Effect of Residual Interactions on Coulomb Energy Shifts in Analogue States*

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The effect of residual interactions on the Coulomb energy separation between analogue states is examined in the same spirit and accuracy with which residual interactions are handled in nuclear physics. A closed-form expression allows for simple calculational procedures. The dependence on the form of the residual interaction can then be easily explored. A specific calculation is made for the Ca isotopes; the effect of configuration mixing is studied.

I. INTRODUCTION

IN the past many authors have examined the relation between Coulomb energy separations and nuclear structure or size. This subject has been revitalized by the recent flurry of activity in analogue states. Nolen, Schiffer, Williams, and Von Ehrenstein[1] have used the Coulomb energy separation between analogue states to determine a neutron radius for the Ca isotopes. Similar investigations[2] have been carried out for the Pb region of the Periodic Table. These latter studies have relied on an extreme single-particle model of the nucleus. This paper is principally concerned with the effect on this Coulomb energy separation of the residual interaction and the correlations introduced by such interactions.

To illustrate the general procedure, we limit ourselves here to the light Ca isotopes. The problem divides naturally into two separate lines: the single-particle orbitals on the one hand, and the changes resulting from interaction between particles placed in these orbitals on the other hand. In the present paper, we wish to consider only the latter aspect. However, for orientation we begin with a brief description of the Coulomb energy difference in the Sc^{41}-Ca^{41} pair of nuclei and the difficulties which have arisen[3] in treating this difference.

II. Sc^{41}-Ca^{41}

To study the Coulomb energy in an independent-particle context, one requires the experimental charge distribution and a particle wave function. The latter is obtained from an appropriate nuclear model. Nolen *et al.*[1,2] deduce the wave function from a single-particle Woods-Saxon potential; the radius and depth of the potential are adjusted to simultaneously match the Coulomb energy and the $f_{7/2}$ neutron binding energy in Ca^{41}. In the selection of this single-particle well, one may begin by assuming that the entire measured energy separation $\Delta E_{\text{expt}}=7.28$ MeV, between the ground states of Sc^{41} and Ga^{41} could be attributed to a direct Coulomb energy

$$\Delta E_C=\int d\tau\, \psi_{f_{7/2}}{}^* V_C \psi_{f_{7/2}}, \tag{2.1}$$

where $\psi_{f_{7/2}}$ is the single-neutron wave function and V_C is the one-particle electrostatic potential arising from the experimentally determined charge distribution. There is no need to compute only to lowest order in V_C, but for a reasonably bound level it is of little importance to improve on this.

Of course, one should remove from ΔE_{expt} a number of corrections of both a single-particle and a correlational nature. Such corrections would lead to a readjustment of the well parameters, and hence to a change in the rms radius of the valence neutron orbital. Indeed, the classic relation

$$\Delta E_C=(Ze^2/R)\,(\tfrac{3}{2}-\tfrac{1}{2}\langle r^2\rangle/R^2) \tag{2.2}$$

for a particle in a uniform charge distribution of radius R indicates the rather direct connection between Coulomb energy and valence radius. To extract physical information from the experimentally measured Sc^{41}-Ca^{41} energy separation, one must contend with the sensitivity to small theoretical corrections apparent in Eq. (2.2). Using values of $\langle r^2\rangle$ appropriate to a $f_{7/2}$ orbit and of $R=4.52$ fm (corresponding to an rms radius 3.50 fm, fixed by electron scattering experiments) one sees a 1% change in ΔE_C is magnified into a 3 or 4% change in the physically more interesting $\langle r^2\rangle$. The corrections which have already been considered[3] are (a) the Coulomb exchange energy, (b) the electromagnetic spin-orbit interaction, and (c) the kinetic energy arising from the neutron-proton mass difference. The calculated estimates for these

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[1] J. A. Nolen, Jr., J. P. Schiffer, N. Williams, and D. von Ehrenstein, Phys. Rev. Letters **18**, 1140 (1967).

[2] J. A. Nolen, Jr., J. P. Schiffer, and N. Williams, Phys. Letters **27B**, 1 (1968).

[3] J. P. Schiffer, in *Proceedings of the Second Conference on Nuclear Isospin*, edited by J. D. Anderson, S. D. Bloom, J. Cerny, and W. W. True (Academic Press Inc., New York, 1969).

corrections suggest that one should take for the direct Coulomb term in Eq. (2.1) a figure some few hundred keV higher than ΔE_{expt}. However, this value for the direct term implies a valence neutron rms radius of 3.50 fm; this is somewhat surprising in view of the previously quoted value 3.50 fm for the average proton radius in Ca^{40}.

Clearly, one must approach the problem of extracting absolute valence neutron radii with caution, taking care to include all sizable theoretical corrections to the Sc^{41}-Ca^{41} energy separation. In a subsequent work, we will treat this essentially distinct problem in detail, including estimates of core polarization effects and of an important neutron-proton energy difference which arises from breaking the isobaric-spin invariance of the Ca^{40} core wave function.

III. Sc^{42}-Ca^{42}

The effects of valence nucleon interaction on a study of Coulomb energies and neutron radii can be delineated by considering only the changes resulting in passing from the $A=41$ to the $A=42$ nuclei. Nolen *et al.*[1,2] treat these nuclei in an equivalent fashion, using as a measure of the neutron binding energy in Ca^{42} the average binding energy of the two valence neutrons. We handle the excess binding energy in Ca^{42} by explicitly considering a residual nuclear interaction between the valence particles without disturbing the shell-model average potential. No doubt some rearrangement in the core occurs, resulting in a small change in the average valence-core potential, but we expect this to be smaller still.

We wish to answer the question as to how interaction changes the conclusions about Coulomb energies drawn on the basis of noninteracting orbits. Further, we want to do this in the same spirit as one performs normal shell-model calculations. As a first consideration, let us treat the neutron configuration $(1f_{7/2})^n$ with a residual interaction V_{res} between the valence nucleons. Suppose for the moment that mixing with $2p_{3/2}$, $2p_{1/2}$, $f_{5/2}$, and other configurations can be ignored. Then the increase in ground-state energy due to the presence of residual interaction is in the usual, lowest order given by the expectation of V_{res} in the unperturbed wave function ψ_0,

$$\psi_0(T, T_3) = \mid (1f)^n JM;\, TT_3\rangle, \qquad T=T_3=\tfrac{1}{2}n. \quad (3.1)$$

The first-order correction to the wave function resulting from the residual interaction $\psi^{(1)}(T, T_3)$ is given by the first order of perturbation theory

$$\psi^{(1)}(T, T_3) = \sum_{I\neq 0} (E_0-E_I)^{-1} \mid I\rangle \langle I \mid V_{\text{res}} \mid \psi_0(T, T_3)\rangle. \quad (3.2)$$

The Coulomb energy separations ΔE_C between analogue states is then given by

$$\langle\psi_0(T, T-1) \mid V_C \mid \psi_0(T, T-1)\rangle + 2\langle\psi_0(T, T-1) \mid V_C \mid \psi^{(1)}(T, T-1)\rangle = \Delta E_C{}^{(0)} + \Delta E_C{}^{(1)}. \quad (3.3)$$

The first term $\Delta E_C{}^{(0)}$ is the proton Coulomb energy which when calculated for $(1f)^1$ remains constant for the whole set of $(1f)^n$ nuclei.

The change with neutron number is contained in the second term $\Delta E_C{}^{(1)}$, involving both the Coulomb potential and residual interaction. One might point out at this stage that we find V_{res} produces only a small alteration in the wave function and hence including terms quadratic in V_{res} is unlikely to prove fruitful. The entire effect of the extra binding energy as well as the correlations produced by V_{res} are contained in this term. That V_{res} brings in correlations is obvious.

It is also true that it contains the binding-energy correction to the asymptotic dependence, within the order of perturbation theory considered, thus removing any need for special or *ad hoc* prescriptions to wrench the asymptotic dependence into agreement with the binding energy.[4] To see this, we have only to look at the first-order perturbation equation

$$(H_0-E_0)\psi^{(1)} = (\Delta E - V_{\text{res}})\psi_0 \quad (3.4)$$

in the asymptotic region and outside any of the interactions, it becomes

$$(T-E_0)\psi_{\text{asym}}{}^{(1)} = \Delta E \psi_{0\ \text{asym}}. \quad (3.5)$$

Now, in the asymptotic region $\psi_{0\ \text{asym}}$ will, in general, consist of pieces of different angular and radial dependences, subject however to over-all angular momentum and energy conservation. The radial dependences that appear in $\psi_{0\ \text{asym}}$ are of the form

$$\prod_i r_i^{-1} \exp\{-[(2M_i/\hbar^2)E_0{}^{(i)}]^{1/2} r_i\}, \qquad \sum_i E_0{}^{(i)} = E_0. \quad (3.6)$$

Then it can be shown directly from the asymptotic form of the differential equation (3.5), that $\psi_{\text{asym}}{}^{(1)}$ has this same form but with radial dependence replaced by

$$\left\{\prod_i r_i^{-1} \exp\left[-\left(\frac{2M_i}{\hbar^2} E_0{}^{(i)}\right)^{1/2} r_i\right]\right\} \times \left[-\tfrac{1}{2}\sum_j \left(\frac{2M_i}{\hbar^2}\right)^{1/2} \frac{\Delta E^{(j)}}{(E_0{}^{(j)})^{1/2}} r_j\right], \quad (3.7)$$
$$\sum \Delta E^{(j)} = \Delta E.$$

[4] Such effects have been considered for nucleon-transfer calculations by N. Austern, Phys. Rev. **136**, B1743 (1964); W. T. Pinkston and G. R. Satchler, Nucl. Phys. **72**, 641 (1965).

This can be seen as a first-order expansion of

$$\prod_i r_i^{-1} \exp\{[-(2M_i/\hbar^2)(E_0^{(i)}+\Delta E^{(i)})]^{1/2} r_i\}, \quad (3.8)$$

which guarantees the correct asymptotic dependence governed by

$$\sum_i (E^{(i)}+\Delta E^{(i)}) = E_0+\Delta E.$$

The calculation of $\Delta E_C^{(1)}$ is particularly simple, if we make the quite good approximation that V_C can be written as a sum of one-body operators

$$V_C = \sum_i [\tfrac{1}{2}(1-\tau_3^{(i)})] v_C(r_i), \quad (3.9)$$

the $v_C(r_i)$ being the potential appropriate to an assumed charge distribution. In writing V_C in this simplified way we are approximating the irrelevant isoscalar pieces and the important isovector portion, and throwing away the small isotensor. Once this useful assumption is made, one may include the one-body operators in (3.9) along with the one-body potential defining the neutron orbits. Just how this goes can be seen by straightforward manipulation of the form of $\Delta E_C^{(1)}$

$$\Delta E_C^{(1)} = 2\langle \psi_0(T,T) \mid n^{-1} \sum_n v_C(r_i) \mid \psi^{(1)}(T,T)\rangle, \quad T=\tfrac{1}{2}n. \quad (3.10)$$

Using the form (3.2) for $\psi^{(1)}$, we obtain

$$\Delta E_C^{(1)} = 2 \sum_{I\neq 0} \langle \psi_0(T,T) \mid n^{-1} \sum_i v_C(r_i) \mid I\rangle \langle I \mid V_{\rm res} \mid \psi_0(T,T)\rangle/(E_0-E_I) \quad (3.11)$$

$$= 2\langle \psi_C^{(1)}(T,T) \mid V_{\rm res} \mid \psi_0(T,T)\rangle \quad (3.12)$$

with

$$\psi_C^{(1)}(T,T) = \sum_{I\neq 0} \mid I\rangle \langle I \mid n^{-1} \sum_i v_C(r_i) \mid \psi_0(T,T)\rangle/(E_0-E_I). \quad (3.13)$$

Now, $\psi_C^{(1)}$ is just the first-order perturbational change in the nuclear wave function produced by adding the one-particle potential $v_C(r)/n$ to the potential determining the orbit. Instead of computing the sum we can go back and recalculate the single-particle orbits $(1f)\rightarrow(1f)'$, which are quasiproton orbits, produced by apportioning one-nth of the single-particle Coulomb potential to each neutron. Then the wave function $\psi_0'(T,T)$, which is formed of $(1j)'$ orbitals in the same fashion as $\psi_0(T,T)$ of $(1f)$ orbitals,

$$\psi_0'(T,T) = \mid (1f)'^n, JM; T, T\rangle \quad (3.14)$$

agrees to first order with

$$\psi_0(T,T)+\psi_C^{(1)}(T,T). \quad (3.15)$$

Finally, to the order of interest,

$$\Delta E_C^{(1)} = \langle \psi_0'(T,T) \mid V_{\rm res} \mid \psi_0'(T,T)\rangle - \langle \psi_0(T,T) \mid V_{\rm res} \mid \psi_0(T,T)\rangle. \quad (3.16)$$

The problem is, thereby, reduced to the level of the simplest calculations with residual interactions. The energy $\Delta E_C^{(1)}$ calculated from this equation may be viewed as the reduction in the nuclear binding energy produced by the more extended proton wave function. Finally, we should note that the residual interaction $V_{\rm res}$ is purely the $T=1$ part of the internucleon force; the $T=0$ part appears only if we keep terms of higher order in V_C.

IV. NUMERICAL RESULTS FOR Sc^{42}-Ca^{42}

To evaluate $\Delta E_C^{(1)}$ for the case of two valence neutrons one should properly construct the ground-state wave function

$$\psi_0 = \mid (f^2)J=0, T=1T_3=1\rangle \quad (4.1)$$

from the single-particle functions of a realistic potential—say, a Woods-Saxon well. However, it has been shown[5] that the deeply bound $f_{7/2}$ level of such a Woods-Saxon well is adequately described by a single oscillator function with $\hbar\omega=12$ MeV. We may exploit this feature of the Ca^{41} orbitals to evaluate $\Delta E_C^{(1)}$ in a purely harmonic-oscillator basis. If we are willing, in addition, to use for v_C the potential appropriate to a charge distributed uniformly in a sphere of radius R

$$v_C(r_i) = (Ze/R)[\tfrac{3}{2}-\tfrac{1}{2}(r_i^2/R^2)], \quad (4.2)$$

then the calculation of $\Delta E_C^{(1)}$ becomes trivial. Actually, this should be used only in the domain $r_i \leq R$, but for well-bound wave functions it might as well be used everywhere. Then, the nonconstant term in v_C proportional to r^2 simply modifies the oscillator parameter for each quasiproton, i.e.,

$$\omega^2 \rightarrow \omega^2 - \tfrac{1}{2}(Ze^2/2R^3)2/m.$$

A useful relation can be obtained that connects the Coulomb energy shift $\Delta E_C^{(1)}$ with the change in the valence neutron mean-square radius

$$\Delta\bar{r}^2 = 2\langle \psi_0 \mid \{n^{-1}\sum_i r_i^2\} \mid \psi^{(1)}\rangle \quad (4.3)$$

[5] C. K. Scott, Ph.D. thesis, McGill University, 1969 (unpublished).

TABLE I. Coulomb energy shifts due to the various components of the V_{res} of Eq. (4.5). The ground-state matrix element and $\Delta\bar{r}^2/\bar{r}^2$ for these components are also given.

Force component	$-G_\alpha \exp(-\alpha^2 r^2)$	$G_\alpha[f/(mc)^2]p^2\delta(\mathbf{r})$	$-G_\beta \exp[-(\beta r)^2]$	Total
$\langle (f^2)^0 \mid v \mid (f^2)^0 \rangle$ (MeV)	−2.118	+0.399	−1.28	−3.00
$\Delta\bar{r}^{-2}/\bar{r}^2$ (%)	−1.07	+0.92	−0.46	−0.63
$\Delta E_C^{(1)}$ (MeV)	0.026	−0.022	0.011	0.015

due to V_{res}. The relation which reads

$$\Delta E_C^{(1)} = -\tfrac{1}{2}(Ze^2/R^3)\Delta r^2 \qquad (4.4)$$

results from comparing (4.3) with (3.3). This relation clearly depends on the approximate form we have chosen for v_C but does not involve our restrictive choice of single-particle potential.

For a residual interaction we use a purely S-wave, $T=1$ force taken over from realistic force calculations, whose form is

$$v_{res} = -G_\alpha\{\exp(-\alpha^2 r^2) - [\tfrac{1}{2} f/(mc)^2] \times [\bar{p}^2\delta(\mathbf{r}) + \delta(\mathbf{r})\bar{p}^2]\} - \{G_\beta \exp[-(\beta)^2 r^2]\}, \qquad (4.5)$$

$$G_\alpha = 15.93 \text{ MeV}, \quad \alpha^{-1} = 2.804 \text{ fm}, \quad f = 41.1 \text{ fm}^{-3},$$

$$G_\beta = 3.38 \text{ MeV}, \quad \beta^{-1} = 2(3.72) \text{ fm}.$$

The range and strength parameters in the first bracket were chosen by equating this term to the free reaction matrix $K(\epsilon)$, in the fashion discussed by Kahana, Lee, and Scott.[6] Strictly speaking, this reaction matrix has a dependence on an energy-like parameter ϵ, but in the case of the $^1S_0$ matrix, to which we are here limiting ourselves, adequate accuracy is obtained by approximating this rather weak dependence by the value obtained when $\epsilon = \infty$. This portion of our realistic S-wave force is very much like the similar component in any of the other existing "realistic" forces. The salient feature of the first bracketed term, in V_{res}, for our purposes is that it contains a p^2 momentum dependence which simulates the repulsive, very short-ranged component of the nuclear force. The second bracket in V_{res} is intended to be a longer-ranged[7] component replacing a core-polarization contribution. We are not considering the inherent Coulomb structure of the core polarization. In practice, we adjusted G_β so as to guarantee that the total matrix element of V_{res} in $\psi_0(T, T)$ fits the empirical value of approximately -3 MeV. The other f^2, $J \neq 0$ states of Ca^{42} are then also approximately fitted. The numerical results are displayed in Table I.

The net Coulomb shift obtained by summing up the contributions of the various components of V_{res} is

$$\Delta E_C^{(1)} \approx (0.026 - 0.022) + (0.11) = 0.015 \text{ MeV}.$$

Why is this shift so small? First, it is clear from our basic equation (4.6) that a very long-ranged force gives a vanishing shift; in general, the longer the range of the force the smaller the Coulomb shift. This explains the smallness of the contribution for the core-polarization component. Moreover, this range effect also permits a significant cancellation to occur between the contributions of the attractive and repulsive pieces of the first bracket in V_{res} by magnifying the shift due to the zero-range momentum-dependent repulsive piece relative to that due to the finite-ranged attractive piece. The lesson to be learned from this calculation is that the Coulomb energy shift is not determined simply by a fitting of the energy levels; one must also know the detailed nature of the residual force. The details of the matrix-element calculations are contained in the Appendix.

The sign of $\Delta E_C^{(1)}$ deduced from our over-all attractive force is opposite to that observed empirically. As we shall show in Sec. V, configuration mixing in the $1f$-$2p$ major shell can bring in effects of opposite sign. However, a magnitude for $\Delta E_C^{(1)}$ much larger than that given in Eq. (4.6) would probably imply an unreasonable amount of such configuration mixing.

V. MIXED CONFIGURATIONS

Calculations with residual interactions frequently include configuration mixing. It is usual to work in a truncated subspace, but within this small space to diagonalize exactly. The method outlined above is readily extended to this situation. Roughly, one has only to include the single-particle Coulombic interaction in the calculation of the orbitals, and use these modified orbitals in diagonalizing the residual interaction. The inclusion of admixed configurations can be important in calculating mean radii or Coulomb energies, say a $(f_{5/2}^2)$, $(p_{3/2}^2)$, or $(p_{1/2}^2)$ part mixed into a dominantly $(f_{7/2}^2)$ configuration. In the presence of configuration mixing and to lowest order in the Coulomb force, we have again a simple result

$$\Delta E_C^{(1)} = \langle \psi_{(T/T)}' \mid V_{res} \mid \psi'(T, T) \rangle - \langle \psi(T, T) \mid V_{res} \mid \psi(T, T) \rangle,$$

[6] S. Kahana, H. C. Lee, and C. K. Scott, Phys. Rev. (to be published).

[7] Somewhat large "range" parameters β^{-1} and α^{-1} result when Gaussian forms for the residual interaction are used.

where for a mixture of two configurations denoted by ψ_0 and ψ_1

$$\psi(T,T)=a_0\psi_0(T,T)+a_1\psi_1(T,T),$$
$$\psi'(T,T)=a_0\psi_0'(T,T)+a_1\psi_1'(T,T).$$

We also, of course, must realize that the zero-order Coulomb energy is changed by configuration mixing:

$$\Delta E_C^{(0)}=\langle\psi(T,T)\mid V_C\mid\psi(T,T)\rangle.$$

The amplitudes a_0 and a_1 are those obtained in the absence of the Coulomb force. As in our earlier presentation the wave function $\psi'(T,T)$ is obtained by dividing the Coulomb single-particle potential equally between the two valence nucleons. We do not show any detail, but it follows also for mixed configurations that the asymptotic behavior is well represented by perturbation theory. Finally, it should be noted that the obvious generalization works for more than two configurations.

We have previously stated that the Coulomb energy shift in the Ca^{41}-Sc^{41} pair is 7.28 MeV. The corresponding number for $A=42$ is 7.21 MeV. Part of this difference presumably is due to a proton radius increase between the $A=41$ and 42 nuclei. We have estimated by interpolating the $A=41$ radius fiom measured radii for Ca^{40} and Ca^{42}, that 0.03 MeV of the change in Coulomb energy is so obtained. Since the observed Coulomb energies for the $A=41$, $p_{3/2}$, $p_{1/2}$, or $f_{5/2}$ levels are all some 300 MeV less than that for the $f_{7/2}$ level, the remaining 0.05 MeV in the difference in Coulomb energies for the $A=41$ and $A=42$ nuclei implies an admixture of some 15% non-$f_{7/2}$ levels in the ground state of Ca^{42}. This conclusion can be arrived at because again for the mixed configurations we find $\Delta E_C^{(1)}$ to be small, and so only $\Delta E_C^{(0)}$ is significant. We do not view the latter calculation as more than schematic. This is especially true in view of our neglect of non-Coulombic isotopic-spin-breaking components of the residual force.

VI. FURTHER REMARKS

For the particular nucleus we chose to consider, $\Delta E_C^{(1)}$ turned out to be small. There were two reasons for this. First the specific features of our force played a role: If the p^2 dependence of our $^1S_0$-wave force were absent a somewhat larger $\Delta E_C^{(1)}$ would result. In addition, the inclusion of higher relative-partial-wave interaction is important: A calculation with the ^3P-wave force included produced a small but negative $\Delta E_C^{(1)}$. Second, $\Delta E_C^{(1)}$ was reduced in magnitude by the deep binding energy and high angular momentum of our main $f_{7/2}$ configuration. One might expect $\Delta E_C^{(1)}$ to be sizably larger for the $2s$-$1d$ shell because the $2s$ proton and neutron single-particle wave functions differ considerably.

TABLE II. The range dependence of $\Delta E_C^{(1)}$ and $\Delta\bar{r}^2/\bar{r}^2$ produced by a Gaussian force $-G\exp(-\alpha^2r^2)$ normalized to yield $\langle(f_{7/2}^2)^0\mid\{-G\exp(-\alpha^2r^2)\}\mid(f_{7/2}^2)^0\rangle=-3$ MeV. The variation is in terms of the dimensionless parameter $2\alpha^2/\mu^2$.

$2\alpha^2/\mu^2$	$\Delta\bar{r}^2/\bar{r}^2$ (%)	$\Delta E_C^{(1)}$ (MeV)
100	4.04	0.098
12	3.67	0.089
4	1.70	0.041
1	1.56	0.038
0.5	1.64	0.040
0.125	1.07	0.026

APPENDIX

To perform the calculations in Secs. III and IV of this paper we require the matrix elements in $J=0$ states, constructed from f^2 or p^2 configurations, of an s-wave force. A simple calculation is presented here for the state $(f_{7/2}^2)_{J=0}$ and the results for the other states are then stated. Using the standard 9-j [8] and Brody-Moshinsky[9] algebra for a relative $^1S_0$-wave force, we have

$$\langle(f_{7/2}^2)^0\mid V_{\rm res}\mid(f_{7/2}^2)^0\rangle$$
$$=\sum_{n=0}^{3}\{(3\tfrac{1}{2})^{7/2}(3\tfrac{1}{2})^{7/2}\mid(33)^0(\tfrac{1}{2}\,\tfrac{1}{2})^0\}_0^2\times(03030\mid n0N00)^2x(n0\mid v\mid n0) \quad \text{(A1)}$$
$$=0.037797[v_{00}+v_{30}]+0.264515[v_{10}+v_{20}], \quad \text{(A2)}$$

where n refers to the numbers of radial nodes in the relative oscillator wave function. Using a Gaussian force $G\exp(-\alpha^2r^2)$, we then obtain

$$\langle(f_{7/2}^2)^0\mid V_{\rm res}\mid(f_{7/2}^2)^0\rangle$$
$$=G\sigma^{3/2}[1.0134-3.2742\sigma+6.6971\sigma^2-8.2680\sigma^3+6.6971\sigma^4-3.2742\sigma^5+1.0134\sigma^6], \quad \text{(A3)}$$

where

$$\sigma=(1+2\alpha^2/\mu^2)^{-1},\qquad \mu^2=m\omega/\hbar$$

is the oscillator "size" parameter. One can easily check that the correct limits are obtained for zero-range or infinite-range forces.

In addition, one has for the $1f$-$2p$ shell

$$\langle(f_{7/2}^2)^0V_{\rm res}\mid(f_{7/2}^2)^0\rangle=\tfrac{4}{3}\langle(f_{5/2}^2)^0\mid V_{\rm res}\mid(f_{5/2}^2)^0\rangle$$
$$=(\tfrac{4}{3})^{1/2}\langle(f_{5/2}f_{7/2})^0\mid v_{\rm res}\mid(f_{5/2}f_{7/2})^0\rangle, \quad \text{(A4)}$$

and finally that

$$\langle(p_{3/2}^2)^0\mid v_{\rm res}\mid(p_{3/2}^2)^0\rangle$$
$$=G\sigma^{3/2}\{0.56726-2.32628\sigma+8.09056\sigma^2-16.8927\sigma^3+2.1383\sigma^4-15.1593\sigma^5+4.6921\sigma^6\}, \quad \text{(A5)}$$
$$\langle(p_{3/2}^2)^0\mid v\mid(f_{7/2}^2)^0\rangle$$
$$=G\sigma^{3/2}\{0.3049-1.1756\sigma+3.6902\sigma^2-7.2183\sigma^3+8.8166\sigma^4-6.1252\sigma^5+1.8959\sigma^6\}. \quad \text{(A6)}$$

[8] J. M. Kennedy and M. J. Cliff, Atomic Energy of Canada Limited Report, Chalk River, Canada, 1955 (unpublished).

[9] J. A. Brody and M. Moshinsky, *Tables of Transformation Brackets* (Monografras del Instituto de Fisica, Mexico, 1960).

Matrix elements of the p^2-dependent δ force are easily evaluated:

$$\begin{aligned}&\langle (f_{7/2}{}^2)^0 \mid v_{\text{repul}} \mid (f_{7/2}{}^2)^0 \rangle \\ &\quad = [Gf/(mc)^2] \langle (f_{7/2}{}^2)^0 \mid p^2\delta(r) \mid (f_{7/2}{}^2)^0 \rangle \\ &\quad = m[Gf/(mc)^2] \sum c_n \epsilon_{n0} (n0 \mid \delta(r) \mid n0) \qquad \text{(A7)} \\ &\quad = 2.143 \times 10^4 (\hbar\omega/mc^2)^{5/2} \text{ MeV},\end{aligned}$$

where c_n are the coefficients in (A1) and $\epsilon_{n0} = (2n+\frac{3}{2})\hbar\omega$.

The dependence on ω of the repulsive component of our realistic force may be used directly to evaluate the Coulomb shift produced by this component. Equations (A3) can be similarly employed to deduce the Coulomb shifts and/or changes in radius resulting from the Gaussian terms in our force. Table II summarizes the dependence of these quantities on the range of the force.

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Yields of Radionuclides Produced in Thick Targets Bombarded with 3.0-GeV Electrons*

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Thick targets of aluminum, iron, and lead were bombarded with 3.0-GeV electrons. γ-ray spectra of the bombarded samples were measured with a Ge(Li) spectrometer. These spectra were used to determine yields of individual radionuclides as a function of depth in the targets. The target thickness at which the yield curves peak decreases with the number of charged particles emitted from the target nucleus. The target thicknesses at which the yield curves peak were used to obtain estimates of the dominant photon energies for the production of each nuclide.

I. INTRODUCTION

HIGH-ENERGY electrons in passing through matter lose energy principally by bremsstrahlung. Many of the photons produce second-generation electrons by Compton scattering and by pair production. The result is an electron-photon cascade shower. Nuclear reactions are induced by the photons and by the electrons. In 1924, Fermi[1] pointed out the close relationship between interactions produced by moving charged particles and incident electromagnetic waves. Subsequently, Weizsacker[2] and Williams[3] showed, by making a Fourier analysis of the field produced at a given point by a passing relativistic electron, that the field of the electron contains predominantly transverse components and thus may be regarded as a beam of virtual photons that can produce nuclear reactions.

Information concerning the nuclear reactions induced by electron-photon cascade showers can be obtained by measuring the yields of radionucludes produced in a target. The high-resolution Ge(Li) γ-ray spectrometer makes such measurements relatively easy. In the work reported here, thick targets of aluminum, iron, and lead were bombarded with 3.0-GeV electrons. Yields of several radionuclides were measured as a function of depth in the targets.

II. EXPERIMENTAL

Bombardments were made with a 3.0-GeV beam of the Cambridge Electron Accelerator. The targets were positioned in air, 5 ft downstream from a secondary-emission beam-current monitor. Integrated beams of $\sim 10^{15}$ electrons were used for each bombardment. Targets consisted of thin foils sandwiched between slabs of absorber material; each target stack was several

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