Saskawa Theory of Reactions

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An iterative nonrelativistic theory of reactions is discussed. It generalizes Sasakawa's theory of elastic scattering, published in 1963. A key feature of the theory is its use of zero-order wave functions that allow self-consistent treatment of the asymptotic boundary conditions. This allows the theory to be applied both for inelastic scattering and for rearrangement. The analysis of rearrangement is a special case of the work by Efimenko, Zakhariev, and Zhigunov. The present paper emphasizes a zero-order wave function based on the "surface-δ model" of channel coupling. This wave function gives a zero-order theory (no iteration) that has the nature of a renormalized distorted-wave Born approximation. This zero-order theory may be adequate for most applications in nuclear physics.

1. INTRODUCTION

IN 1963 Sasakawa presented a nonlinear iterative theory of nonrelativistic elastic scattering.[1] The present paper discusses the extension of this theory to inelastic scattering and rearrangement. Techniques of practical calculation are discussed.

The Sasakawa method allows the development of controlled approximations to the solutions of coupled differential equations. These enable it to meet the admonition of Hüfner and Lemmer,[2] who remark that coupled-channel calculations in nuclear physics ought not to be conducted to an accuracy that exceeds the accuracy of the initial nuclear model. In fact, most coupled-channel theories are solved exactly,[3] in violation of the Hüfner-Lemmer criterion. This excessive accuracy is achieved at the expense of lengthy numerical calculations, whose difficulty inhibits the application of coupled-channel theories. By contrast, in zero order the Sasakawa method generates a simple "renormalized distorted-wave Born approximation"[4] that seems quite easy to apply. When greater accuracy is required it is achieved systematically, by iteration of the zero-order solution. It may be hoped that this method will allow the development of coupled-channel analyses for larger classes of physical problems.

There is considerable flexibility in the construction of trial functions with which to begin the iterative calculation. A trial function based on the surface-δ model of channel coupling[5] is proposed in this paper. This trial function may well be of sufficient accuracy so that most applications of the Sasakawa method in nuclear physics can be performed in zero order.

Section 2 discusses elastic scattering of spinless particles by a central potential, to introduce the Sasakawa method. The trial function based on δ-function coupling is described. Section 3 discusses inelastic scattering. Section 4 discusses a two-channel stripping reaction, to illustrate how the Sasakawa method can be applied for rearrangement. Section 5 is a summary. It is noted that the analysis in Sec. 4 is in large measure a special case of a general theory of rearrangement reactions, given by Efimenko, Zakhariev, and Zhigunov,[6] that successfully bypasses the well-known convergence difficulties.[7]

2. ELASTIC SCATTERING

Accurate calculations of scattering by a central potential are performed by expanding the wave function in partial waves. If the z axis is taken along the direction of the incident momentum $\mathbf{k}$, the partial-wave expansion has the form

$$\chi^{(+)}(\mathbf{r}) = (kr)^{-1} \sum_L i^L (2L+1) f_L(r) P_L(\hat{k}\cdot\hat{r}). \quad (2.1)$$

For convenience, the Coulomb interaction has been

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[6] T. G. Efimenko, B. N. Zakhariev, and V. P. Zhigunov, Ann. Phys. (N.Y.) **47**, 275 (1968).

[7] For a survey of the literature see R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Co., New York, 1966). A particularly clear discussion of the convergence difficulties is given by R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961).

ignored. In partial-wave expansion, the Schrödinger equation for $\chi^{(+)}(\mathbf{r})$ reduces to a set of ordinary differential equations for the radial wave functions $f_L(r)$,

$$\{(d^2/dr^2)-[L(L+1)/r^2]+k^2\}f_L(r)=\tilde{U}f_L(r), \quad (2.2)$$

where

$$\tilde{U}(r)\equiv(2\mu/\hbar^2)U(r). \quad (2.3)$$

To solve Eq. (2.2), we introduce $F_L(kr)$ and $G_L(kr)$, the usual regular and irregular functions governed by the left-hand side of Eq. (2.2). The regular function obeys

$$\{(d^2/dr^2)-[L(L+1)/r^2]+k^2\}F_L(kr)=0 \quad (2.4)$$

with a corresponding equation for $G_L(kr)$. At the origin, $F_L(kr)$ satisfies the boundary condition

$$F_L(0)=0. \quad (2.5)$$

Additional boundary conditions are imposed asymptotically, by adjusting the normalization and relative phase of F_L and G_L so that these functions satisfy

$$F_L\longrightarrow\sin\theta_L, \qquad G_L\longrightarrow\cos\theta_L, \quad (2.6)$$

where

$$\theta_L=kr-(\tfrac{1}{2}L\pi). \quad (2.7)$$

Of course, the functions F_L and G_L defined here are just kr times the familiar spherical Bessel and Neumann functions. They are real standing-wave solutions of Eq. (2.4). An outgoing-wave solution is obtained by constructing the linear combination

$$H_L=G_L+iF_L. \quad (2.8)$$

This is ikr times the spherical Hankel function.

To derive Sasakawa's solution of Eq. (2.2), we construct a corresponding integral equation[1]

$$f_L(r)=F_L(kr)+C_LH_L(kr)+k^{-1}\left\{H_L(kr)\int_r^\infty F_L(kr')-F_L(kr)\int_r^\infty H_L(kr')\right\}\times\tilde{U}(r')f_L(r')dr', \quad (2.9)$$

using the auxiliary functions defined in the preceding paragraph. The coefficient C_L is the scattered amplitude for the partial wave of order L. It is related to the phase shift δ_L by the equation

$$C_L=\exp(i\delta_L)\sin\delta_L. \quad (2.10)$$

The amplitude C_L is computed by means of the well-known integral formula

$$C_L=-k^{-1}\int_0^\infty F_L(kr)\tilde{U}(r)f_L(r)dr, \quad (2.11)$$

which may be verified, say, by requiring that Eq. (2.9) fulfill the boundary condition $f_L(0)=0$.

Equation (2.9) does not possess the form of the more usual integral equations derived from the radial Schrödinger equation. We obtain Eq. (2.9) by rearranging the Green's function of one of the standard integral equations, to display the scattered-wave term $C_LH_L(kr)$. By this step, and by proper choice of C_L, the first two terms of Eq. (2.9) are made to give an exact and explicit representation of the eigenfunction $f_L(r)$, everywhere outside the range of the potential $\tilde{U}(r)$. These two terms thereby isolate all those parts of $f_L(r)$ that are asymptotically nonvanishing. The third term of Eq. (2.9) generates only short-ranged parts of $f_L(r)$, effective in the region $\tilde{U}(r)\neq0$. The third term makes no contribution asymptotically.

We note at this point that for partial waves with $L>0$, the second and third terms of Eq. (2.9) have poles at the origin. Hence in the neighborhood of the origin there is strong cancellation between these two terms.

Sasakawa[1] advocates the iterative solution of Eq. (2.9), using as a zero-order form for $f_L(r)$ the expression

$$f_L^{(0)}=F_L(kr)+C_LH_L(kr). \quad (2.12)$$

In the iteration based on Eq. (2.12), the coefficient C_L is carried as an undetermined parameter; in effect, the two terms of Eq. (2.12) are iterated independently. After the iteration is terminated, C_L then is computed self-consistently (this is the nonlinear step in the theory) by substituting the ith iterate into Eq. (2.11):

$$C_L^{(i)}=-k^{-1}\int_0^\infty F_L(kr)\tilde{U}(r)f_L^{(i)}(r)dr. \quad (2.13)$$

Equation (2.13) determines an appropriate linear combination of the two independently iterated parts of $f_L^{(i)}(r)$. Even in zero order, Eq. (2.13) gives

$$C_L^{(0)}=-k^{-1}\int_0^\infty F_L^2(kr)\tilde{U}(r)dr\Big/\left(1+k^{-1}\int_0^\infty F_L(kr)H_L(kr)\tilde{U}(r)dr\right), \quad (2.14)$$

a result that goes beyond the Born approximation. Because the trial function is designed to be accurate outside the range of $\tilde{U}(r)$, the principal benefit of the iteration procedure evidently is the generation of successive short-ranged improvements to the wave function. The Sasakawa iteration procedure converges[1] for all short-ranged potentials $\tilde{U}(r)$ no matter what their strength.

The basic idea of the Sasakawa procedure is that $f_L(r)$ is computed by iteration of a trial wave function that contains C_L, so that subsequent self-consistent calculation of C_L using Eq. (2.11), controls the accuracy

of the asymptotic parts of the trial wave function. The particular technique used for the iteration, e.g., substitution in Eq. (2.9), is of lesser importance. For example, it is frequently best to discard Eq. (2.9) and to iterate directly with the radial Schrödinger equation, by substituting $f_L^{(i)}$ on the right-hand side of Eq. (2.2) and then solving for $f_L^{(i+1)}$. Once again C_L is carried throughout the iteration procedure as an undetermined parameter, and is only determined self-consistently, from Eq. (2.13), after the iteration has been terminated.

Whatever the method of iteration, it is presumably possible to speed up the convergence by replacing Eq. (2.12) by a zero-order form that is a better approximation to the exact $f_L(r)$. Although, any alternative zero-order wave function must reduce asymptotically to the same expression given by Eq. (2.12), it can depart from Eq. (2.12) in the neighborhood of the origin. At the very least we may use this freedom to introduce a zero-order form that does not diverge at $r=0$. One simple form that possesses this property is

$$f_L^{(0)\prime}=F_L(kr)+C_L\exp(i\theta_L). \tag{2.15}$$

However, the second term of Eq. (2.15) ignores centrifugal repulsion. More accurate trial wave functions are desirable.

A quite accurate zero-order wave function would seem to be

$$f_L^{(0)\prime\prime}=F_L(kr)+C_LJ_L(r,r_0), \tag{2.16}$$

where

$$\begin{aligned}J_L(r,r_0)&=H_L(kr), \quad \text{for} \quad r>r_0\\ &=F_L(kr)[H_L(kr_0)/F_L(kr_0)], \quad \text{for} \quad r<r_0.\end{aligned} \tag{2.17}$$

Here the function $J_L(r,r_0)$ is a conveniently normalized multiple of the Green's function for Eq. (2.4), for a source at the point r_0. This means that $J_L(r,r_0)$ satisfies the boundary conditions of the exact scattered wave, both asymptotically and at the origin. In addition, the function $J_L(r,r_0)$ is a solution of Eq. (2.4) everywhere except at r_0. Under these conditions, if $\tilde{U}(r)$ were a zero-range potential located at the radius r_0, Eq. (2.16) would be the corresponding *exact eigenfunction* governed by this potential. It seems plausible that with potentials of finite range Eq. (2.16) should serve as a quite accurate starting point for iteration, provided r_0 is chosen somewhere in the region $\tilde{U}(r)\neq 0$. In typical nuclear reaction applications, such as will be discussed in Secs. 3 and 4, r_0 would be chosen at the "nuclear surface," where channel couplings are strongest.

Although the choice of r_0 in Eq. (2.17) is reasonably arbitrary, it does seem best that any values chosen for r_0 should avoid the zeros of $F_L(kr_0)$, where the trial wave function has resonances.[8] Fortunately, in nuclear-physics applications the use of complex potentials reduces the importance of this question.

TABLE I. Square-well (attractive) s-wave phase shift in rad. With the well radius R and the well depth V_0, the dimensionless parameters A and B are $A=(2MR^2V_0/\hbar^2)$, $B=kR=(2MR^2E/\hbar^2)^{1/2}$. To calculate δ^J we use $r_0=\frac{1}{2}R$.

B ↓	$A\rightarrow$	0.5	1.0	1.5	2.0
0.5	δ_0	0.096	0.243	0.481	0.861
	δ_0^B	0.079	0.157	0.233	0.307
	δ_0^S	0.103	0.285	0.654	1.322
	δ_0^J	0.100	0.266	0.567	1.093
1.0	δ_0	0.155	0.351	0.587	0.845
	δ_0^B	0.136	0.266	0.388	0.499
	δ_0^S	0.164	0.399	0.717	1.079
	δ_0^J	0.160	0.377	0.655	0.966
1.5	δ_0	0.167	0.347	0.530	0.704
	δ_0^B	0.158	0.308	0.445	0.566
	δ_0^S	0.177	0.387	0.619	0.850
	δ_0^J	0.173	0.369	0.576	0.778
2.0	δ_0	0.148	0.291	0.425	0.547
	δ_0^B	0.148	0.289	0.419	0.536
	δ_0^S	0.155	0.320	0.486	0.643
	δ_0^J	0.152	0.306	0.455	0.593

Tables I–III illustrate[9] the extent to which Eq. (2.16) is an improvement over Eq. (2.12). These tables present values of s- and p-wave phase shifts for the square-well and Yukawa potential, as functions of the well depth (parameter A) and bombarding energy (parameter B). For each pair of values of A and B these tables give the exact phase shift δ, the Born approximation δ^B, the *zero-order* Sasakawa approximation δ^S derived from Eq. (2.14), and the *zero-order* J approximation δ^J based on Eq. (2.16) and derived by replacing H_L by J_L in Eq. (2.14). It is seen that the phase shifts δ^J are a considerable improvement over δ^S. The J approximation also is greatly superior to the Born approximation, wherever the Born approximation is significantly inaccurate. In other cases, its accuracy is comparable with that of the Born approximation. On the whole, Tables I–III support the view that Eq. (2.16) provides an excellent zero-order approximation. It is interesting to note how the phase shifts δ^J fall off at high energy. This is a consequence of the resonance effect mentioned in the preceding paragraph.

3. INELASTIC SCATTERING

In this section we discuss inelastic scattering that excites bound states of a target nucleus. The Sasakawa

[8] I am grateful to John Truelove for calling this question to my attention.

[9] Tables I–III are obtained from calculations performed by John Truelove.

TABLE II. Yukawa-well (attractive) s-wave phase shift in rad. With the potential $\tilde{U}=-K^2e^{-\mu r}/\mu r$, the dimensionless parameters A and B are $A=K^2/\mu^2$, $B=(k/\mu)$. To calculate δ^J we use $r_0=\mu^{-1}$.

B ↓	$A\rightarrow$	0.5	1.0	1.5	2.0
0.5	δ_0	0.204	0.496	0.900	1.369
	$\delta_0{}^B$	0.172	0.334	0.479	0.606
	$\delta_0{}^S$	0.278	1.016	1.901	2.260
	$\delta_0{}^J$	0.215	0.533	0.938	1.324
1.0	δ_0	0.218	0.475	0.764	1.072
	$\delta_0{}^B$	0.199	0.383	0.543	0.678
	$\delta_0{}^S$	0.271	0.734	1.297	1.704
	$\delta_0{}^J$	0.216	0.449	0.676	0.876
1.5	δ_0	0.201	0.420	0.655	0.898
	$\delta_0{}^B$	0.190	0.366	0.522	0.655
	$\delta_0{}^S$	0.238	0.582	0.993	1.356
	$\delta_0{}^J$	0.189	0.365	0.519	0.649
2.0	δ_0	0.182	0.374	0.575	0.782
	$\delta_0{}^B$	0.175	0.340	0.488	0.616
	$\delta_0{}^S$	0.209	0.487	0.813	1.127
	$\delta_0{}^J$	0.163	0.297	0.405	0.493

method is generalized, and is seen to provide a practical procedure for the evaluation of the coupled-channel theory of inelastic scattering.[3] For simplicity, only spinless projectiles are considered; these give a sufficient illustration of the method.

For spinless projectiles the wave function in a coupled-channel inelastic scattering theory has the form

$$\Psi=r^{-1}\sum_{L_0JM}A_{L_0JM}\sum_{nL}R_{L_0JnL}(r)\{i^LY_L(\hat{r}),\psi_{I(n)}\}_{JM}, \tag{3.1}$$

where the curly brackets indicate vector coupling. In Eq. (3.1), $\psi_{I(n)}$ is the wave function of the nth bound state of the target nucleus, with angular momentum $I(n)$. The variable $\mathbf{r}$ is the displacement of the projectile from the c.m. of the target nucleus. The radial wave functions $R_{L_0JnL}(r)$ describe the relative motion of the projectile and the target nucleus. The Schrödinger equation for Ψ reduces to a set of coupled differential equations for these radial functions (see below).

The interactions in the Schrödinger equation couple the various terms in the nL sum in Eq. (3.1), but they, of course, do not couple terms with different JM. Hence each term in the JM sum is an independent eigenfunction of the Schrödinger equation, and is governed by an independent set of coupled equations.

At the origin, the $R_{L_0JnL}(r)$ satisfy the boundary condition

$$R_{L_0JnL}(0)=0. \tag{3.2}$$

Asymptotically they satisfy the condition that only radial functions R_{L_0J0L} associated with the target-nucleus ground state contain incoming parts, the radial functions associated with excited states are purely outgoing. The additional index L_0 in Eqs. (3.1) and (3.2) is required to define unique sets of eigenfunctions; it indicates which individual ground-state partial wave contains a nonzero incoming part.

It is clear that the nL sum in Eq. (3.1) is needed to construct eigensolutions of the Schrödinger equation, and that the L_0JM sum is needed to construct from these Schrödinger eigenfunctions that one linear combination whose incoming part correctly matches the incoming plane-wave part of Ψ. Because the present paper only concerns techniques for calculating the Schrödinger eigenfunctions, there is no need to develop detailed properties of the L_0JM sum. These indices will shortly be dropped.

The Schrödinger equation for Ψ is reduced to sets of coupled equations by utilizing the orthogonality properties of the functions $\{i^LY_L(\hat{r}),\psi_{I(n)}\}_{JM}$. The number of coupled equations for each JM is equal to the number of terms in the associated nL summation. We note that this number $N(J)$ is determined both by the number of bound states $\psi_{I(n)}$ incorporated in Ψ, and by the angular momenta $I(n)$ of each of these states. The number $N(J)$ is ordinarily much larger than the number of bound states $\psi_{I(n)}$.

Having disposed of the above preliminaries, we turn to the coupled differential equations for the $R_{L_0JnL}(r)$.

TABLE III. Yukawa-well (attractive) p-wave phase shifts in rad. Remainder of caption as for Table II.

B ↓	$A\rightarrow$	0.5	1.0	1.5	2.0
0.5	δ_1	0.021	0.043	0.067	0.094
	$\delta_1{}^B$	0.020	0.040	0.060	0.079
	$\delta_1{}^S$	0.024	0.062	0.127	0.269
	$\delta_1{}^J$	0.022	0.049	0.082	0.124
1.0	δ_1	0.054	0.112	0.174	0.243
	$\delta_1{}^B$	0.052	0.103	0.154	0.204
	$\delta_1{}^S$	0.062	0.153	0.299	0.548
	$\delta_1{}^J$	0.056	0.122	0.199	0.291
1.5	δ_1	0.070	0.143	0.222	0.305
	$\delta_1{}^B$	0.068	0.135	0.201	0.265
	$\delta_1{}^S$	0.079	0.188	0.343	0.567
	$\delta_1{}^J$	0.072	0.151	0.237	0.331
2.0	δ_1	0.076	0.154	0.237	0.322
	$\delta_1{}^B$	0.074	0.147	0.219	0.289
	$\delta_1{}^S$	0.085	0.195	0.341	0.532
	$\delta_1{}^J$	0.076	0.156	0.239	0.325

A typical member of this set of equations is

$$\left\{k_n^2+\frac{d^2}{dr^2}-\frac{L(L+1)}{r^2}-\tilde{V}_J[LI(n),LI(n)]\right\}R_{L_0JnL}(r)$$
$$=\sum_{n'L'}{}'\tilde{V}_J[LI(n),L'I(n')]R_{L_0Jn'L'}(r), \quad (3.3)$$

where

$$\tilde{V}_J[LI(n),L'I(n')]\equiv(2\mu/\hbar^2)$$
$$\times\langle\{i^LY_L(\hat{r}),\psi_{I(n)}\}_{JM}\mid V\mid\{i^{L'}Y_{L'}(\hat{r}),\psi_{I(n')}\}_{JM}\rangle \quad (3.4)$$

with V as the projectile-nucleus interaction. This interaction both scatters the projectile and couples different states of the target nucleus. The prime on the summation in Eq. (3.3) merely means that the diagonal term of the interaction has been transferred to the other side of the equation, where it will be treated as a distorting potential. We also note that a distinct wave number k_n is associated with each nuclear state $\psi_{I(n)}$ through the relation

$$k_n^2=(2\mu/\hbar^2)[E-\epsilon_n], \quad (3.5)$$

where ϵ_n is the excitation energy for the state $\psi_{I(n)}$.

For brevity, now that the coupled equations have been formulated, the indices L_0J will be omitted. These indices play no role in any process for solving the coupled equations. Hence the radial functions, for example, will be denoted $R_{nL}(r)$.

It is next convenient to introduce sets of distorted waves $F_{nL}(r)$, $G_{nL}(r)$, and $H_{nL}(r)$, that generalize the set of free radial functions $F_L(kr)$, $G_L(kr)$, and $H_L(kr)$ defined in Sec. 2. The regular distorted waves obey

$$\left\{k_n^2+\frac{d^2}{dr^2}-\frac{L(L+1)}{r^2}-\tilde{V}[LI(n),LI(n)]\right\}F_{nL}(r)=0 \quad (3.6)$$

with corresponding equations for G_{nL} and H_{nL}. The functions F_{nL}, G_{nL}, and H_{nL} satisfy the usual boundary conditions

$$F_{nL}(0)=0, \quad (3.7)$$

$$F_{nL}\longrightarrow\sin\theta_{nL}, \qquad G_{nL}\longrightarrow\cos\theta_{nL}, \quad (3.8)$$

$$H_{nL}=G_{nL}+iF_{nL}. \quad (3.9)$$

In the present case the argument θ_{nL} takes the form

$$\theta_{nL}=k_nr-(\tfrac{1}{2}L\pi)+\eta_n\ln k_nr+\delta_{nL}, \quad (3.10)$$

with the phase shift δ_{nL} produced by the diagonal potential $\tilde{V}[LI(n),LI(n)]$, and with η_n the usual Coulomb parameter

$$\eta_n=Z_aZ_A\mu e^2/\hbar^2k_n. \quad (3.11)$$

Here Z_a and Z_A are the charge numbers of the projectile and the target nucleus, respectively. Of course, the Coulomb phase shift in the radial wave nL has been incorporated in δ_{nL}.

In attempting the iterative solution of Eq. (3.3) it is as well to begin with our best available trial wave function. Hence we define the distorted-wave equivalents of the free scattered waves $J_L(r,r_0)$ of Eq. (2.17):

$$J_{nL}(r,r_0)=H_{nL}(r), \quad \text{for} \quad r>r_0$$
$$=F_{nL}(r)[H_{nL}(r_0)/F_{nL}(r_0)], \quad \text{for} \quad r<r_0. \quad (3.12)$$

The functions $J_{nL}(r,r_0)$ are subject to the same interpretations as the free waves $J_L(r,r_0)$. We use these functions to construct zero-order radial functions from which to begin the iterative calculation:

$$R_{L_0nL}^{(0)}(r)=\delta_{0n}\delta_{LL_0}F_{nL}(r)+C_{L_0nL}J_{nL}(r,r_0). \quad (3.13)$$

In defining these trial functions the index L_0 has been restored, to select a unique solution of the coupled equations. Only the trial radial function for the one ground-state channel $n=0$, $L=L_0$ contains a regular distorted wave, and this is defined, arbitrarily, to have unit amplitude. We note that this step, by implication, completes the definition of the coefficients A_{L_0JM} of Eq. (3.1).

The scattered amplitudes C_{L_0nL} in Eq. (3.13) are to be determined self-consistently, as in the discussion of elastic scattering. We compute these amplitudes from the integral formula

$$C_{L_0nL}=-k_n^{-1}\int_0^\infty F_{nL}(r)\sum_{n'L'}{}'\tilde{V}[LI(n),L'I(n')]$$
$$\times R_{L_0n'L'}(r)dr, \quad (3.14)$$

that is derived from the standard Green's-function solution of each of Eq. (3.2). Equation (3.14) expresses the amplitude for channel nL in terms of the radial functions in all other channels. Self-consistent approximate solutions of the set of Eq. (3.14) are obtained by substituting the ith iterates of the $R_{L_0nL}^{(0)}$, giving

$$C_{L_0nL}^{(i)}=-k_n^{-1}\int_0^\infty F_{nL}(r)\sum_{n'L'}{}'\tilde{V}[LI(n),L'I(n')]$$
$$\times R_{L_0n'L'}^{(i)}(r)dr. \quad (3.15)$$

Because the functions $R_{L_0n'L'}^{(i)}(r)$ contain the C_{L_0nL} as parameters, Eq. (3.15) is a set of linear equations from which the ith approximants $C_{L_0nL}^{(i)}$ may be determined.

The lowest-order set of amplitudes $C_{L_0nL}^{(0)}$ is obtained by substituting Eqs. (3.13) in Eq. (3.15), and solving the resulting linear algebraic equations. We note that this procedure involves effects that go far beyond the distorted-wave Born approximation (DBWA), but it does not involve any integrals of greater difficulty than in the usual DWBA calculations. Moreover, we recall (see Sec. 2) that this lowest-order application of the Sasawaka method gives an exact solution for the surface-δ-function model of channel coupling, in agreement with

early work by Yoshida.[5] Because this model often gives good descriptions of nuclear properties, it is plausible that many applications of the Sasakawa method in nuclear physics can be kept in zero order.

Improved accuracy is obtained by using Eq. (3.3) to develop higher iterates of Eq. (3.13). In practical calculations the iteration must be done numerically, and it is probably best performed by the numerical integration of Eq. (3.3), equation by equation, with the trial forms of the right-hand sides carried as known inhomogeneities. Integral-equation equivalents of this procedure are not as suitable for numerical calculation. However, we must recall that each time a differential equation is solved a suitable multiple of the homogeneous function $F_{nL}(r)$ must be added to the solution, to ensure that the appropriate asymptotic boundary conditions are fulfilled.

In principle, the right-hand side of Eq. (3.3) contains the amplitudes C_{L_0nL} as unknown parameters, and it is necessary to iterate separately on each term of each right-hand side, so that the C_{L_0nL} may be determined self-consistently, after iteration. However, this procedure would require the computer to carry an $N\times N$ matrix of functions, at each stage of iteration. To reduce this complication it probably is best to iterate in the following sequence of steps: (a) Solve for the C_{L_0nL} in zero order. (b) Insert these zero-order amplitudes in the right-hand sides of Eqs. (3.3), to reduce the right-hand sides to single-term explicit functions, that contain no unknown parameters. (c) Solve the inhomogeneous differential equations. (d) Use the solutions so obtained to define an improved set of scattered waves, to replace the $J_{nL}(r, r_0)$. (e) Repeat steps (a)–(d), using the improved scattered waves.

Iteration according to the explicit procedure just described undoubtedly converges more slowly than iteration according to the self-consistent Sasakawa procedure, because in the explicit procedure the C_{L_0nL} used in the wave functions of order (i) are determined only in order $(i-1)$. In return for this slight disadvantage, at any given step of iteration the explicit procedure never requires the computer to store more functions than (a) the best set of solution functions $R_{L_0nL}{}^{(i)}(r)$ and (b) the set of homogeneous functions $F_{nL}(r)$. (The latter set of functions must always be retained, so that boundary conditions may be fulfilled.)

The above-described iteration procedure, of course, is complicated. However, it must be compared with standard coupled-channel calculations.[3] The standard technique for solving N coupled differential equations requires that the differential equations be solved simultaneously N separate times, and if wave functions are needed it requires that all N^2 solution functions so generated be stored in the computer until the boundary conditions are imposed. By contrast, the iteration procedure never requires the computer to solve simultaneous differential equations, and it never requires the computer to store more than $2N$ functions.

One further practical question may be mentioned. In principle, the distorting potentials in Eq. (3.6) depend on L and J, and this complicates the determination of the distorted waves. To alleviate this complication we should retain on the left-hand side of Eq. (3.3) only the *average* diagonal terms of the projectile-nucleus interaction. The state-dependent diagonal interactions can be transferred to the right-hand side and carried in the iteration procedure. We also note that there is no need to treat all N coupled equations in every stage of iteration.

4. REARRANGEMENT

The method of the preceding sections may be extended to rearrangement reactions simply by introducing a self-consistent treatment of the amplitudes in all open channels, irrespective of arrangement. Just such a theory has been presented already by Efimenko *et al.*[6] An independent discussion of rearrangement will nevertheless be given here for completeness and because there will be several differences of physical emphasis.

The Sasakawa method is most easily applied to rearrangement theories that contain only a limited number of two-body channels. Such theories are frequently of interest. For example, for a (d, p) stripping reaction it is sometimes considered sufficient to treat just the d-A and p-B channels, where A and B are the target and residual nuclei.[10,11] To illustrate this application of the Sasakawa method, let us consider a two-channel (d, p) theory, in which the "nucleons" are assumed spinless, and in which nuclei A and B each have zero angular momentum. Hence an s-wave neutron is transferred. We also assume that nucleus A has infinite mass.

The complete model wave function for this (d, p) theory is

$$\Psi(\mathbf{r}_p, \mathbf{r}_n)=\phi(r)\xi(\mathbf{R})+\psi(r_n)\chi(\mathbf{r}_p), \qquad (4.1)$$

where $\mathbf{r}_p$ and $\mathbf{r}_n$ are the displacements of the proton and neutron from the c.m. of nucleus A, $\mathbf{r}=\mathbf{r}_p-\mathbf{r}_n$ and $\mathbf{R}=\frac{1}{2}(\mathbf{r}_p+\mathbf{r}_n)$ are the relative coordinates for the deuteron channel, and $\phi(r)$ and $\psi(r_n)$ are the deuteron bound state and neutron bound state, respectively. The relative wave functions $\xi(\mathbf{R})$ and $\chi(\mathbf{r}_p)$ must be adjusted so that Ψ fulfills scattering boundary conditions and so that Ψ is as nearly as possible an eigenfunction of the three-body Hamiltonian

$$H=T_p+T_n+U_n(r_n)+U_p(r_p)+V(r), \qquad (4.2)$$

in which T_p and T_n are the proton and neutron kinetic-energy operators. Of course, Ψ cannot be an exact eigen-

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function of Eq. (4.2), because H couples Ψ to parts of Hilbert space that are not described by functions of the form of Eq. (4.1). It is usual to avoid this problem by projecting $(E-H)\Psi$ on to the bound states ϕ and ψ, obtaining the pair of coupled equations

$$[E-\epsilon_n-T_p-U_p-\bar{V}]\chi(\mathbf{r}_p)=\int\psi V\phi\xi\, d^3r_n -[E-\epsilon_n-T_p-U_p]\int\psi\phi\xi\, d^3r_n, \quad (4.3)$$

$$[E-\epsilon_d-T_R-\bar{U}]\xi(\mathbf{R})=\int\phi[U_n+U_p]\psi\chi\, d^3R -[E-\epsilon_d-T_R]\int\phi\psi\chi\, d^3r, \quad (4.4)$$

where T_R is the kinetic-energy operator associated with $\mathbf{R}$, and where

$$\bar{V}(r_p)\equiv\int V(r)\mid\psi(r_n)\mid^2 d^3r_n, \quad (4.5)$$

$$\bar{U}(R)\equiv\int[U_n(r_n)+U_p(r_p)]\mid\phi(r)\mid^2 d^3r. \quad (4.6)$$

Here we have utilized the information that ϕ and ψ are eigenfunctions of parts of H, with ϵ_d and ϵ_n the respective eigenenergies. The relative wave functions ξ and χ may now be computed by solution of Eqs. (4.3) and (4.4). These two equations, in a sense, complete the definition of the model problem for which Eq. (4.1) is the solution.

The nonorthogonality of the deuteron and proton channels causes the coupling terms in Eqs. (4.3) and (4.4) to be considerably more complicated than the corresponding terms in the theory of inelastic scattering. Some clarification of these complications is obtained if we define the overlap functions

$$\rho(\mathbf{r}_p)\equiv\int\psi\phi\xi\, d^3r_n, \quad (4.7)$$

$$\zeta(\mathbf{R})\equiv\int\phi\psi\chi\, d^3r, \quad (4.8)$$

and if we utilize these functions to arrange Eqs. (4.3) and (4.4) in the form

$$[E-\epsilon_n-T_p-U_p-\bar{V}](\chi+\rho)=\int\psi[V-\bar{V}]\phi\xi\, d^3r_n, \quad (4.9)$$

$$[E-\epsilon_d-T_R-\bar{U}](\xi+\zeta)=\int\phi[U_n+U_p-\bar{U}]\psi\chi\, d^3r. \quad (4.10)$$

In this form of the coupled equations we see that the principal effect caused by nonorthogonality is the introduction of short-range corrections to the relative wave functions χ and ξ. We see that Eqs. (4.9) and (4.10) may be regarded as coupled equations for the linear combinations $(\chi+\rho)$ and $(\xi+\zeta)$, with comparatively simple coupling terms standing on the right-hand sides of these equations. In an iterative solution of Eqs. (4.9) and (4.10), it is straightforward to treat the corrections ρ and ζ.

To solve Eqs. (4.9) and (4.10) the relative wave functions ξ and χ must be expanded in partial waves. Because ϕ and ψ are assumed to be scalars, the angular momenta associated with ξ and χ are conserved, and the expansion of the complete wave function Ψ has the form

$$\Psi=\sum_L i^L[4\pi(2L+1)]^{1/2}\left[\frac{f_L(r_p)}{k_p r_p}\psi(r_n)Y_L^0(\hat{r}_p) +\frac{x_L(R)}{k_d R}\phi(r)Y_L^0(\hat{R})\right]. \quad (4.11)$$

Here the z axis is taken along the direction of the incident beam. Radial functions f_L and x_L of different multipole orders are not coupled, hence the coefficient $i^L[4\pi(2L+1)]^{1/2}$ plays no role in the solution of Eqs. (4.9) and (4.10). This coefficient has been chosen so that a unit-normalized deuteron or proton plane wave is associated, as usual, with a unit-normalized deuteron or proton regular radial wave function. This normalization has significance when cross sections are computed.

Insertion of Eq. (4.11) in Eqs. (4.9) and (4.10) yields coupled equations for the radial functions $f_L(r_p)$ and $x_L(R)$:

$$\left[k_p^2+\frac{d^2}{dr_p^2}-\frac{L(L+1)}{r_p^2}-\frac{2M}{\hbar^2}(U_p+\bar{V})\right](f_L+\rho_L) =W_{pd}{}^L x_L, \quad (4.12)$$

$$\left[k_d^2+\frac{d^2}{dR^2}-\frac{L(L+1)}{R^2}-\frac{4M}{\hbar^2}\bar{U}\right](x_L+\zeta_L)=W_{dp}{}^L f_L. \quad (4.13)$$

The coupling terms in these equations are

$$W_{pd}{}^L x_L=(2M/\hbar^2)(k_p/k_d)\int\int Y_L^0(\hat{r}_p)Y_L^0(\hat{R})\phi\psi[V-\bar{V}] \times[r_p/R]x_L(R)\,d^3r_n\,d\Omega_p, \quad (4.14)$$

$$W_{dp}{}^L f_L=(4M/\hbar^2)(k_d/k_p)\int\int Y_L^0(\hat{r}_p)Y_L^0(\hat{R}) \times\phi\psi[U_n+U_p-\bar{U}][R/r_p]f_L(r_p)\,d^3r\,d\Omega_R. \quad (4.15)$$

Of course, these coupling terms involve complicated nonlocal operations on the functions $x_L(R)$ and $f_L(r_p)$. However, these complications are in no way new in the coupled-channel theory. The same complications are found in the finite-range distorted-wave theory, and a variety of methods for the treatment of these complications has already been developed.[12] There is no need to discuss these methods here.

The two terms that correct Eqs. (4.12) and (4.13) for nonorthogonality are

$$\rho_L=(k_p/k_d)\int\int Y_L^0(\hat{r}_p)Y_L^0(\hat{R})\phi\psi[r_p/R]x_L(R)\,d^3r_n\,d\Omega_p, \quad (4.16)$$

$$\zeta_L=(k_d/k_p)\int\int Y_L^0(\hat{r}_p)Y_L^0(\hat{R})\phi\psi[R/r_p]f_L(r_p)\,d^3r\,d\Omega_R. \quad (4.17)$$

These short-range correction terms possess much the

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same complications as the coupling terms of Eqs. (4.14) and (4.15).

Equations (4.12) and (4.13) now are to be solved by iteration, according to the Sasakawa method. We may use the same distorted-wave trial functions as in Sec. 3. Hence the zero-order radial wave functions, assuming incident waves in the deuteron channel, are

$$[f_L+\rho_L]^{(0)}=C_{pL}J_{pL}(r_p, r_0), \tag{4.18}$$

$$[x_L+\zeta_L]^{(0)}=F_{dL}(R)+C_{dL}J_{dL}(R, R_0). \tag{4.19}$$

Here the nonorthogonality correction terms are included in the trial wave functions, as shown, on the grounds that these trial functions are intended to be good-quality zero-order solutions of Eqs. (4.12) and (4.13). The amplitudes C_{pL} and C_{dL} as usual are to be determined self-consistently. We compute these amplitudes from the internal formulas

$$C_{pL}=-k_p^{-1}\int_0^\infty F_{pL}(r_p)W_{pd}{}^L x_L\, dr_p, \tag{4.20}$$

$$C_{dL}=-k_d^{-1}\int_0^\infty F_{dL}(R)W_{dp}{}^L f_L\, dR, \tag{4.21}$$

that are derived from the Green's-function solutions of Eqs. (4.12) and (4.13).

It is interesting to apply the Sasakawa method in zero order, to see what kinds of corrections to the distorted-wave theory are implied. Hence Eqs. (4.18) and (4.19) are substituted directly into Eqs. (4.20) and (4.21), without any iteration. We obtain

$$C_{pL}{}^{(0)}=-k_p^{-1}\int_0^\infty F_{pL}W_{pd}{}^L(F_{dL}+C_{dL}{}^{(0)}J_{dL}-\zeta_L{}^{(0)})\,dr_p, \tag{4.22}$$

$$C_{dL}{}^{(0)}=-k_d^{-1}\int_0^\infty F_{dL}W_{dp}{}^L(C_{pL}{}^{(0)}J_{pL}-\rho_L{}^{(0)})\,dR. \tag{4.23}$$

Because the nonorthogonality corrections do not lend themselves to simple algebraic manipulation, we now omit them from the present preliminary discussion and obtain

$$C_{pL}{}^{(0)}\approx k_p^{-1}\int_0^\infty F_{pL}W_{pd}{}^L F_{dL}\,dr_p \Big/ \left(1-k_p^{-1}k_d^{-1}\int_0^\infty F_{dL}W_{dp}{}^L J_{pL}\,dR\cdot\int_0^\infty F_{pL}W_{pd}{}^L J_{dL}\,dr_p\right), \tag{4.24}$$

$$C_{dL}{}^{(0)}\approx -C_{pL}{}^{(0)}k_d^{-1}\int_0^\infty F_{dL}W_{dp}{}^L J_{pL}\,dR. \tag{4.25}$$

In DWBA, only the numerator of Eq. (4.24) survives; the denominator of Eq. (4.24) reduces to unity, and the amplitude $C_{dL}{}^{(0)}$ vanishes. Evidently Eqs. (4.24) and (4.25) indicate the corrections that arise, because coupling to the proton channel modifies the radial wave functions in the deuteron channel. These corrections may be regarded as "radiation reaction" effects caused by the channel coupling.

Other studies of "radiation reaction" effects in deuteron stripping[11] have been conducted by straightforward numerical integration of coupled differential equations. Large short-range effects are found. On the other hand, the coupled equations used in these studies differ from Eqs. (4.12) and (4.13) by the omission of all short-range nonorthogonality terms. It is not clear that calculations of such a nature can give satisfactory understanding of reaction effects in rearrangement collisions. The method given in the present paper allows the development of less accurate calculations that treat a wider range of physical effects. Presumably the simplest calculation of physical interest would need to be based on Eqs. (4.22) and (4.23), which carry the nonorthogonality corrections.

5. CONCLUSIONS

We have seen that the Sasakawa method gives simple approximate solutions of coupled-channel problems, both for inelastic scattering and for rearrangement. It thereby allows the treatment of multiple-excitation effects and radiation-reaction effects. It may be used for preliminary treatment of these effects in zero order, or it may be iterated if greater accuracy is required.

The basic aspect of the Sasakawa method would seem to be its use of trial wave functions that contain parameters that are handled self-consistently. Throughout the present paper the parameters treated in this fashion are the scattering amplitudes in the open channels. The self-consistent treatment of these parameters maximizes the accuracy with which the asymptotic boundary conditions are handled.

Because the Sasakawa treatment of rearrangement reactions starts with a trial wave function that contains amplitudes in every open channel, it does not impose on the iteration process the burden of generating asymptotic wave functions in exit channels in terms of incoming waves in rearranged entrance channels. Hence it may well be immune from the convergence difficulties generally associated with rearrangement theories.[7]

It is even more apparent in the work of Efimenko *et al.*[6] that use of a trial wave function with amplitudes in all the open channels should solve the convergence problems of rearrangement reactions. In Ref. 6, a term Φ that contains the asymptotic parts of the full wave-function Ψ, in all arrangements except one, is subtracted from Ψ, as in the Sasakawa method. Then the

remainder $(\Psi-\Phi)$ is expanded in channel eigenstates for the one remaining arrangement, and is computed by usual coupled-channel methods. Then self-consistency is imposed. This theory of rearrangement seems to be more general and more orderly than the one given in Sec. 4.

Other simple generalizations of the present paper are developed (a) by the introduction of antisymmetrization, using the methods developed for rearrangement[6] and (b) by the use of trial wave functions that contain bound-state terms, for the construction of "unified" reaction theories.

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Effect of ΛN Tensor Force on ${}_{\Lambda}{}^{5}\mathrm{He}$ †

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The effect of a tensor component in the ΛN force on the binding B_Λ of a Λ particle in ${}_{\Lambda}{}^{5}\mathrm{He}$ is investigated using the Hartree-Fock and rigid-α models. Second-order effects of the ΛN tensor force are taken into account in an approximate way. With a variety of ΛN noncentral forces that fit the low-energy Λp scattering data, overbinding of ${}_{\Lambda}{}^{5}\mathrm{He}$ persists. For B_Λ calculations in this hypernucleus, the rigid-α model is found to be as good as the more elaborate Hartree-Fock scheme.

1. INTRODUCTION

In the past two years, there has been a significant increase in the data on Λp scattering,[1–3] and the low-energy parameters (scattering length and effective range) for the singlet and triplet Λp force have been extracted from the scattering data by several authors.[1–5] No experimental information on the noncentral part of the lambda-nucleon (ΛN) force is as yet available. It has been consistently found[4–6] that if the experimental low-energy data are fitted by purely central (one-channel) potentials, then ${}_{\Lambda}{}^{5}\mathrm{He}$ is overbound by about 2 MeV or more, although the calculated binding energies of the other s-shell hypernuclei are compatible with their experimental values.[4] Various plausible reasons for this overbinding have been suggested,[6–11] which we will not repeat here. One of the mechanisms which might reduce the calculated B_Λ in ${}_{\Lambda}{}^{5}\mathrm{He}$ could be an appreciable tensor force in the triplet component of the ΛN potential, whose contribution to the binding of Λ would be suppressed in a spherically symmetric system like ${}^{4}\mathrm{He}$. The purpose of this paper is to consider this effect alone, in detail, on the binding B_Λ of ${}_{\Lambda}{}^{5}\mathrm{He}$. The disadvantage is, of course, that no experimental data from scattering are available about the tensor force.

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