 0$, Eq. (2.6) yields a $(2s+1)$-degree equation for the energy eigenvalue E_n, and hence $2s+1$ values for it, which we denote E_{nl}. If $n<0$, the order of the determinant, and therefore the number of solutions, is reduced. In this case there are $2s+1+n$ solutions. In all cases we denote the corresponding eigenvectors $\mathbf{U}_{nl}$ and the corresponding wave function as

$$\psi_{nl}(z^*)=\sum_{j=1}^{2s+1}U_{nl}{}^{(j)}(z^*)^{n+j-1}\epsilon_j. \quad (2.7)$$

For $n=0, 1, 2, \ldots$, we have $l=1, 2, \ldots, 2s+1$. For $n<0$, $U_{nl}{}^{(j)}=0$ for $j=1, 2, \ldots, |n|$; and $l=1, 2, \ldots, 2s+1+n$.

Since the solutions of Eq. (2.1) must be entire functions, the complete solution must be arbitrary combinations of the solutions (2.7). We therefore expect those solutions to be a complete set. Using this property, a certain completeness relation can be shown to exist between the $\mathbf{U}_{nl}$. Because these vectors are not in general

orthogonal, this relation can be expressed more simply after a transformation to be applied in the next section.

We will see in the next section that n corresponds to the smallest number of photons we might measure in the state $|\psi_{nl}\rangle$; that if $n\geq 0$, we might find all values of s_3 from $-s$ to s, but that the negative values of n correspond to states in which there are not enough photons to excite the higher s_3 values even if all are "absorbed"; and that, in particular, the case $n=-2s$ corresponds to the "vacuum" combined with all particles in their lowest state. We will also see a slightly more symmetric form in which to pose the eigenvalue problems (2.4) and (2.5).

We postpone the normalization to the next section.

III. TRANSFORMATION TO OTHER REPRESENTATIONS

In this section we transform the solutions (2.7) to more familiar representations, first to the Fock representation, then to the Schrödinger representation.

A. Fock Representation

Using the resolution of the identity for the Fock representation,

$$|\psi_{nl}\rangle=\sum_{m=0}^{\infty}\langle m|\psi_{nl}\rangle|m\rangle\equiv\sum_m \phi_{nl}{}^{(m)}|m\rangle\,;$$

then using Eqs. (1.2)-(1.4), we obtain

$$\phi_{nl}{}^{(m)}=\pi^{-1}\int\langle m|z\rangle\langle z|\psi_{nl}\rangle d^2z$$
$$=\pi^{-1}\int e^{-|z|^2}|z|^m(m!)^{-1/2}\psi_{nl}(z^*)d^2z.$$

Using Eq. (2.7) for ψ_{nl}, we obtain

$$\phi_{nl}{}^{(m)}=\pi^{-1}(m!)^{-1/2}\sum_j U_{nl}{}^{(j)}\epsilon_j\int e^{-|z|^2}z^m(z^*)^{n+j-1}d^2z.$$

We use polar coordinates in the complex plane; $z=re^{i\theta}$, $d^2z=rdrd\theta$. The angle integral yields $2\pi\delta_{m(n+j-1)}$. Using this δ function in the r integral gives $m!/2$ for that integral. Combining, we have

$$\phi_{nl}{}^{(m)}=(m!)^{-1/2}\sum_j \delta_{m(n+j-1)}U_{nl}{}^{(j)}\epsilon_j,$$

or

$$|\psi_{nl}\rangle=\sum_{m=1}^{2s+1}\phi_{nl}{}^{(m)}|m\rangle=\sum_{j=1}^{2s+1}[(n+j-1)!]^{1/2}$$
$$\times U_{nl}{}^{(j)}\epsilon_j|n+j-1\rangle,$$

with $U_{nl}{}^{(j)}\equiv 0$ for $n+j-1<0$. (The state $\epsilon_j|k\rangle$ is perhaps more commonly written as $|k,s_z\rangle$ where the s_z value corresponding to the particular value of j is used. We will continue with the notation we have been using.) The states $|\psi_{nl}\rangle$ above are not normalized. We have

$$\langle\psi_{nm}|\psi_{lk}\rangle=\sum_j\sum_p[(n+j-1)!(l+p-1)!]^{1/2}$$
$$\times U_{nm}{}^{(j)*}U_{lk}{}^{(p)}\tilde{\epsilon}_j\epsilon_p\langle n+j-1|l+p-1\rangle,$$

where $\tilde{\epsilon}_j$ is the transpose (row) vector corresponding to the column vector ϵ_j, and from the orthonormality of the ϵ_j and the Fock states we get

$$\langle\psi_{nm}|\psi_{lk}\rangle=\sum_j(n+j-1)!U_{nm}{}^{(j)*}U_{nk}{}^{(j)}\delta_{nl}.$$

Since the Hamiltonian is Hermitian, the eigenvectors $|\psi_{nm}\rangle$ must be orthogonal—assuming the corresponding energy eigenvalues are not degenerate. If we define new vectors $\mathbf{W}_{nm}$ by

$$W_{nm}{}^{(j)}=[(n+j-1)!]^{1/2}U_{nm}{}^{(j)}/$$
$$[\sum_{j'}(n+j'-1)!|U_{nm}{}^{(j')}|^2]^{1/2},\quad(3.1)$$

we have the orthonormal condition

$$\sum_{j=1}^{2s+1}W_{nm}{}^{(j)*}W_{nk}{}^{(j)}=\delta_{mk}.\quad(3.2)$$

If, furthermore, we normalize the $\mathbf{U}_{nm}$ so that

$$\sum_{j=1}^{2s+1}(n+j-1)!|U_{nm}{}^{(j)}|^2=1,$$

the states $|\psi_{nm}\rangle$ will be orthonormal. Assuming this has been done, we have

$$|\psi_{nl}\rangle=\sum_{j=1}^{2s+1}W_{nl}{}^{(j)}\epsilon_j|n+j-1\rangle.\quad(3.3)$$

In addition to the orthogonality of the vectors $\mathbf{W}_{nl}$, we have a completeness relation among them. This can be shown to be true because of the completeness of the states $|\psi_{nl}\rangle$ which we write as

$$\sum_{n,l}|\psi_{nl}\rangle\langle\psi_{nl}|=\mathbf{1}.$$

Multiply from the left by $\langle n'+j-1|\tilde{\epsilon}_j$ and from the right by $\epsilon_{j'}|n'+j'-1\rangle$ and obtain, using the orthogonality of the ϵ_j and $|n\rangle$ states,

$$\sum_l W_{n'l}{}^{(j)}W_{n'l}{}^{(j')}=\delta_{jj'}.\quad(3.4)$$

For $n<0$ the vectors $\mathbf{W}_{nl}$ are complete only in the space of dimension $2s+1+n$.

The state $|\psi_{nl}\rangle$ can be interpreted as a sum of states with n, $n+1$, ..., $n+2s$ photons each paired with atomic states which have a decreasing component of angular momentum parallel to the static field. Different combinations of these possible states will have different energies. We should keep in mind that the states and energies really have an s dependence which has been largely suppressed.

There can be no "negative photon states." This provides a physical basis for the fact that $n+j-1\geq 0$, as discussed in obtaining the solution in the last section.

Note that if we substitute the definition (3.1) for $W_{nm}^{(j)}$ into (2.3), the matrix (2.5) takes on the symmetric form

$$\mathbf{C}_n'=\begin{bmatrix} g_n & h_{n1} & 0 & 0 & \cdots & 0 \\ h_{n1} & g_n+\Delta\omega & h_{n2} & 0 & & \vdots \\ 0 & h_{n2} & g_n+2\Delta\omega & \ddots & & \\ 0 & 0 & \ddots & & & \\ \vdots & & & & & h_{n,2s} \\ 0 & \cdots & & & h_{n,2s} & g_n+2s\Delta\omega \end{bmatrix}, \tag{3.5}$$

where

$$h_{nj}=K[(n+j)j(2s-j+1)]^{1/2},$$

corresponding to the eigenvalue problem for $\mathbf{W}$ analogous to (2.4),

$$(\mathbf{C}_n'-E_n\mathbf{I})\mathbf{W}=0. \tag{3.6}$$

The matrix equation (3.6) and (3.5) can also be obtained by an extension of the method of Tavis and Cummings.[6,10] If we write the Hamiltonian (1.1) in the form $\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2$, where

$$\mathcal{H}_1=\omega(a^\dagger a+S_3),\quad \mathcal{H}_2=K(aS_++a^\dagger S_-)-\Delta\omega S_3,$$

it can be easily shown that

$$[\mathcal{H}_1,\mathcal{H}_2]=0.$$

Hence

$$[\mathcal{H}_1,\mathcal{H}]=0,$$

and, therefore, the eigenvectors of $\mathcal{H}$ are just linear combinations of the eigenvectors of $\mathcal{H}_1$, which are just $\epsilon_j|n+j\rangle$ with n arbitrary. If such a solution is assumed—with arbitrary coefficients to be determined from the requirement that the solution also be an eigenvector of $\mathcal{H}_2$—we obtain the same condition on the solution, i.e., Eq. (3.6) with Eq. (3.5).

Note added in proof. It has recently come to my attention that this approach was used by M. Tavis in his dissertation, University of California, Riverside, 1968 (unpublished).

B. Schrödinger Representation

The transformation to the Schrödinger representation could be made from either the Fock states or the coherent-state continuous representation; we choose the latter. Using Eqs. (1.8) and (1.9) along with the eigenstate (2.7), we obtain for the Schrödinger wave function $\phi_{nl}(y)$

$$\phi_{nl}(y)=\tfrac{1}{2}(\pi\hbar)^{-3/4}e^{y^2/2\hbar}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{4y^2+p^2}{2\hbar}\right)\times\sum_{j=1}^{2s+1}U_{nl}^{(j)}\epsilon_j\left(\frac{2y-ip}{\sqrt{2}\hbar^{1/2}}\right)^{n+j-1}dp$$

$$=\tfrac{1}{2}(\pi\hbar)^{-3/4}e^{-y^2/2\hbar}\sum_{j=1}^{2s+1}U_{nl}^{(j)}\epsilon_jI_{n+j-1},$$

where

$$I_k=\int_{-\infty}^{\infty}e^{-p^2/4\hbar}\left(\frac{2y-ip}{\sqrt{2}\hbar^{1/2}}\right)^k dp.$$

We make the substitution $\xi=y/\hbar^{1/2}$ and $\rho=p/\sqrt{2}\hbar^{1/2}$ and use the binomial expansion. Noting that terms with odd powers of ρ yield zero, we obtain after straightforward integration

$$I_k=\sqrt{2}\hbar^{1/2}\sum_{l=0}^{[k/2]}C_{k,2l}\sqrt{2}^{k+1}\xi^{k-2l}(-1)^l\Gamma(l+\tfrac{1}{2}),$$

where $[k/2]$ means "greatest integer in $k/2$," $C_{k,2l}$ is the binomial coefficient, and $\Gamma(l+\frac{1}{2})$ is the gamma function.

If a few of these polynomials are checked, we find, not unexpectedly, that they are closely related to the Hermite polynomials[17] H_k. Thus

$$I_k=(\pi\hbar)^{1/2}\sqrt{2}^{-n+2}H_k(\xi)$$

and

$$\phi_{nl}(\xi)=2^{-n/2}(\pi\hbar)^{-1/4}e^{-\xi^2/2}\sum_{j=1}^{2s+1}U_{nl}^{(j)}\epsilon_jH_{n+j-1}(\xi). \tag{3.7}$$

IV. ANALYTIC SOLUTION FOR GENERAL MATRIX EQUATION—GENERATING-FUNCTION APPROACH

A formal expression for the eigenvector components $U^{(j)}$ or $W^{(j)}$, in terms of the (unknown) eigenvalues E, can be obtained from Eq. (2.3) [or Eq. (2.4)] or Eq. (3.6), respectively. For example, viewing Eq. (2.3) as a recursion relation, we can express the $U^{(j)}$ for higher j successively in terms of $U^{(1)}$. Even without knowing the eigenvalues it can be seen that succeeding equations ($j=1, 2, \ldots, 2s+1$) are independent (up to the last one) since each new equation involves a new $U^{(j)}$. For the general case, however, the expression becomes too unwieldy to provide any insight, especially since it still involves the undetermined eigenvalues.

One approach to the solution for the eigenvalues and eigenvectors in the general case is through the use of a

[17] The normalization is that of L. Pauling and E. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill Book Co., New York, 1935), p. 81.

generating function[18] for the components $U^{(j)}$ or $W^{(j)}$. Consider the solution of the matrix equation

$$d\chi/dt=\mathbf{C}\chi. \tag{4.1}$$

The solution can be written formally as

$$\chi(t)=\exp(\mathbf{C}t)\chi(0). \tag{4.2}$$

Let

$$\chi(0)=\mathbf{y}, \tag{4.3}$$

a column vector whose ith element is y^i (y to the ith power), and call the resultant solution of (4.1) $\chi(t,y)$. Let $\mathbf{x}$ be a row vector with elements x^i. We define the generating function $G(t,x,y)$ as

$$G(t,x,y)=\mathbf{x}\cdot\chi(t,y)=\mathbf{x}\cdot e^{\mathbf{C}t}\mathbf{y}. \tag{4.4}$$

Note that

$$G(0,x,y)=\sum_{j=1}^{2s+1}(xy)^j. \tag{4.5}$$

To see the utility of this function, use the completeness of the row and column eigenvectors of $\mathbf{C}$. The resolution of the identity is,[19] for a nonsingular real square matrix,

$$\textstyle\sum_j \mathbf{U}_j\mathbf{V}_j=\mathbf{1}, \tag{4.6}$$

where $\mathbf{V}_j$ is the left (row) eigenvector and $\mathbf{U}_j$ is the right eigenvector, i.e.,

$$\mathbf{C}\mathbf{U}_j=E_j\mathbf{U}_j, \quad \mathbf{V}_j\mathbf{C}=E_j\mathbf{V}_j. \tag{4.7}$$

(If $\mathbf{C}$ is symmetric, $\mathbf{U}_j=\mathbf{V}_j^T$.) Then

$$G(t,x,y)=\textstyle\sum_{i,j}\mathbf{x}\cdot\mathbf{U}_i\mathbf{V}_i\cdot e^{\mathbf{C}t}\mathbf{U}_j\mathbf{V}_j\cdot\mathbf{y},$$

which becomes upon use of (4.6) and (4.7)

$$G(t,x,y)=\textstyle\sum_j\mathbf{x}\cdot\mathbf{U}_j\mathbf{V}_j\cdot\mathbf{y}e^{E_jt}. \tag{4.8}$$

Thus we see that if G can be found, an expansion which exhibits the exponential t dependence in the form (4.8) will yield the eigenvalues. Furthermore, the x dependence furnishes a generating function for the elements of the column eigenvectors $\mathbf{U}_j$. That is, if we let

$$F(x)=\mathbf{x}\cdot\mathbf{U}_j=\textstyle\sum_k x^kU_j^{(k)}, \tag{4.9}$$

then

$$U_j^{(k)}=\partial^kF/\partial x^k\big|_{x=0}.$$

We want now to obtain the differential equation obeyed by G in terms of the elements of $\mathbf{C}$. It is necessary to eliminate the radicals f_j involved in the coefficients of $U^{(j)}$ in Eq. (2.3). In the manner of Lambropoulos[20] we transform $\mathbf{U}$ by $\mathbf{U}=\mathbf{AP}$, where $\mathbf{A}$ is the diagonal matrix $A_{ij}=\delta_{ij}a_j$, with

$$a_j=[(2s+1-j)!/(j-1)!]^{1/2}. \tag{4.10}$$

The eigenvalue equation becomes $\mathbf{CAP}=E\mathbf{AP}$ or $\mathbf{C}''\mathbf{P}=E\mathbf{P}$, where

$$\mathbf{C}''=\mathbf{A}^{-1}\mathbf{C}\mathbf{A}. \tag{4.11}$$

$\mathbf{C}''$ has the form

$$\mathbf{C}_n''=\begin{bmatrix} g_n & K(n+1) & 0 & \cdots & & & 0 \\ K(2s) & g_n+\Delta\omega & K(n+2) & & 0 & \cdots & \vdots \\ 0 & 2K(2s-1) & g_n+2\Delta\omega & & \ddots & & \\ \vdots & 0 & \ddots & \ddots & & K(n+j) & \\ & & & & g_n+(j-1)\Delta\omega & \ddots & \\ & & & & K(j-1)(2s+2-j) & \ddots & \\ 0 & \cdots & & & \ddots & K(2s) & g_n+2s\Delta\omega \end{bmatrix}, \tag{4.12}$$

where the subscript n has again been added in recognition of the dependence of the matrix on n. The eigenvalues are unchanged by the similarity transformation (4.11). Note that the eigenvectors $\mathbf{P}_j$, like $\mathbf{U}_j$, but unlike $\mathbf{W}_j$, will not be orthogonal in general, since $\mathbf{C}_n'$ of Eq. (3.5) is real symmetric but $\mathbf{C}_n''$, like $\mathbf{C}_n$ of Eq. (2.5), is not.

From Eq. (4.1) and the definition of the generating function, Eq. (4.4), we obtain

$$\frac{\partial G}{\partial t}=\mathbf{x}\cdot\mathbf{C}_n''\chi(t,y)=\textstyle\sum_{j,l}x^jC_{njl}''\chi_l(t,y).$$

By using $\mathbf{C}_n''$, we can write

$$\textstyle\sum_l\mathbf{C}_{njl}''\chi_l=[(j-1)\Delta\omega+g_n]\chi_j+K(j-1)(2s+2-j)\chi_{j-1}+K(n+j)\chi_{j+1}.$$

The right-hand side can be converted by differentiation with respect to x and redefinition of summation indices to yield the parabolic partial differential equation

$$\frac{\partial G(t,x,y)}{\partial t}=\frac{\partial^2G}{\partial x^2}+p(x)\frac{\partial G}{\partial x}+q(x)G,$$

where

$$p(x)=-(1/Kx^3)(2sKx^2+\Delta\omega x+K),$$
$$q(x)=-(1/Kx^4)[(g_n-\Delta\omega)x+K(n-1)],$$

with the Dirichlet data

$$G(0,x,y)=\sum_{j=1}^{2s+1}(xy)^j,$$

and (from the definition of G)

$$G(t,0,y)=0.$$

[18] See, e.g., P. M. Mathews, I. I. Shapiro, and D. L. Falkoff, Phys. Rev. **120**, 1, (1960).

[19] F. B. Hildebrand, *Method of Applied Mathematics* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965), 2nd ed., p. 76.

[20] P. Lambropoulos, Phys. Rev. **156**, 286 (1967).

V. ANALYTIC SOLUTIONS FOR SPECIAL CASES

The eigenvalues and eigenvectors of any 2×2 matrix are easily found. Since the characteristic equation for a 3×3 matrix is a cubic equation in the eigenvalue, these solutions can also be written out explicitly. For the resonant case the matrix $\mathbf{C}$ from Eq. (3.5) can be solved simply for any case of dimension up to 5×5. This includes not only the cases $s=\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $n\geq0$, but also the cases $s\geq2$, $n=-2s+k$, $k=0$, 1, 2, 3, 4. The latter states will be—with exceptions to be noted in the next section—the lowest-lying states for a given value of s. Thus for a system of up to four particles interacting with the radiation field the exact solution for the resonant case can be written in simple form. The system for a single particle for the nonresonant case is also simply displayed. The nonresonant solutions for two particles ($s=1$, a 3×3 matrix) and the resonant solutions for five particles (including an s value of $\frac{5}{2}$ and hence a 6×6 matrix—but a simple one) can also be written as solutions of a cubic equation.

The simpler solutions will be given herein in the Fock representation. The cubic solutions are not given, since they are too unwieldy to provide much insight without further work. Note that the eigenvectors are orthogonal (except in the unlikely case of degeneracy) since $\mathbf{C}_n'$ is real symmetric.

A. Single-Particle Case

For the single particle the only allowed states are for the representation $s=\frac{1}{2}$, $n\geq-1$. Using Eq. (3.5), we obtain two solutions for each $n\geq0$:

$$\begin{aligned} &E_{n\pm}=(n+\tfrac{1}{2})\omega\pm K\eta_n\,,\\ &|\psi_{n\pm}\rangle=\Theta_{n\pm}\epsilon_1|n\rangle\pm\Theta_{n\mp}\epsilon_2|n+1\rangle\,, \end{aligned} \tag{5.1}$$

where

$$\eta_n\equiv(\alpha^2+n+1)^{1/2},\quad \alpha\equiv-\Delta\omega/2K,$$
$$\Theta_{n\pm}\equiv[(\eta_n\pm\alpha)/2\eta_n]^{1/2}.$$

The case $n=-2s$ yields a single "vacuum" state

$$E_{-2s}=-s\omega_0\,,\quad |\psi_{-2s}\rangle=\epsilon_{2s+1}|0\rangle \tag{5.2}$$

for general s. For the single-particle case, $s=\frac{1}{2}$, we have

$$E_{-1}=-\tfrac{1}{2}\omega_0\,,\quad |\psi_{-1}\rangle=\epsilon_2|0\rangle. \tag{5.3}$$

B. Two-Particle Case

The state $|\psi\rangle$ will be an arbitrary combination of states for which $s=0$ (which are essentially just pure photon states, since there can be no interaction) and states for which $s=1$. In the latter case the eigenvalues and eigenvectors can be written explicitly, using standard solutions for cubic equations. In the simple resonant case $\Delta\omega=0$, the three energies E_{nl} are

$$E_{nl}=(n+1)\omega-\lambda_{nl}\,, \tag{5.4}$$

with

$$\lambda_{n2}=0,\quad \lambda_{n\pm}=\pm K[2(2n+3)]^{1/2}.$$

The corresponding normalized states are

$$\begin{aligned} |\psi_2\rangle=&(2n+3)^{-1/2}\\ &\times[(n+2)^{1/2}\epsilon_1|n\rangle-(n+1)^{1/2}\epsilon_3|n+2\rangle], \end{aligned} \tag{5.5}$$

$$\begin{aligned} |\psi_\pm\rangle=&[2(2n+3)]^{-1/2}[(n+1)^{1/2}\epsilon_1|n\rangle\\ &\mp(2n+3)^{1/2}\epsilon_2|n+1\rangle+(n+2)^{1/2}\epsilon_3|n+2\rangle]. \end{aligned} \tag{5.6}$$

$|\psi_+\rangle$ will be the lowest-lying state of the three for $K>0$.

For the case $n=0$, these results were checked with those of Tavis and Cummings.[2] [$n=0$ corresponds to their $c=1$; their r is our s. We set their $\phi=0$. There is some ambiguity about their Eq. (2.11b) for A_0 for this case, but it can be determined from their Eq. (2.8) after A_1 and A_2 are found.]

It is interesting to note in this case that the strength of the coupling constant does not enter into the distribution among the states of different photon number, etc. However, examination of Eq. (5.1) will reveal that for the single-atom case the constant K does enter. Since this case was calculated for nonresonant conditions, we expect that the resonant restriction in the $s=1$ case causes the independence. It can be demonstrated that this is indeed so by examining the nonresonant case in detail. There is a simple nonresonant case which will illustrate the point.

The case $n=-2s+1$, $s\geq1$ has a 2×2 matrix which can be solved for the nonresonant case. The result is quite similar to the single-particle solution. We have

$$\begin{aligned} E_{nl}\equiv E_{(-2s+1)_\pm}=&(-s+1)\omega\\ &+(s-\tfrac{1}{2})\Delta\omega\mp K\eta_{2s-1}, \end{aligned} \tag{5.7}$$

$$|\psi_{(-2s+1)\pm}\rangle=\Theta_{(-2s+1)\pm}\epsilon_{2s}|0\rangle\pm\Theta_{(-2s+1)\mp}\epsilon_{2s+1}|1\rangle, \tag{5.8}$$

where η_k and $\Theta_{k\pm}$ have the same meaning as in Eq. (5.1). It is therefore clear that K is involved in the distribution of states in general.

Setting $s=1$ above yields the solution for $n=-1$ for the two-particle case (and for the case of a higher number of particles with $s=1$ also).

C. Three-Particle Case (at Resonance)

Solutions are allowed for $s=\frac{1}{2}$, $n\geq-1$; $s=\frac{3}{2}$, $n\geq-3$. The solutions for $s=\frac{1}{2}$ have been given in Sec. IV A. The solutions for $s=\frac{3}{2}$, $n\geq0$ are

$$\begin{aligned} &E_{nl}=(n+\tfrac{3}{2})\omega-\lambda_{nl}\,,\\ &\lambda=\pm K[5(n+2)]^{1/2}\\ &\quad\times\left\{1\pm\left[1-\left(\frac{3}{5}\right)^2\frac{(n+3)(n+1)}{(n+2)^2}\right]^{1/2}\right\}^{1/2}, \end{aligned} \tag{5.9}$$

where all combinations of the signs yield the four values. Take $\lambda_{n1}=++$, $\lambda_{n2}=+-$, $\lambda_{n3}=--$, $\lambda_{n4}=-+$ to order the energies in ascending or descending order

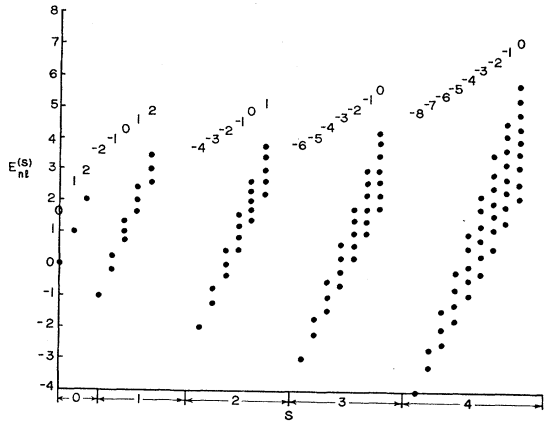

FIG. 1. Energies of low-lying states for eight particles. Numbers at the top are the values of n. $\delta=0$, $K=0.1$.

depending on the sign of K. The normalized eigenvectors are

$$|\psi_{nl}\rangle=(\beta_{nl})^{-1}[h_1h_2\lambda\epsilon_1|0\rangle-h_2\lambda^2\epsilon_2|1\rangle +\lambda(\lambda^2-h_1^2)\epsilon_3|2\rangle-h_3(\lambda^2-h_1^2)\epsilon_4|3\rangle], \quad (5.10)$$

where $h_j=K[j(4-j)(n+j)]^{1/2}$ and the normalization factor β_{nl} is given by

$$\beta^2=4K^4[\lambda^2(26n^2+109n+143) -9K^2(n+1)(n+3)(2n+7)].$$

The solutions for $s=\frac{3}{2}$, $n=-3$, -2 are contained in the previous sections as special cases. The case $s=\frac{3}{2}$, $n=-1$ is contains in the general solution for $n=-2s+2$, which is

$$E_{(-2s+2)l}=(-s+2)\omega-\lambda_{(-2s+2)l}, \quad (5.11)$$

with the three values for λ

$$\lambda_2=0, \quad \lambda_\pm=\pm K[2(4s-1)]^{1/2}.$$

The normalized eigenvectors are

$$|\psi_{(-2s+2)2}\rangle=(4s-1)^{-1/2}[(2s)^{1/2}\epsilon_{2s-1}|0\rangle -(2s-1)^{1/2}\epsilon_{2s+1}|2\rangle], \quad (5.12)$$

$$|\psi_{(-2s+2)\pm}\rangle=[2(4s-1)]^{-1/2}[(2s-1)^{1/2}\epsilon_{2s-1}|0\rangle \mp(4s-1)^{1/2}\epsilon_{2s}|1\rangle+(2s)^{1/2}\epsilon_{2s+1}|2\rangle], \quad (5.13)$$

with $\psi_\pm$ corresponding to $\lambda_\pm$.

D. Four-Particle Case (at Resonance)

The cases which allow solutions are $s=0$, $n\geq 0$; $s=1$, $n\geq -2$; $s=2$, $n\geq -4$. All these are contained in the earlier sections except for $s=2$, $n\geq -1$. The solutions for $s=2$, $n\geq 0$ are

$$E_{nl}=(n+2)\omega-\lambda_{nl}, \quad (5.14)$$

$\lambda_{n3}=0$. The other four values of λ are obtained from

$$\lambda=\pm K[5(2n+5)]^{1/2}\left\{1\pm\frac{3}{5}\left[1+2\left(\frac{2}{2n+5}\right)^2\right]^{1/2}\right\}^{1/2}.$$

With the choice of signs $\lambda_{n1}=++$, $\lambda_{n2}=+-$, $\lambda_{n4}=--$, $\lambda_{n5}=-+$, the energies are ordered ascending or descending. The eigenvectors are

$$|\psi_{nl}\rangle=(\beta_{nl})^{-1}[h_1h_2h_3\epsilon_1|0\rangle-\lambda h_2h_3\epsilon_2|1\rangle+h_3(\lambda^2-h_1^2)\epsilon_3|2\rangle -\lambda(\lambda^2-\sigma^2)\epsilon_4|3\rangle+h_4(\lambda^2-\sigma^2)\epsilon_5|4\rangle], \quad (5.15)$$

where $h_j=K[(j+n)j(5-j)]^{1/2}$, $\sigma^2=24K^2(n+1)(n+2)$, and the normalization factor β_{nl} is given by

$$\beta^2=4[18K^4\lambda^2(2n^2+15n+29)-72K^6(8n^2+40n+41)].$$

The solutions for $s=2$, $n=-1$ are contained in the solutions for $s\geq\frac{3}{2}$, $n=-2s+3$:

$$E_{(-2s+3)l}=(-s+3)\omega-\lambda_{(-2s+3)l},$$
$$\lambda=\pm K[5(2s-1)]^{1/2} \times\left\{1\pm\left[1-\left(\frac{3}{5}\right)^2\frac{2s(2s-2)}{(2s-1)^2}\right]^{1/2}\right\}^{1/2}, \quad (5.16)$$

$$|\psi_{(-2s+3)l}\rangle=(\beta_{(-2s+3)l})^{-1}[h_1h_2\lambda\epsilon_{2s-2}|0\rangle-h_2\lambda^2\epsilon_{2s-1}|1\rangle +\lambda(\lambda^2-h_1^2)\epsilon_{2s}|2\rangle-h_3(\lambda^2-h_1^2)\epsilon_{2s+1}|3\rangle], \quad (5.17)$$

where $l=1, 2, 3, 4$ corresponding to the four values of λ as before;

$$h_j=K[j(2s-3+j)(4-j)]^{1/2}$$

and

$$\beta^2=8K^4[\lambda^2(52s^2-37s+10)-18K^2(s-1)s(4s+1)].$$

E. Five-Particle Case (at Resonance)

The cases which allow solutions are $s=\frac{1}{2}$, $n\geq -1$; $s=\frac{3}{2}$, $n\geq -3$; $s=\frac{5}{2}$, $n\geq -5$. The $s=\frac{1}{2}$ and $\frac{3}{2}$ cases have been given, as have the cases $s=\frac{5}{2}$, $n=-5$, -4, -3, -2. The solution for $s=\frac{5}{2}$, $n=-1$ is contained in the solutions for $s\geq 2$, $n=-2s+4$:

$$E_{(-2s+4)l}=(-s+4)\omega-\lambda_{(-2s+4)l},$$
$$\lambda_3=0,$$

or

$$\lambda=\pm K[5(4s-3)]^{1/2} \times\left\{1\pm\frac{3}{5}\left[1+\frac{1}{2}\left(\frac{4}{4s-3}\right)^2\right]^{1/2}\right\}^{1/2}. \quad (5.18)$$

For λ_3,

$$|\psi_{(-2s+4)3}\rangle=(\beta_{(-2s+4)3})^{-1}\{[12s(s-1)]^{1/2}\epsilon_{2s-3}|0\rangle -[4s(2s-3)]^{1/2}\epsilon_{2s-1}|2\rangle+[3(2s-3)(2s-1)]^{1/2} \times\epsilon_{2s+1}|4\rangle\}, \quad (5.19)$$

where $\beta^2=32s^2-48s+9$. The states for the other four values of λ are given by

$$|\psi_{(-2s+4)l}\rangle =(\beta_{(-2s+4)l})^{-1}[h_1h_2h_3\epsilon_{2s-3}|0\rangle-\lambda h_2h_3\epsilon_{2s-2}|1\rangle +h_3(\lambda^2-h_1^2)\epsilon_{2s-1}|2\rangle-\lambda(\lambda^2-\sigma^2)\epsilon_{2s}|3\rangle +h_4(\lambda^2-\sigma^2)\epsilon_{2s+1}|4\rangle], \quad (5.20)$$

with

$$h_j=K[j(2s-4+j)(5-j)]^{1/2},\quad \sigma^2=4K^2(5s-6),$$

and

$$\beta^2=72K^4[\lambda^2(8s^2-2s+1)-4K^2(32s^2-48s+9)].$$

The solutions for the case $s=\frac{5}{2}$, $n\geq 0$ involve a cubic equation.

F. One Solution for All Odd-Order Matrices

It is possible to prove that $E_n=g_n$ is an eigenvalue for all odd-order matrices for the resonant case by induction (g_n is not a root in general for the nonresonant case nor for half-integral s). The proof is as follows.

The characteristic equation (3.5) increases in order by 2 as s increases by unity. Suppose we have a determinant $A_{2s-1}=0$, with a root $E_n=g_n$. The determinant A_{2n+1} will be

$$A_{2n+1}=\begin{vmatrix} & & & & 0 & 0 \\ & A_{2n-1} & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & h_{2n-1} & 0 \\ 0 & \cdots & 0 & h_{2n-1} & 0 & h_{2n} \\ 0 & \cdots & 0 & 0 & h_{2n} & 0 \end{vmatrix}, \tag{5.21}$$

with E_n set equal to g_n. If we expand by the Laplace method about the last two rows or columns, we find $A_{2n+1}=-h_{2n}{}^2A_{2n-1}=0$. Now we know $E_n=g_n$ is a root for the 3×3 characteristic equation. Hence it is a root for the 5×5, the 7×7, ..., equation.

The fact that $E_n=g_n$ is a root for integer s is mentioned without proof by Tavis and Cummings.[6]

Examination of Eq. (5.21), incidentally, reveals that the eigenvalues do not depend on the sign of K, since $h^2\propto K^2$ and the same sort of inductive proof as above goes through.

VI. NUMERICAL RESULTS AND DISCUSSION

A digital computer was used with a standard subroutine in the IBM library called EIGEN to compute eigenvalues and eigenvectors for the symmetric matrix $\mathbf{C}_n'$ of Eq. (3.5). The method used seems to be efficient for the tridiagonal matrix involved. As the size of the off-diagonal elements increases relative to the diagonal ones the accuracy decreases, but single-precision arithmetic was found sufficient to yield answers accurate to four or five significant figures for matrices as large as 26×26 for any of the parameter values used herein. In some cases the ratio of the off-diagonal to diagonal elements was of the order of unity. It seems entirely practical to incorporate this calculation directly into numerical computation of other quantities—such as the density matrix—for matrices of the above order. Thus problems involving, e.g., thermal distributions of the solutions for this Hamiltonian can be solved exactly for 25 to 50 particles.

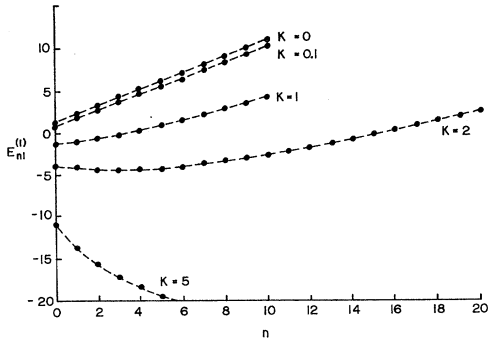

FIG. 2. Energy of lowest state for $s=1$, showing minimum for strong K. $\delta=0.01$.

Parameters varied in the solutions were the coupling constant K, the off-resonance parameter $\delta\equiv\Delta\omega/\omega$, the minimum photon number n, and the angular momentum s. In the results presented below, E_{nl} and K have been normalized by dividing by ω, or, equivalently, unit frequency has been used.

We consider first the behavior of the eigenvalues $E_{nl}{}^{(s)}$ (the superscript s is used only where it lends clarity). Figure 1 shows the lowest-lying states for integral values of s through 4. There is one state for each value of $n\geq 0$ for $s=0$, two for each $n\geq 0$ for $s=\frac{1}{2}$, three for each $n\geq 0$ for $s=1$, etc. Values of $n<0$ have a smaller number of states as discussed previously. The energies vary somewhat for different δ, the variation being greatest for small n and s. For a given s and n, the states become displaced symmetrically from $(n+s)\omega$ as δ approaches zero. Larger values of K spread the $2s+1$ states more about this mean value. The larger s values are lower-lying, but it is not always true—contrary to what one might expect from Fig. 1—that for a given s, the lowest n value has the lowest energy. Figure 2 demonstrates a case for which higher n have lower energy for the lowest of the three states belonging to $s=1$, $E_{n1}{}^{(1)}$ (the value of δ does not significantly affect these results). The effect occurs here for only very large K, but for larger s the value of K required decreases. This can be seen by examination of the analytic cases in the previous section. From Eqs. (5.2) and (5.7), we obtain

$$E_{(-2s+1)1}-E_{(-2s)1}=\omega-|K|\sqrt{2}s^{1/2}.$$

We see that under conditions of strong coupling and/or coherent actions of many particles, a state with a higher mean photon number and also some material excitation is lower in energy than a state with less photons and no (or less) material excitation. This is a reflection of the difference in "binding" energy between the field and material system.

Figure 3 shows the behavior of the eigenvalues as a function of the departure from resonance for various values of K and n. In practice, of course, K will be a function of δ, but here it is treated as an independent

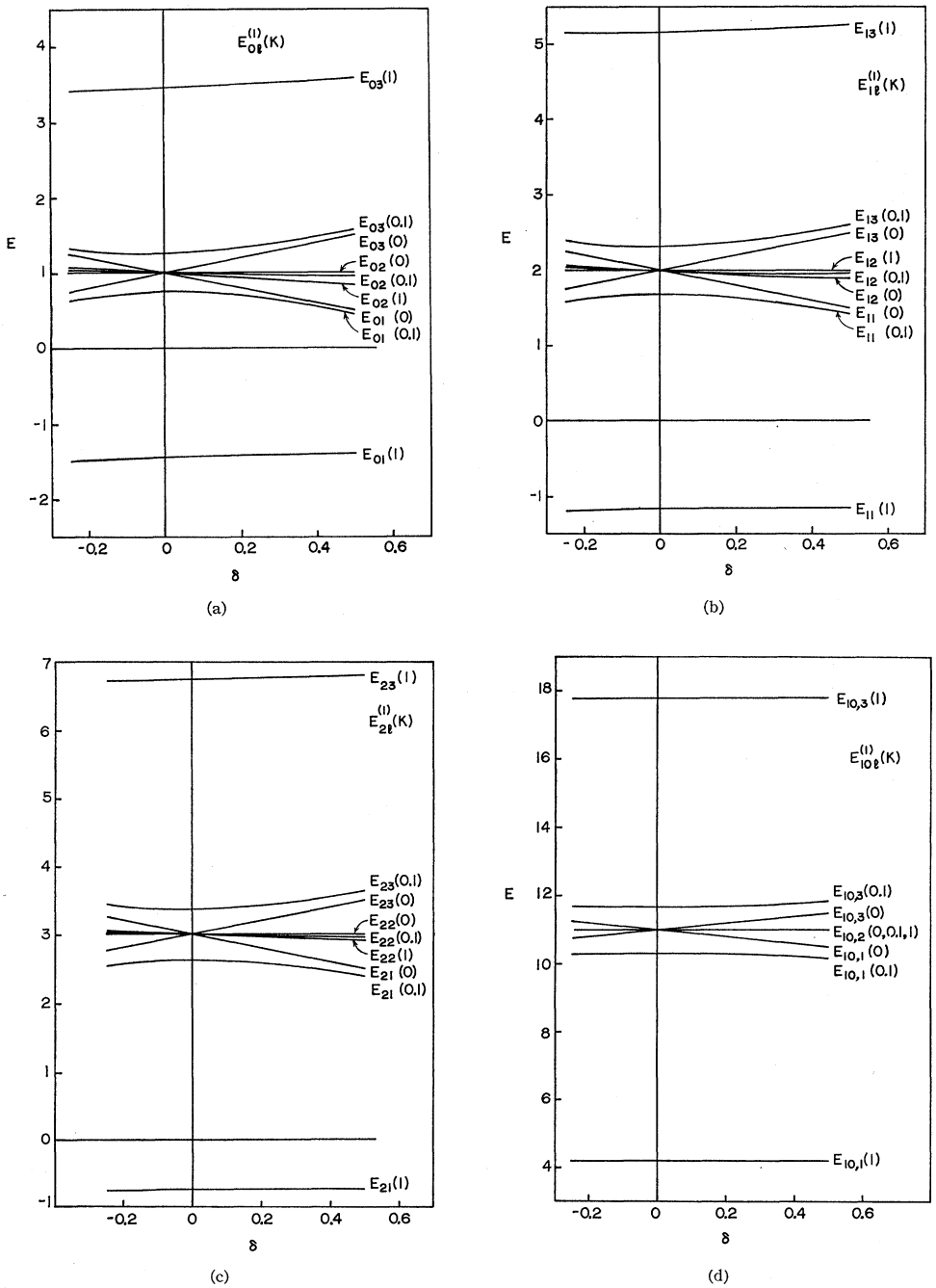

FIG. 3. Energy behavior versus δ and K for different values of n. (a) $n=0$, (b) $n=1$, (c) $n=2$, (d) $n=10$.

parameter. It is clear that the value of δ becomes of less and less importance with increasing minimum photon number n. That is, if δ causes a change ΔE, then the fractional change $\Delta E/E$ decreases as n increases. Note also that larger K causes less dependence on δ. Although not investigated extensively, it is suggested by Eq. (5.7) and by the structure of $\mathbf{C}_n'$ that larger values of s also decrease the relative effects of δ. Let us compare the results shown in Fig. 3 with those obtained by first-order perturbation theory.

Write the Hamiltonian (1.1) as $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1$, where

$$\mathcal{H}_0=\omega(a^\dagger a+S_3)+K(aS_++a^\dagger S_-),$$
$$\mathcal{H}_1=-\Delta\omega\, S_3=-\delta\omega\, S_3.$$

Then standard first-order perturbation theory gives for

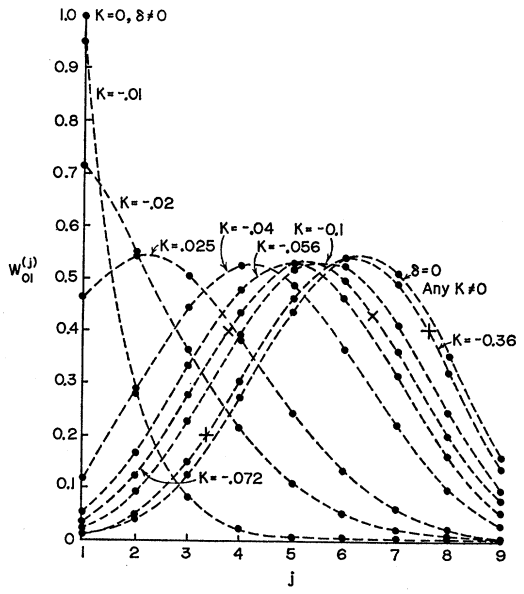

FIG. 4. Eigenvector variation with K/δ. $\delta=0.1$ unless otherwise indicated. $s=4$, $n=0$. The two symbols + indicate sections of the same curve, and similarly for the symbols $\times$.

the change in energy with departure from resonance

$$dE_{nl}^{(1)}/d\delta=[2(2n+3)]^{-1}$$

for $l=1$ and 3 (called $\pm$ in Sec. V) and

$$dE_{nl}^{(1)}/d\delta=-(2n+3)^{-1}$$

for $l=2$. For $n=0$, these become $\frac{1}{6}$ and $-\frac{1}{3}$, respectively, in agreement with the data of Fig. 3(a). Note that the slopes for both $l=1$ and $l=3$ are positive at $\delta=0$, but that the slope for $l=1$ changes sign when $\delta\approx|K|$.

This point of breakdown appears in the eigenvector distributions also. Figure 4 shows the eigenvector components for the lowest state corresponding to $s=4$, $n=0$. For $\delta=0$ the distribution among the components is independent of K. For $K\neq0$ and fixed $\delta\neq0$ the components shift with decreasing K to a "pure" state $W_{01}^{(1)}=1$; $W_{01}^{(j)}=0$, $j\neq1$ (for $\delta=0$ this state is degenerate with other such states as $W_{01}^{(i)}=1$; $W_{01}^{(j)}=0$, $j\neq i$). The first-order perturbation theory above applied to the eigenvectors for $s=1$ predicts that the mixing in $|\psi_{nl'}\rangle$ with $|\psi_{nl}\rangle$ is proportional to δ/K so that the solution for $|\delta/K|\gtrsim1$ cannot be obtained by perturbation theory from the resonant solution. Qualitatively the same results are obtained by reducing both K and δ a factor of 10.

TABLE I. Mean photon number for $s=1$, $\delta=0$, any $K\neq0$.

n / l	0	1	2	10
1	1.167	2.100	3.072	11.022
2	0.667	1.800	2.857	10.956
3	1.167	2.100	3.072	11.022

TABLE II. Mean photon number for $s=1$, $\delta=0.1$, $K=-0.1$.

n / l	0	1	2	10
1	0.743	1.780	2.803	10.876
2	0.789	1.850	2.884	10.959
3	1.467	2.370	3.313	11.167

As n increases, the behavior of the eigenvector distribution depends less and less on K and δ. Larger K always gives a smaller slope magnitude $|dW/d\delta|$, at $\delta=0$, and for any fixed K this slope decreases as n increases.

Tavis and Cummings[6] have described the appearance of the $2s+1$ eigenvectors in the Fock representation for a given s and n. Those states having the slowest variation or smallest number of nodes among the components (i.e., in a graph such as Fig. 4) belong to the lowest energy for $K<0$. A change in sign of K has the effect of keeping the magnitudes of the components the same but switching the signs of the higher- and lower-lying states. For $\delta=0$, in particular, the magnitudes $|W^{(j)}|$ and $|W^{(2s+1-j)}|$ are equal, and changing the sign of K just inverts the ordering of the states with respect to the eigenvalues.

A few values of the mean photon number for $s=1$ are given in Tables I and II to illustrate the effect of δ on some low-lying states for $s=1$. Only a positive value of δ is illustrated. The eigenvector is sensitive to the sign as well as the magnitude of δ.

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