where the factor $(-)^{l+1}$ arises from the easily derived relation

$$\varphi_l(-k^*, r) = (-)^{l+1}\varphi_l^*(k, r). \quad \text{(A19)}$$

Substituting (A18) and (A19) into (A7″) gives the desired result

$$R_{l,\alpha}(r, r') = \varphi_l(k_\alpha, r)\varphi_l^*(-k_\alpha^*, r')/[2k_\alpha I_{l,\alpha\alpha}\{\varphi_l\}]. \quad \text{(A20)}$$

We have obtained (A2) in terms of $\varphi_l(k, r)$, however the result is obviously independent of the normalization of the Gamow-state wave function. It is well-known that (A2) holds at bound-state momenta and the results of this appendix indicate that the definition (2.18) provides a method for extending (A2) to consider a wider class of Gamow states. Romo[3] has derived the same result for his definition of the Gamow-state inner product.

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Partial-Wave Analysis for Certain Simple Systems with Application to the Reaction $^{16}O(d, \alpha_1)^{14}N^*$ †

P. L. Jolivette and H. T. Richards

University of Wisconsin, Madison, Wisconsin 53706

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The angular distributions of spin-zero even-parity particles emerging from a compound nucleus formed by a 1^+ and 0^+ pair are displayed in a simple form which facilitates parametrizing the differential cross section in terms of partial waves. The method is applied to the $^{16}O(d,\alpha_1)^{14}N^*$ data of Jobst *et al.* to display the resonant partial waves corresponding to ^{18}F states. Intrinsically ambiguous solutions of number $2^{l_{max}}$ are found empirically. The problem of selecting the correct solution is discussed. Data with polarized deuterons are of no help since the analyzing powers are energy-independent (cf. Jacobsohn and Ryndin, who first noted that the reaction is an absolute detector of deuteron alignment).

INTRODUCTION

As pointed out by Jobst, Messelt, and Richards,[1] angular momentum and parity conservation result in simple and unique angular distributions for a pair of spin-zero even-parity particles emerging from a single state J^π of a compound nucleus formed initially by a 1^+ and 0^+ pair. (For other interesting properties of such spin and parity combinations, see Bohr,[2] and Peshkin,[3] and Jacobsohn and Ryndin.[4]) By use of $\bar{Z}$ coefficients, the theoretical angular distributions can be evaluated in a straightforward but tedious manner. The resultant angular distributions are so simple, however, that even qualitative considerations give unambiguous J^π assignments to relatively isolated resonances. Jobst *et al.* used these considerations to assign J^π to a number of ^{18}F states which appear as resonances in the $^{16}O(d, \alpha_1)^{14}N^*$ reaction where the ^{14}N residual nucleus is left in a $J^\pi = 0^+$ excited state. However, sizable asymmetries about 90° in the c.m. system in the data also imply various amounts of interference with neighboring states. The next step of quantitatively evaluating the amplitudes of the interfering partial waves becomes a much more difficult operation. Since the angular distributions computed for each isolated state turn out to be so simple, we sought also a simple analytic expression that exhibited the interference effects in a form more convenient for analysis and for physical understanding.

RESULTS

This simple analytic form turns out to be as follows:

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{12}\left|\sum_{l=1}^{\infty}\frac{2l+1}{[l(l+1)]^{1/2}}S_l\frac{dP_l(\cos\Theta)}{d\Theta}\right|^2,$$

where S_l is the complex amplitude of the lth partial wave, and Θ is measured in the c.m. system. An isolated resonance of the compound system involves only a single S_l and corresponds to a $J^\pi = l^{(-)^l}$ level of the compound nucleus. The above formula can be obtained by a tedious reduction (for our special system) of the general equations 3.13-3.16 of Ref. 5. Its

† Work supported in part by the U.S. Atomic Energy Commission.

[1] J. Jobst, S. Messelt, and H. T. Richards, Phys. Rev. **178**, 1663 (1969).

[2] A. Bohr, Nucl. Phys. **10**, 486 (1959).

[3] M. Peshkin, Phys. Rev. **133**, B428 (1964).

[4] B. A. Jacobsohn and R. M. Ryndin, Nucl. Phys. **24**, 505 (1961).

[5] J. M. Blatt and L. C. Biedenharn, Rev. Mod. Phys. **24**, 258 (1952).

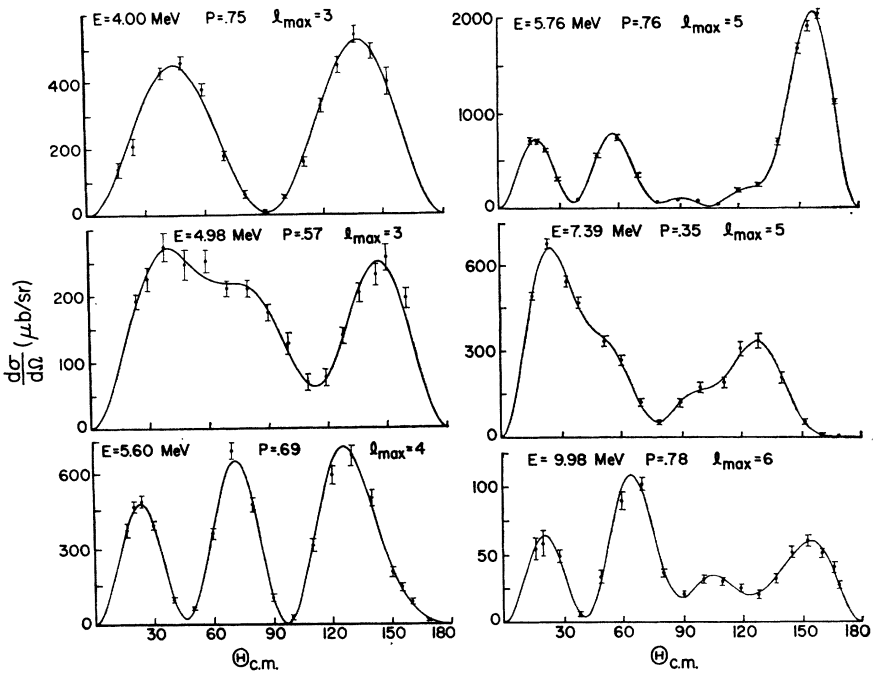

FIG. 1. Representative samples of the fit obtained for Figs. 2, 4, and 5 of Ref. 1. P is the index for Pearson's χ^2 test (Ref. 7).

simple character can, however, be directly understood as follows: For a reaction $(0^+)+(1^+)\rightarrow(0^+)+(0^+)$, $J=l_f$. Parity conservation requires $l_i=l_f(\equiv l)$ and excludes $l=0$. For the z axis along the incident-particle direction, $l_z=0$ so that $J_z=s_z$ and $s_z=0, \pm1$. But $\mathbf{l}_i+\mathbf{s}=\mathbf{j}=\mathbf{1}$ means that $\mathbf{l}_i$ is perpendicular to $\mathbf{s}$ and the plane of the $s_z=0$ substate is not perpendicular to $\mathbf{l}_i$. This statement corresponds to the Clebsch-Gordan coefficient $(l100\,|\,l0)=0$. Thus the only outgoing waves for a given l are those with $l_z=\pm1$, both of whose amplitudes are proportional to $Y_l^{(\pm1)}$. But

$$Y_l^{(\pm1)}\propto\left(\frac{(2l+1)\,(l-1)\,!}{4\pi(l+1)\,!}\right)^{1/2}\frac{dP_l(\cos\Theta)}{d\Theta}\,.$$

The trajectory of the complex amplitude, $S_l=R_l\times\exp(i\phi_l)$, corresponding to a resonance is a circle traced in the counterclockwise direction. The radius of the resonance circle is $(\Gamma_f\Gamma_i)^{1/2}/\Gamma$, and the center is determined by the nonresonant background in the manner discussed in detail by McVoy.[6]

Interference effects can then be visualized in terms of adding of these S_l vectors: a resonant one rapidly rotating, plus one or more slowly varying partial-wave vectors.

APPLICATION TO ^{18}F STATES

The above expression is particularly suitable when programming a computer to do a least-squares fit to the experimental $d\sigma/d\Omega$ in terms of the complex S-matrix elements. Each experimental $^{16}O(d, \alpha_1)^{14}N^*$ point of Jobst *et al.* (Figs. 2, 4, and 5 of Ref. 1) was weighted inversely by its assigned error, and a CDC-3600 computer then searched for a minimum in χ^2 for each angular distribution. The resulting fits were generally quite good (e.g., see samples shown in Fig. 1), and gave values P (from the Pearson χ^2 test[7]) of $0.1\leq P\leq0.9$. The number of permitted parameters (one less than twice the number of partial waves) was, of course, kept less than the number of data points being fitted (usually between 9 and 23). In most cases, good fits could be obtained with only a relatively few partial waves interfering, e.g., Fig. 1, and Table I.

In fitting the data, the summation over l ran from 1 to an $l_{\max}$ required for an acceptable χ^2 ($l=0$ is forbidden). The $l_{\max}$ required was sometimes greater than expected from hard-sphere particles of radius 4.3 fm for the deuteron and 1.5 $A^{1/3}$ fm for ^{16}O. If, after fitting the distribution, $l_{\max}$ was reduced by one, then the new fit was generally not acceptable. If $l_{\max}$ was increased by 1, the approximation of independent fitting parameters usually broke down almost completely and the least-squares matrix became very ill-conditioned, indicating too many free parameters. As the over-all phase of the system is arbitrary, we set the phase of the highest partial wave at zero. The uncertainties in the S_l's, which are a few parts in the last digits in the tables, we estimated from the

[6] K. W. McVoy, in *Fundamentals in Nuclear Theory*, edited by C. Villi and A. DeShalit (International Atomic Energy Agency, Vienna, 1967), p. 478.

[7] R. D. Evans, *The Atomic Nucleus* (McGraw-Hill Book Co., New York, 1955), p. 775.

TABLE I. The eight solutions at $E_d=5.76$ MeV for $l_{max}=5$. $S_l=R_l\exp(i\phi_l)$. Note that R_5 and $\mathrm{Re}(S_4)$ are the same for all solutions. Since the over-all phase is arbitrary, we set $\phi_5=0$. The solution numbers correspond to Fig. 2. Uncertainties are a few parts in the last place for R and a few tenths of a radian for ϕ.

Solution No.	R_1	ϕ_1	R_2	ϕ_2	R_3	ϕ_3	R_4	ϕ_4	$\mathrm{Re}(S_4)$	R_5	ϕ_5
1	0.01	2.0	0.135	2.3	0.055	5.8	0.005	3.7	0.003	0.100	0.0
2	0.03	4.4	0.115	2.7	0.075	5.4	0.020	4.6	0.003	0.100	0.0
3	0.06	3.5	0.110	4.1	0.065	0.9	0.035	4.6	0.003	0.100	0.0
4	0.06	2.8	0.080	3.6	0.095	1.1	0.020	4.5	0.003	0.100	0.0
5	0.08	2.2	0.075	3.8	0.040	1.2	0.080	4.7	0.003	0.100	0.0
6	0.10	2.0	0.110	3.8	0.040	0.3	0.045	4.6	0.003	0.100	0.0
7	0.10	1.8	0.040	3.2	0.075	1.2	0.060	4.7	0.003	0.100	0.0
8	0.13	1.7	0.080	3.4	0.065	0.8	0.025	4.6	0.003	0.100	0.0

likelihood function[8] $\mathfrak{L}$ (which describes the probability of getting the experimentally observed results as a function of the free parameters) for each variable (assumed independent). In general, the $l=1$ parameter had the largest error and l_{max} had the smallest.

In a few cases (not shown), in terms of P, satisfactory fits were not obtained. The poor fit may result from an underestimation of the data uncertainties, especially at forward angles, where dead-time corrections and background subtractions were relatively more important. A more likely cause may be undetected shifts in deuteron energy. The small cross sections for the reaction of interest required that the slits defining the deuteron energy be set further open than normal. The beam homogeneity was, however, much better than the energy thus defined by the slits. As a result, changes in tandem operating conditions (but with no changes in analyzing magnet) sometimes shift the average beam energy by as much as 15–20 keV. Since many of the angular distributions required long running periods, such shifts cannot be ruled out, although the general goodness of the fits even for narrow resonances suggests that the energy shifts were usually not important. For one angular distribution where a 16-keV shift was discovered, it was impossible to fit the data with a reasonable l_{max}.

AMBIGUITIES IN SOLUTIONS

It is well known that phase-shift analyses of scattering cross sections at a fixed energy usually have intrinsic ambiguities (e.g., Minami and Fermi-Yang equivalent solutions for spin-0–spin-$\frac{1}{2}$ scattering). In fact, Klepikov[9] shows that for spin-0–spin-$\frac{1}{2}$ scattering (with unknown polarization), there are 2^n solutions, where n is the number of partial waves. More recently, Van Wageningen[10] treated some of the ambiguities for elastic-scattering experiments. To our knowledge, the inelastic scattering and reaction problem has never been discussed, but we would conjecture that there are least an equivalent number of intrinsic ambiguities in the determination of the S-matrix elements.

In our least-squares fitting of the angular distributions by the computer, these intrinsically ambiguous solutions were found empirically by starting with random input parameters and letting the program vary the parameters until a minimum in χ^2 was found. If l partial waves were permitted in fitting an angular distribution, we found empirically a set of 2^l different solutions which gave identically the same cross section at all angles. In two cases, $l_{max}=3$ and $l_{max}=5$, a concerted but unsuccessful effort was made to find more solution than $2^{l_{max}}$.

Of the 2^l solutions, only 2^{l-2} had different amplitudes. The other solutions represented (1) reflections about the real axis (only cosines of the angle with respect to the real axis enter the cross-section formula), and (2) rotation of 180°. The latter operation is equivalent to a change in sign of all amplitudes, which, of course, yields the same cross section.

In order to discuss in more detail the properties of the ambiguous sets of solutions, we note that for our

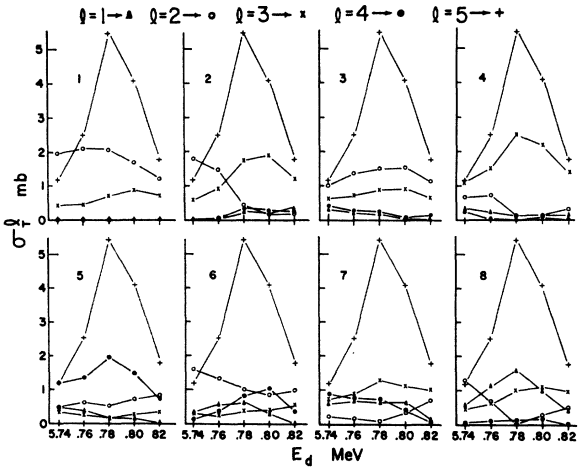

FIG. 2. Energy dependence of the S-matrix elements at the $E_d=5.78$-MeV resonance. The quantity plotted, $\sigma^l{}_T=\frac{1}{2}\pi\lambda^2(2l+1)|\,S_l\,|^2$, is the total cross section for each partial wave.

[8] J. Orear, University of California Radiation Laboratory Report No. UCRL 8417, 1958, p. 8 (unpublished). Available through Superintendent of Documents, U.S. Dept. of Commerce.

[9] N. P. Klepikov, Zh. Eksperim. i Teor. Fiz. **41**, 1187 (1961) [English transl.:Soviet Phys.—JETP **14**, 846 (1962)].

[10] R. Van Wageningen, Ann. Phys. (N.Y.) **31**, 148 (1965).

TABLE II. Splitting at $E_d=4.98$ MeV of the two ambiguous solutions for $l_{max}=3$ as $l_{max}=4$ is permitted. The number pairs are amplitude and phase (in radians) of $S_l=R_l\exp(i\phi_l)$. Uncertainties for the $l_{max}=3$ solutions are a few parts in the last place for R and a few tenths of a radian for ϕ. They are larger for the $l_{max}=4$ solution; in fact S_4 is not inconsistent with zero.

	S_1		S_2		S_3		S_4	
	$l_{max}=3$	$l_{max}=4$	$l_{max}=3$	$l_{max}=4$	$l_{max}=3$	$l_{max}=4$	$l_{max}=3$	$l_{max}=4$
Set I	0.06(2.5)	0.04(3.7) 0.10(1.3)	0.100(1.7)	0.085(3.4) 0.100(0.1)	0.063(0)	0.080(1.7) 0.025(4.3)	0(0)	0.016(0) 0.016(0)
Set II	0.13(5.0)	0.14(3.0) 0.12(6.1)	0.039(4.3)	0.015(2.1) 0.075(0.6)	0.063(0)	0.060(1.7) 0.040(4.5)	0(0)	0.016(0) 0.016(0)

case, Biedenharn's Eq. 102 (Ref. 11) becomes

$$d\sigma/d\Omega=(\lambda^2/12)\sum_\nu A_\nu P_\nu(\cos\Theta),$$

where

$$A_\nu=-\sum_{J_aJ_b}\bar{Z}(l_aJ_al_bJ_b;\,1\nu)\bar{Z}(l_aJ_al_bJ_b;\,0\nu)\,\mathrm{Re}[S_{J_a}{}^*S_{J_b}].$$

But the A_ν can be found empirically by a least-squares fit to the angular distribution. And since $\nu_{max}=2l_{max}$, we see that the empirical A_ν give us $2l_{max}+1$ equations in $2l_{max}-1$ unknowns. The number of unknowns is less because one phase may be arbitrarily picked and because $S_0=0$, since $l=0$ violates parity conservation. The symmetry properties of the $\bar{Z}$ coefficients reduce the $2l_{max}+1$ relations to $2l_{max}-1$ independent equations. The last two are

$$A_{\nu_{max-1}}=-\bar{Z}(lJl'J';\,1\nu_{max-1})\bar{Z}(lJl'J';\,0\nu_{max-1})\times S_{l_{max}}\,\mathrm{Re}[S_{l_{max-1}}],$$

$$A_{\nu_{max}}=\bar{Z}(lJlJ;\,1\nu_{max})\bar{Z}(lJlJ;\,0\nu_{max})\mid S_{l_{max}}\mid^2,$$

where $l=J=\frac{1}{2}\nu_{max}$ and $l'=J'=l-1$, and the arbitrary phase has been set at zero for the l_{max}th partial wave. The above relations show that $S_{l_{max}}$ and $\mathrm{Re}[S_{l_{max-1}}]$ should be the same for all solution sets. The empirical results (Table I) also exhibit these characteristics.

The above set of coupled quadratic equations allows for at most $2l_{max}-1$ choices of sign for square roots.

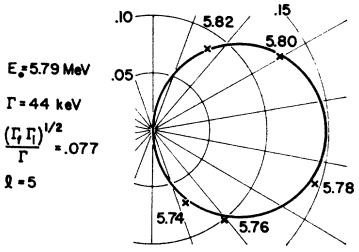

FIG. 3. Fit to S_5 of Fig. 2 with a single-level Breit-Wigner formula with no background contribution.

However, as we saw earlier, not all of these need be made. In fact, only l_{max} are necessary; hence we have $2^{l_{max}}$ different sets of answers.

When a higher partial wave first appears, its S-matrix element is small; therefore, as seen from the above, there are two solutions for $S_{l_{max-1}}$ with the same magnitude and with imaginary parts of opposite sign. As the partial wave becomes more important, two slightly split solutions grow out of each of the solutions that existed before this higher partial wave was introduced. See, for example, Table II, where at $E_d=4.98$ MeV there are two $l=4$ partial-wave solutions identified with each $l=3$ solution.

REMOVAL OF AMBIGUITIES

A "complete" experiment should of course remove the ambiguities. In the better-studied spin-0–spin-$\frac{1}{2}$ elastic-scattering case, polarization data usually serve this purpose. For charged-particle elastic scattering, interference with Coulomb scattering can usually distinguish solutions, or the known low-energy dependence of the partial waves may suffice. In the present case of spinless outgoing particles, polarization data is of no help (see below). If we follow individual solutions as a function of energy, perhaps we can exclude certain solutions as being physically unreasonable. In general, we expect that data from other reaction channels will be needed before we can identify a unique solution.

One might hope that use of polarized incident deuterons would help remove ambiguities. Such is not the case. Jacobsohn and Ryndin[4] have shown that the relevant quantities, the analyzing powers, are for this reaction independent of energy. This result is here easily demonstrated by noting that the channel-spin state $\mid m_s=0\rangle$ does not contribute. The analyzing powers[12] are then (in terms of the spin operators S_i)

$$T_{11}=-\tfrac{1}{2}\sqrt{3}\langle S_+\rangle=0,$$

$$T_{20}=\tfrac{1}{2}\sqrt{2}\langle 3S_z{}^2-2\rangle=\tfrac{1}{2}\sqrt{2},$$

$$T_{21}=-\tfrac{1}{2}\sqrt{3}\langle S_+S_z+S_zS_+\rangle=0,$$

$$T_{22}=\tfrac{1}{2}\sqrt{3}\langle S_+{}^2\rangle=\tfrac{1}{2}\sqrt{3},$$

[11] L. C. Biedenharn, in *Nuclear Spectroscopy*, edited by F. Ajzenberg-Selove (Academic Press Inc., New York, 1960), Part B, p. 787.

[12] W. Lakin, Phys. Rev. **98**, 139 (1955).

as the raising operator S_+ sends both $|-1\rangle$ and $|+1\rangle$ into nonexistent states, while S_+^2 and S_z can have nonzero eigenvalues.

We made an unsuccessful attempt to reduce the ambiguities by determining the energy dependence of the matrix elements of each solutions and eliminating those which appear unphysical. The only resonance for which sufficient data are available is the 5^- level at $E_d=5.78$ MeV (see Fig. 5 of Ref. 1). Here we followed the eight different sets of amplitudes (Table I) across the resonance by using the solutions at one energy as starting values for a neighboring energy. We checked for consistency by using these results as starting values for the first energy again. Figure 2 displays the energy dependence of the resulting $|S_l|^2$ near the resonance for all eight solutions, but no solution displays really unphysical behavior. A further difficulty arises since the relative phase between energies is unknown. For the resonant $|S_5|^2$, we may use a single-level Breit-Wigner formula to fix the relative phase (Fig. 3). These $l=5$ phases should then fix the relative phases of the other partial waves. However, there are several difficulties with this procedure. For one we need more data just above the resonance to preclude the possibility of another $l=5$ peak whose presence would complicate the analysis. Also, the possible 20-keV energy shifts discussed earlier could be of considerable importance since the resonance is only ~50-keV. Therefore, we cannot yet identify the proper set of solutions.

^{18}F Spins and Parities

When the multiple solutions to an angular distribution are similar (as, for example, in Fig. 2) we can often identify the resonating partial wave. These conditions are most likely to occur at low energy when l_{max} is small or where there is a strong relatively isolated level. Also, a resonance in $S_{l_{max}}$ is more apparent since the magnitude is the same in all solutions. The determinations below are made entirely from numerical solutions where all ambiguous solutions were studied and are thus independent of those of Jobst *et al.* In the following discussion, we refer to the data given in Fig. 2 of Ref. 1.

$E_d=3.64$ MeV: $l_{max}=3$. The two solutions are nearly identical and indicate $J^\pi=2^+$ as the resonating partial wave.

$E_d=3.85$ MeV: $l_{max}=3$, $J^\pi=2^+$. However, one solution has comparable amplitudes for $l=1$ and $l=2$ and also a fair amount of $l=3$ amplitude. If S_1 and S_3 have about the same phase, the center maximum can disappear and the result looks very much like $|S_2|^2$. We consider this unlikely here, since the S_2 is large for both solutions.

$E_d=4.00$ MeV: $l_{max}=3$, but $l=2$ is very dominant and hence $J^\pi=2^+$.

$E_d=4.98$ MeV: $l_{max}=3$. The solutions differ markedly and each requires a large amount of all three waves. The $l=3$ is stronger than at 4.92 MeV, but nothing can be said accurately about level positions.

$E_d=5.51$ MeV: $l_{max}=4$. The values fluctuate greatly between solutions with a tendency for $l=2$ to be large.

For the remaining discussion we refer to the data given in Figs. 4 and 5 of Ref. 1.

$E_d=5.60$ MeV: $l_{max}=4$. The solutions are all similar and all partial waves except $l=1$ are large.

$E_d=5.74$–5.82 MeV: $l_{max}=5$. Our Fig. 2 shows for all solutions the energy dependence of the matrix elements displayed as contributions to the total cross section. There is clearly a $J^\pi=5^-$ resonance at $E_d=5.78$ MeV, but no compelling reason to abandon any of the solutions although we may prefer No. 1 because of its simplicity. This solution would also imply a $J^\pi=2^+$ resonance at about $E_d=5.76$ MeV and a $J^\pi=3^-$ resonance at about $E_d=5.80$ MeV.

$E_d=7.39$ MeV: $l_{max}=5$. The solutions are again a mixture of all available partial waves; the mixture changes greatly from one solution to another.

At other energies the search for all possible solutions was not made and so no conclusions can be drawn. The sixth partial wave was necessary first at 7.80 MeV, and $l=7$ was not needed even at 11 MeV, although this does not necessarily mean that it was absent at all energies.

ACKNOWLEDGMENTS

We wish to thank Professor C. H. Blanchard for helping us with a simple physical description of the system.