PHYSICAL REVIEW VOLUME 188, NUMBER 3 15 DECEMBER 1969

New Mathematical Formulation for Piezoelectric Wave Propagation

EDGAR A. KRAUT
Science Center, North American Rockwell Corporation, Thousand Oaks, California 91360
(Received 11 August 1969)

Elastic waves in a general anisotropic piezoelectric medium are discussed in terms of an eight-dimensional vector formalism. The eight-dimensional state vectors have the physical significance that their first three components represent the local particle displacement, the second three components represent the stresses on an arbitrarily selected plane, and the seventh and eight components represent the electric displacement normal to this plane and to the scalar electric field potential, respectively. As an application of the formalism, the dispersion relations for bulk waves and surface waves in $Bi_{12}GeO_{20}$ are derived, and the dispersion relations are given for a system consisting of a vacuum followed by an arbitrary piezoelectric film on an arbitrary semi-infinite piezoelectric substrate. Expressions are given for the propagator in arbitrary heteroepitaxial structures, and the boundary conditions associated with electrical or mechanical excitation of such structures is discussed.

STATEMENT OF PROBLEM

THE standard description of piezoelectric wave propagation[1] leads to a set of four coupled second-order partial-differential equations with constant coefficients in four unknowns, u_1, u_2, u_3, and ϕ. These represent the three components of particle displacement and the scalar electric field potential, respectively. The 4×4 system

$$c_{pqrs}u_{r,sp}+e_{rpq}\phi_{,rp}=\rho\ddot{u}_q\,, \tag{1}$$

$$e_{prs}u_{r,sp}-\epsilon_{pr}\phi_{,rp}=0\,,\quad p,q,r,s=1,2,3 \tag{2}$$

(summation convention understood)

in which c_{pqrs}, e_{rpq}, and ϵ_{pr} are the elastic, piezoelectric, and dielectric tensors, respectively, and ρ is the mass density, is easy to solve in principle. However, the application of boundary conditions at dielectric interfaces or at free surfaces is algebraically complicated. The reason is that the boundary conditions involve the stress τ_{pq} and electrical displacement D_p which must be calculated in terms of $\mathbf{u}$ and ϕ from the equations of state

$$\tau_{pq}=c_{pqrs}u_{r,s}+e_{rpq}\phi_{,r}\,, \tag{3}$$

$$D_p=e_{prs}u_{r,s}-\epsilon_{pr}\phi_{,r}\,. \tag{4}$$

EIGHT-DIMENSIONAL MATRIX FORMULATION

In studying piezoelectric wave propagation in heteroepitaxial structures, there are eight quantities of physical interest; these are the three components of particle displacement which must be continuous across interfaces, three components of stress which must vanish at a free surface or must be continuous across interfaces, and the normal component of electrical displacement, which together with the scalar electric field potential, must pass continuously across dielectric interfaces.

[1] H. F. Tiersten, *Linear Piezoelectric Plate Vibrations* (Plenum Publications Corp., New York, 1969), p. 36.

In view of these considerations it is convenient to recast the 4×4 second-order system (1) and (2) as an 8×8 system of first-order partial-differential equations in which the eight dependent variables are the three particle displacement components, the three stress components, the normal component of electric displacement, and the scalar electric field potential. Let x_1, x_2, x_3 be rectangular Cartesian coordinates chosen so that the x_2 axis is normal to the layers of the heteroepitaxial structure. Define an eight-component column vector w by

$$w=\begin{bmatrix}u_2\\u_1\\u_3\\T_2\\T_6\\T_4\\D_2\\\phi\end{bmatrix}, \tag{5}$$

where u_2, u_1, and u_3 are the displacement components along x_2, x_1, and x_3, respectively; $T_2=\tau_{22}$, $T_6=\tau_{12}$, $T_4=\tau_{23}$ are the components of stress on a surface normal to the x_2 axis; ϕ is the scalar electric potential; and D_2 is the component of electric displacement along x_2.

The system (1) and (2) can be rewritten as a single 8×8-matrix partial-differential equation which is of first order in the variable x_2, i.e.,

$$\frac{\partial w}{\partial x_2}=Q\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3},\frac{\partial}{\partial t}\right)w\,. \tag{6}$$

This elegant canonical form greatly simplifies the analysis of many problems of current interest. The remainder of this paper will list the 64 matrix elements of the (8×8) matrix Q and describe how (6) is derived and applied.

The matrix Q may be represented as

$$Q=RPR^{-1}, \tag{7}$$

where the 8×8 matrix P is partitioned into submatrices so that

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} & P_{38} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} & P_{47} & P_{48} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} & P_{57} & P_{58} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} & P_{67} & P_{68} \\ P_{71} & P_{72} & P_{73} & P_{74} & P_{75} & P_{76} & P_{77} & P_{78} \\ P_{81} & P_{82} & P_{83} & P_{84} & P_{85} & P_{86} & P_{87} & P_{88} \end{bmatrix} = \begin{bmatrix} -\lambda & \mu & -\nu G \\ -\beta & -\alpha & -\gamma G \\ \cdots & \cdots & \cdots \end{bmatrix}. \tag{8}$$

The matrices α, β, γ, λ, μ, ν, and G are defined in the Appendix. The last two rows of (8) are defined as follows: The row vectors P_{71}, P_{72}, P_{73} and P_{74}, P_{75}, P_{76} are each a sum of row vectors divided by a scalar and are given by

$$P_{71}, P_{72}, P_{73} = (\mathbf{M}_1\partial_1+\mathbf{M}_3\partial_3-\mathbf{M}_2\lambda-\mathbf{L}_2\beta)(\mathbf{N}_2\cdot\mathbf{G}_1)^{-1}, \tag{9}$$

$$P_{74}, P_{75}, P_{76} = (\mathbf{L}_1\partial_1+\mathbf{L}_3\partial_3+\mathbf{M}_2\mu-\mathbf{L}_2\alpha)(\mathbf{N}_2\cdot\mathbf{G}_1)^{-1}. \tag{10}$$

The vectors and matrices on the right-hand sides of (9)–(12) are defined in the Appendix and $\partial_1=\partial/\partial x_1$, $\partial_3=\partial/\partial x_3$. The matrix elements P_{77} and P_{78} are given by

$$P_{77}=-(\mathbf{N}_2\cdot\mathbf{G}_1)^{-1}[(\mathbf{N}_1\cdot\mathbf{G}_1)\partial_1 +(\mathbf{N}_3\cdot\mathbf{G}_1)\partial_3+(\mathbf{M}_2\cdot\mathbf{V}_2) +(\mathbf{L}_2\cdot\mathbf{V}_1)+(\mathbf{N}_2\cdot\mathbf{G}_2)], \tag{11}$$

$$P_{78}=-(\mathbf{N}_2\cdot\mathbf{G}_1)^{-1}[(\mathbf{N}_1\cdot\mathbf{G}_2)\partial_1 +(\mathbf{N}_3\cdot\mathbf{G}_2)\partial_3+(\mathbf{M}_2\cdot\mathbf{W}_2)+(\mathbf{L}_2\cdot\mathbf{W}_1)]. \tag{12}$$

The last row of P is given by

$$P_{8j}=\delta_{7j}, \quad j=1, 2, \cdots, 8. \tag{13}$$

Finally, the matrix R is defined except for the seventh row by

$$R_{ij}=\delta_{ij}\ (i\neq 7), \quad i=1, \cdots, 8;\ j=1, \cdots, 8. \tag{14}$$

The seventh row of R is given by

$$\begin{gathered} R_{71}=0, \quad R_{72}=M_{22}, \quad R_{73}=M_{23}, \\ R_{74}=L_{21}, \quad R_{75}=L_{22}, \quad R_{76}=L_{23}, \\ R_{77}=-N_{22}, \quad R_{78}=-N_{21}\partial_1-N_{23}\partial_3, \end{gathered} \tag{15}$$

where again N_{ij} and L_{ij} are matrix elements defined in the Appendix.

The inverse R^{-1} is defined, except for $i=7$, $j=7$, by $(R^{-1})_{ij}=-R_{ij}(R_{77})^{-1}$,

$$i=1, \cdots, 8; \quad j=1, \cdots, 8; \quad i\neq 7, \quad j\neq 7. \tag{16}$$

The element $(R^{-1})_{77}$ is given by

$$(R^{-1})_{77}=R_{77}^{-1}. \tag{17}$$

We have thus completely described the matrix Q in (6) in terms of the matrices α, β, γ, λ, μ, ν, G, L, M, and N. These are related to the elastic, piezoelectric, and dielectric constants of the material as shown in the Appendix.

DERIVATION AND APPLICATIONS

In order to illustrate how the general expression for Q was derived, we consider a simple example of physical interest. The general case, though tedious, is handled in exactly the same manner.

The simplest piezoelectric crystal we can consider is that of bismuth germanium oxide ($Bi_{12}GeO_{20}$) which belongs to the cubic class 23. Its only nonzero elastic, piezoelectric, and dielectric constants are[2] c_{11}, c_{12}, c_{44}, e_{14}, and ϵ_{11}, respectively.

Let the x_2 axis be normal to a cube face and let us assume that all the field variables are independent of the x_3 coordinate (x_2 cut, x_1 propagation). Equation (3) becomes

$$\begin{aligned} T_2&=c_{11}\partial_2u_2+c_{12}\partial_1u_1, & T_1&=c_{12}\partial_2u_2+c_{11}\partial_1u_1, \\ T_6&=c_{44}(\partial_2u_1+\partial_1u_2), & T_3&=c_{12}\partial_2u_2+c_{12}\partial_1u_1, \\ T_4&=c_{44}\partial_2u_3+e_{14}\partial_1\phi, & T_5&=c_{44}\partial_1u_3+e_{14}\partial_2\phi, \end{aligned} \tag{18}$$

while conservation of momentum requires that

$$\begin{aligned} \partial_2T_2&=\rho\ddot{u}_2-\partial_1T_6, \\ \partial_2T_6&=\rho\ddot{u}_1-\partial_1T_1, \\ \partial_2T_4&=\rho\ddot{u}_3-\partial_1T_5. \end{aligned} \tag{19}$$

Equation (4) becomes

$$\begin{aligned} D_1&=e_{14}\partial_2u_3-\epsilon_{11}\partial_1\phi, \\ D_2&=e_{14}\partial_1u_3-\epsilon_{11}\partial_2\phi, \\ D_3&=e_{14}(\partial_1u_2+\partial_2u_1), \end{aligned} \tag{20}$$

and charge conservation requires that

$$\partial_2D_2=-\partial_1D_1=-e_{14}\partial_1\partial_2u_3+\epsilon_{11}\partial_1^2\phi. \tag{21}$$

[2] M. Onoe, A. W. Warner, and A. A. Ballmann, IEEE Trans. Sonics Ultrasonics **SU-14**, 165 (1967).

From (18), we obtain

$$\partial_2 u_2 = c_{11}^{-1} T_2 - (c_{12}/c_{11})\partial_1 u_1\,,$$
$$\partial_2 u_1 = c_{44}^{-1} T_6 - \partial_1 u_2\,, \qquad (22)$$
$$\partial_2 u_2 = c_{44}^{-1} T_4 - (e_{14}/c_{44})\partial_1 \phi\,,$$

and, from the second equation in (20),

$$\partial_2 \phi = (e_{14}/\epsilon_{11})\partial_1 u_3 - \epsilon_{11}^{-1} D_2\,. \qquad (23)$$

Equations (19 and (21)–(23) can be placed in the form of (6) provided one first expresses $\partial_1 T_1$ and $\partial_1 T_5$ in (19) in terms of the variables of w given in (5). This is done by using T_1 as given in (18) and eliminating $\partial_2 u_2$ with the aid of (22). Similarly, one uses T_5 as given in (18) and then eliminates $\partial_2\phi$ with the aid of (23). The result is that

$$\partial w/\partial x_2 = Q(\partial/\partial x_1, \partial/\partial t)w\,, \qquad (24)$$

where the only nonzero matrix elements of the 8×8 matrix Q are the following:

$$\begin{aligned}
&Q_{12} = -(c_{12}/c_{11})\partial_1\,, && Q_{52} = \partial^2/\partial t^2 - (c_{11}^2 - c_{12}^2)c_{11}^{-1}\partial_1^2\,,\\
&Q_{14} = c_{11}^{-1}\,, && Q_{54} = -(c_{12}/c_{11})\partial_1\,,\\
&Q_{21} = -\partial_1\,, && Q_{63} = \partial^2/\partial t^2 - (c_{44} + \epsilon_{11}^{-1} e_{14}^2)\partial_1^2\,,\\
&Q_{25} = c_{44}^{-1}\,, && Q_{67} = (e_{14}/\epsilon_{11})\partial_1\,,\\
&Q_{36} = c_{44}^{-1}\,, && Q_{76} = -(e_{14}/c_{44})\partial_1\,,\\
&Q_{38} = -(e_{14}/c_{44})\partial_1\,, && Q_{78} = (\epsilon_{11} + c_{44}^{-1} e_{14}^2)\partial_1^2\,,\\
&Q_{41} = \rho\partial^2/\partial t^2\,, && Q_{83} = (e_{14}/\epsilon_{11})\partial_1\,,\\
&Q_{45} = -\partial_1\,, && Q_{87} = -\epsilon_{11}^{-1}\,.
\end{aligned} \qquad (25)$$

The derivation for the case of complete anisotropy (21 elastic constants, 18 piezoelectric constants, and six dielectric constants) proceeds in exactly the same fashion.

DISPERSION RELATION FOR x_2 CUT x_1 PROPAGATING BULK MODES IN $Bi_{12}GeO_{20}$

A solution of (24) and (25) of the form

$$w = w_0 \exp[s(t + px_1 - \eta x_2)] \qquad (26)$$

yields the dispersion relations

$$\eta^4 - K_1(p)\eta^2 + K_2(p) = 0 \qquad (27)$$

and

$$\eta^4 - K_3(p)\eta^2 + K_4(p) = 0\,, \qquad (28)$$

where

$$\begin{aligned}
K_1(p) &= (c_{11}c_{44})^{-1}[(c_{12}^2 - c_{11}^2 + 2c_{12}c_{44})p^2 + \rho(c_{11} + c_{44})]\,,\\
K_2(p) &= p^4 - \rho(c_{11}c_{44})^{-1}(c_{11} + c_{44})p^2 + \rho^2(c_{11}c_{44})^{-1}\,,\\
K_3(p) &= c_{44}^{-1}\rho - 2[1 + 2(\epsilon_{11}c_{44})^{-1}e_{14}^2]p^2\,,\\
K_4(p) &= p^2(p^2 - \rho c_{44}^{-1})\,.
\end{aligned} \qquad (29)$$

Equation (27) contains no piezoelectric coupling terms and corresponds to the pure acoustic bulk modes of an ordinary cubic crystal. The piezoacoustic bulk modes come from (28). When $p=0$ in (28), $\eta^2=0$ gives one pair of nonpropagating solutions, the other pair of solutions of (28) for $p=0$ is $\eta = \pm(\rho/c_{44})^{1/2}$. For values p such that $0 \leqslant p^2 \leq \rho/c_{44}$, (28) has two real roots and two complex-conjugate pure imaginary roots. The two real roots correspond to piezoelectrically stiffened bulk acoustic modes propagating in the directions $(p, \pm\eta)$ at slightly greater than the bulk shear speed $(c_{44}/\rho)^{1/2}$. Finally, for $p^2 > \rho/c_{44}$, all four roots of (28) are pure imaginary.

SURFACE WAVES

In order to examine Rayleigh-type surface waves propagating in the x_1 direction on the surface $x_2=0$ of a half-space $(x_2 \geqslant 0)$ of $Bi_{12}GeO_{20}$, we select four of the eight eigenvectors of (24) corresponding to nonincreasing amplitudes as $x_2 \to +\infty$. Let

$$w = \sum_{j=1}^{4} A_j w_j^{(0)}(p) \exp\{s[t + px_1 - \eta_j(p)x_2]\} \qquad (30)$$

be a linear combination of eigencolumns of Q with arbitrary coefficients A_j and with the four branches $\eta_j(p)$ of (27) and (28) chosen so that

$$\mathrm{Re}[\eta_j(p)] \geqslant 0\,. \qquad (31)$$

Inside the crystal, the electric field potential is

$$\phi = \sum_{j=1}^{4} A_j \phi_j^{(0)}(p) \exp\{s[t + px_1 - \eta_j(p)x_2]\}\,. \qquad (32)$$

Outside the crystal, ϕ must be a solution of Laplace's equation which matches (32) at the boundary $x_2=0$. The potential

$$\phi = \sum_{j=1}^{4} A_j \phi_j^{(0)}(p) \exp[s(t + px_1)] \times(\cos spx_2 + n \sin spx_2) \qquad (33)$$

has the required form. The constant parameter n in (33) is the ratio of the vertical to the horizontal electric field components at the crystal surface, i.e.,

$$n = (E_2/E_1)_{x_2=0^-}\,. \qquad (34)$$

Continuity of D_2 across the boundary $x_2=0$ requires

$$D_2 = \sum_{j=1}^{4} A_j D_2^{(0)}(p) \exp[s(t + px_1)] = -\epsilon_0\left(\frac{\partial\phi}{\partial x_2}\right)_{x_2=0}, \qquad (35)$$

with ϕ given by (33). Consequently,

$$\sum_{j=1}^{4} A_j[D_{2j}^{(0)}(p) + n\epsilon_0 sp\phi_j^{(0)}(p)] = 0\,, \qquad (36)$$

and to this we must adjoin the conditions that stress components T_2, T_6, and T_4 vanish at the free surface

$x_2=0$:

$$\sum_{j=1}^{4} A_j T_{2j}^{(0)}(p)=0, \quad \sum_{j=1}^{4} A_j T_{6j}^{(0)}(p)=0, \quad \sum_{j=1}^{4} A_j T_{4j}^{(0)}(p)=0. \tag{37}$$

The dispersion relation for Rayleigh-type surface waves arises from the requirement that the 4×4 system (36) and (37) have a nontrivial solution. It is given by

$$\begin{vmatrix} T_{21}^{(0)}(p) & T_{22}^{(0)}(p) & T_{23}^{(0)}(p) & T_{24}^{(0)}(p) \\ T_{61}^{(0)}(p) & T_{62}^{(0)}(p) & T_{63}^{(0)}(p) & T_{64}^{(0)}(p) \\ T_{41}^{(0)}(p) & T_{42}^{(0)}(p) & T_{43}^{(0)}(p) & T_{44}^{(0)}(p) \\ G_{21}^{(0)}(p) & G_{22}^{(0)}(p) & G_{23}^{(0)}(p) & G_{24}^{(0)}(p) \end{vmatrix} = 0, \tag{38}$$

where

$$G_{2j}^{(0)}(p)=D_{2j}^{(0)}(p)+n\epsilon_0 sp\phi_j^{(0)}(p), \quad j=1,\cdots,4. \tag{39}$$

If the surface of the crystal is grounded, then (36) is replaced by

$$\sum_{j=1}^{4} A_j\phi_j^{(0)}(p)=0, \tag{40}$$

and (38) is altered accordingly.

The surface-wave problem has an interesting structure in the present formulation. Given the 8×8 matrix $Q(p,s)$ in (24), one forms the corresponding eigenvector matrix $E(p,s)$. From $E(p,s)$ one selects a 4×4 submatrix whose determinant vanishes at the surface-wave poles $p=p_r$. Real values of p_r correspond to pure surface waves; complex p_r produce "leaky" surface waves.

DISPERSION RELATION IN HETEROEPITAXIAL STRUCTURES

Consider now a general piezoelectric film $0\leqslant x_2\leqslant H$, on a piezoelectric substrate $H\leqslant x_2<\infty$, and let the region $-\infty<x_2\leqslant 0$ be occupied by vacuum or air. In the film we have

$$\partial w_F/\partial x_2=Q_F w_F, \tag{41}$$

and in the substrate,

$$\partial w_S/\partial x_2=Q_S w_S. \tag{42}$$

The eight eigenvalues of (41) are complex functions of p and a linear combination of the eight corresponding eigenvectors yields a solution in the film of the form

$$\mathbf{w}_F=\sum_{j=1}^{4} A_j\mathbf{w}_{Fj}^{-}(p)\exp\{s[t+px_1-\eta_{Fj}(p)x_2]\} +\sum_{j=1}^{4} B_j\mathbf{w}_{Fj}^{+}(p)\exp\{s[t+px_1+\hat{\eta}_{Fj}(p)x_2]\}, \tag{43}$$

where $\hat{\eta}_{Fj}(p)$ is defined by

$$\hat{\eta}_{Fj}(p)=-\eta_{F(j+4)}(p), \quad j=1,2,3,4. \tag{44}$$

In the substrate we have

$$\mathbf{w}_S=\sum_{j=1}^{4} c_j\mathbf{w}_{Sj}^{-}(p)\exp\{s[t+px_1-\eta_{Sj}(p)x_2]\}, \tag{45}$$

where Re $[\eta_{Sj}(p)]\geqslant 0$, $j=1, 2, \cdots, 4$, to prevent growth of $\mathbf{w}_S$ as $x_2\to\infty$.

The boundary condition at the interface $x_2=H$, between the film and the substrate, is simply $\mathbf{w}_F=\mathbf{w}_S$, and when applied to (43) and (45) yields a set of eight homogeneous equations in the 12 unknowns $A_1,\cdots,A_4$, $B_1,\cdots,B_4$, and $c_1,\cdots,c_4$.

The remaining four homogeneous equations are provided by requiring continuity of ϕ and D_2 across the vacuum-film interface $x_2=0$, and vanishing of the stress on $x_2=0$. These equations are

$$\sum_{j=1}^{4} A_jG_j^{-}+\sum_{j=1}^{4} B_jG_j^{+}=0, \quad \sum_{j=1}^{4} A_j\mathbf{T}_{Fj}^{-}+\sum_{j=1}^{4} B_j\mathbf{T}_{Fj}^{+}=\mathbf{0}, \tag{46}$$

where

$$G_j^{\pm}=D_{2Fj}^{\pm}(p)+n\epsilon_0 sp\phi_{Fj}^{\pm}(p), \quad j=1,2,3,4. \tag{47}$$

The set of 12 homogeneous equations in the 12 unknowns A_i, B_i, and c_i consists of (46) and the eight equations

$$\sum_{j=1}^{4} A_j\mathbf{w}_{Fj}^{-}\exp(-s\eta_{Fj}H)+\sum_{j=1}^{4} B_j\mathbf{w}_{Fj}^{+}\exp(s\hat{\eta}_{Fj}H) -\sum_{j=1}^{4} C_j\mathbf{w}_{Sj}^{-}\exp(-s\eta_{Sj}H)=\mathbf{0}, \tag{48}$$

generated by the condition $\mathbf{w}_F=\mathbf{w}_S$ at $x_2=H$.

The required dispersion relation is obtained from the condition that the 12th-order determinant of the system (46)–(48) be zero.

PROPAGATORS FOR GENERAL PIEZOELECTRIC HETEROEPITAXIAL STRUCTURES

Consider a structure having many layers of piezoelectric material, generally of different symmetry and layered in the x_2 directions. Assuming solutions of the form

$$\mathbf{w}=\mathbf{w}(x_2)\exp\{s(t+px_1)\}, \tag{49}$$

one obtains

$$\partial\mathbf{w}/\partial x_2=Q(x_2,p,s)\mathbf{w} \tag{50}$$

for the determination of the amplitude $\mathbf{w}(x_2)$. The matrix $P(x_2,x_2')$ is called a propagator for the structure if it has the property that when $\mathbf{w}(x_2')$ is a solution of (50) at point x_2', then $\mathbf{w}(x_2)$, given by

$$\mathbf{w}(x_2)=P(x_2,x_2')\mathbf{w}(x_2'), \tag{51}$$

is a solution of (50) at x_2.

In order to construct the propagator P we need only find eight linearly independent column vector solutions w of (50). If these form a nonsingular 8×8 matrix $E(x_2)$, then the propagator $P(x_2,x_2')$ is given by

$$P(x_2,x_2')=E(x_2)E^{-1}(x_2'), \tag{52}$$

as may be verified by substitution of (51) and (52) into (50), remembering that each column of $E(x_2)$ satisfies (50).

The point to note is that the propagator from any point x_2' can be calculated immediately by matrix inversion once eight linearly independent column vector solutions of (50) are known.

REMARKS ON INPUT-OUTPUT PROBLEM

Energy is introduced into the layered structure by either electrically or mechanically exciting the vacuum-film interface. Electrical excitation can be accomplished by placing a pair of parallel line source electrodes on a free surface a distance λ apart with their axes along the x_3 direction. When these electrodes are connected to a generator which puts out a potential $\phi(t)$, the horizontal component of electric field between the electrodes is approximated by

$$E_1(x_1,0^-,t)=-\phi(t)\lambda^{-1}[H(x_1+\lambda/2)-H(x_1-\lambda/2)], \tag{53}$$

and from (34), the vertical component of electric displacement on the vacuum side of the free surface is

$$D_2=n\epsilon_0E_1(x_1,0^-,t). \tag{54}$$

Equation (54) fixes D_2 as the vacuum-film interface is approached from the interior of the film and (54) leads, after Fourier transformation, to a corresponding modification of (47).

Mechanical excitation of the heteroepitaxial structure can be accomplished by placing an electrically driven mechanical transducer, such as a $BaTiO_3$ or quartz crystal on the vacuum-film interface. If the transducer generates a stress component along the x_2 axis acting normal to the free surface at the origin, then the appropriate boundary condition is

$$T_2=\tau_{22}=-F(t)\delta(x_1)\delta(x_3), \tag{55}$$

where $F(t)$ is the time dependence of the applied stress. Equation (55) introduces a modification of the boundary conditions in the second set of (46), to account for the applied stress at the vacuum-film interface $x_2=0$. For either electrical or mechanical input, the boundary equations (46)–(48) form an inhomogeneous set of algebraic relations for the field amplitudes. These amplitudes become infinite at the normal modes of the system defined by vanishing of the 12×12 determinant of (46)–(48).

RELATION TO OTHER INVESTIGATIONS

The description of the problem (1) and (2) in the matrix form (6) is based on some early investigations of Volterra[3] on systems of linear differential equations. Volterra's ideas have been applied by others to elastic wave and vibration problems in isotropic and anisotropic *nonpiezoelectric*, layered media.[4–6] The present contribution presents the extension of this matrix description to include arbitrary piezoelectric coupling as well as elastic and optical anisotropy effects. Wave propagation in piezoelectric crystals has been considered by a number of people and many of their contributions are discussed in the recent text by Mason.[7] The excitation and propagation of surface waves on piezoelectric media is the topic of several recent papers[8–17] and appears to be a subject of increasing interest in view of potential device applications.[10]

APPENDIX

The matrices α, β, γ, λ, μ, ν, and G appearing in (8) are defined as follows:

$$\alpha=A_3+A_{13}c_{153}{}^{246}B_{264}{}^{246}, \tag{A1}$$

$$\beta=A-A_{13}c_{153}{}^{246}B_{264}{}^{246}H_{264}+A_{13}H_{153}, \tag{A2}$$

$$\gamma=A_{13}(e_{153}{}^{T123}-c_{153}{}^{246}B_{264}{}^{246}e_{264}{}^{T123}), \tag{A3}$$

$$\lambda=JB_{264}{}^{246}H_{264}, \tag{A4}$$

$$\mu=JB_{264}{}^{246}, \tag{A5}$$

$$\nu=JB_{264}{}^{246}e_{264}{}^{T123}, \tag{A6}$$

$$G=\begin{bmatrix}0 & \partial_1\\ 1 & 0\\ 0 & \partial_3\end{bmatrix}, \tag{A7}$$

where

$$A_3=\begin{bmatrix}0 & \partial_1 & \partial_3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{A8}$$

$$A_{13}=\begin{bmatrix}0 & 0 & 0\\ 0 & \partial_1 & \partial_3\\ \partial_1 & \partial_3 & 0\end{bmatrix}, \tag{A9}$$

[3] V. Volterra, Rend. Accad. Lincei **3**, 393 (1887); Mem. Soc. Ital. Sci. **6**, 1 (1887).
[4] F. Gilbert and G. Backus, Geophysics **31**, 326 (1966).
[5] F. Abramovici, Bull. Seism. Soc. Am. **58**, 427 (1968).
[6] K. A. Ingebrigtsen and A. Tonning, Phys. Rev. Letters **23**, A12 (1969).
[7] W. P. Mason, *Crystal Physics of Interaction Processes* (Academic Press Inc., New York, 1966).
[8] J. J. Campbell and W. R. Jones, IEEE Trans. Sonics Ultrasonics **SU-15**, 209 (1968).
[9] S. G. Joshi and R. M. White, J. Appl. Phys. **39**, 5819 (1968).
[10] R. M. White, IEEE Trans. Electron. Devices **ED-14**, 181 (1967).
[11] H. F. Tiersten, J. Acoust. Soc. Am. **35**, 234 (1963).
[12] G. A. Coquin and H. F. Tiersten, J. Acoust. Soc. Am. **41**, 921 (1966).
[13] H. F. Tiersten, J. Appl. Phys. **40**, 770 (1969).
[14] C. C. Tseng and R. M. White, J. Appl. Phys. **38**, 4274 (1967).
[15] C. C. Tseng, J. Appl. Phys. **38**, 4281 (1967).
[16] J. L. Bleustein, J. Acoust. Soc. Am. **45**, 614 (1969).
[17] K. A. Ingebrigtsen and A. Tonning, Electronics Research Laboratory Report No. TE74, Norwegian Institute of Technology, Trondheim, Norway (unpublished).

$$c_{153}{}^{246}=\begin{bmatrix} c_{12} & c_{14} & c_{16} \\ c_{52} & c_{54} & c_{56} \\ c_{32} & c_{34} & c_{36} \end{bmatrix}, \tag{A10}$$

$$c_{264}{}^{246}=\begin{bmatrix} c_{22} & c_{24} & c_{26} \\ c_{62} & c_{64} & c_{66} \\ c_{42} & c_{44} & c_{46} \end{bmatrix}, \tag{A11}$$

$$B_{264}{}^{246}=(c_{264}{}^{246})^{-1}, \tag{A12}$$

$$A=-\rho J(\partial^2/\partial t^2), \tag{A13}$$

$$H_{264}=c_{264}{}^{246}H_3+c_{264}{}^{135}H_{13}, \tag{A14}$$

$$H_3=\begin{bmatrix} 0 & 0 & 0 \\ \partial_3 & 0 & 0 \\ \partial_1 & 0 & 0 \end{bmatrix}, \tag{A15}$$

$$H_{13}=\begin{bmatrix} 0 & \partial_1 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_1 \end{bmatrix}, \tag{A16}$$

$$H_{153}=(c_{153}{}^{246}H_3+c_{153}{}^{135}H_{13}), \tag{A17}$$

$$c_{264}{}^{135}=\begin{bmatrix} c_{21} & c_{23} & c_{25} \\ c_{61} & c_{63} & c_{65} \\ c_{41} & c_{43} & c_{45} \end{bmatrix}, \tag{A18}$$

$$c_{153}{}^{135}=\begin{bmatrix} c_{11} & c_{13} & c_{15} \\ c_{51} & c_{53} & c_{55} \\ c_{31} & c_{33} & c_{35} \end{bmatrix}, \tag{A19}$$

$$e_{153}{}^{T123}=\begin{bmatrix} e_{11} & e_{21} & e_{31} \\ e_{15} & e_{25} & e_{35} \\ e_{13} & e_{23} & e_{33} \end{bmatrix}, \tag{A20}$$

$$e_{264}{}^{T123}=\begin{bmatrix} e_{12} & e_{22} & e_{32} \\ e_{16} & e_{26} & e_{36} \\ e_{14} & e_{24} & e_{34} \end{bmatrix}, \tag{A21}$$

$$J=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{A22}$$

The matrix elements c_{ij} and e_{ij} represent the 21 elastic constants and 18 piezoelectric constants of an arbitrary linear piezoelectric solid. The quantities appearing in (9)–(17) are defined via the matrices L, M, N, where

$$L=e_{123}{}^{246}B_{264}{}^{246}, \tag{A23}$$

$$M=(e_{123}{}^{135}-e_{123}{}^{246}B_{264}{}^{246}c_{264}{}^{135})H_{13}, \tag{A24}$$

$$N=(\epsilon_{123}{}^{123}+e_{123}{}^{246}B_{264}{}^{246}e_{264}{}^{T123}). \tag{A25}$$

In (A25), $\epsilon_{123}{}^{123}$ is the dielectric constant matrix given by

$$\epsilon_{123}{}^{123}=\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}, \tag{A26}$$

and

$$e_{123}{}^{246}=\begin{bmatrix} e_{12} & e_{14} & e_{16} \\ e_{22} & e_{24} & e_{26} \\ e_{32} & e_{34} & e_{36} \end{bmatrix}, \tag{A27}$$

$$e_{123}{}^{135}=\begin{bmatrix} e_{11} & e_{13} & e_{15} \\ e_{21} & e_{23} & e_{25} \\ e_{31} & e_{33} & e_{35} \end{bmatrix}. \tag{A28}$$

The vectors $\mathbf{L}_j$, $\mathbf{M}_j$, $\mathbf{N}_j$, $\boldsymbol{\gamma}_j$, $\mathbf{v}_j$, and $\mathbf{G}_j$ are defined by

$$\begin{aligned} \mathbf{L}_j &= \text{row } (j) \text{ of matrix } L, \\ \mathbf{M}_j &= \text{row } (j) \text{ of matrix } M, \\ \mathbf{N}_j &= \text{row } (j) \text{ of matrix } N, \\ \boldsymbol{\gamma}_j &= \text{row } (j) \text{ of matrix } \gamma, \\ \mathbf{v}_j &= \text{row } (j) \text{ of matrix } \nu, \\ \mathbf{G}_j &= \text{column } (j) \text{ of matrix } G. \end{aligned} \tag{A29}$$

The column vectors $\mathbf{V}_j$ and $\mathbf{W}_j$ in (11) and (12) are defined by

$$\begin{aligned} \mathbf{V}_1 &= \begin{bmatrix} \boldsymbol{\gamma}_1\cdot\mathbf{G}_1 \\ \boldsymbol{\gamma}_2\cdot\mathbf{G}_1 \\ \boldsymbol{\gamma}_3\cdot\mathbf{G}_1 \end{bmatrix}, \quad \mathbf{V}_2 = \begin{bmatrix} \mathbf{v}_1\cdot\mathbf{G}_1 \\ \mathbf{v}_2\cdot\mathbf{G}_1 \\ \mathbf{v}_3\cdot\mathbf{G}_1 \end{bmatrix}, \\ \mathbf{W}_1 &= \begin{bmatrix} \boldsymbol{\gamma}_1\cdot\mathbf{G}_2 \\ \boldsymbol{\gamma}_2\cdot\mathbf{G}_2 \\ \boldsymbol{\gamma}_3\cdot\mathbf{G}_2 \end{bmatrix}, \quad \mathbf{W}_2 = \begin{bmatrix} \mathbf{v}_1\cdot\mathbf{G}_2 \\ \mathbf{v}_2\cdot\mathbf{G}_2 \\ \mathbf{v}_3\cdot\mathbf{G}_2 \end{bmatrix}. \end{aligned} \tag{A30}$$

The parameter ρ is the scalar mass density and all the quantities necessary for the computation of Q in (7) have now been defined.