Theory of a Two-Dimensional Ising Model with Random Impurities. III. Boundary Effects

BARRY M. MCCOY
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790
(Received 2 June 1969)

We study the effects which immobile random impurities may have on magnetic properties and spin-spin correlation functions of a ferromagnetic system near the Curie temperature. This is done within the context of the model introduced in the first paper of this series by computing the spin correlation functions on the boundary of a half-plane of random Ising spins, where the boundary row is allowed to intereact with a magnetic field $\mathfrak{H}$. We find that as $T \to T_c-$ the average boundary spontaneous magnetization vanishes as $T_c - T$. However, the average boundary magnetic susceptibility is shown not to exist for a finite range of temperature about T_c. Furthermore, at T_c the average boundary magnetization behaves as $-N^{-1}\,\mathrm{sign}(\mathfrak{H})[\ln N\beta_c|\mathfrak{H}|]^{-1}$, where N^{-1} is a measure of the width of the distribution of random bonds. Whenever $T-T_c=O(N^{-2})$, the average spin-spin correlation function for two spins on the boundary is shown to approach its limit at infinite separation as some inverse power of the separation instead of as an exponential. At T_c this average correlation function behaves asympotically as $N^{-2}\,(\ln mN^{-2})^{-1}$ when m, the separation between boundary spins, is large. Finally, we make the probabilistic nature of the boundary spontaneous magnetization more precise by computing its probability distribution function.

1. INTRODUCTION

IN the preceding two papers of this series[1,2] we have introduced a modification of the two-dimensional Ising model that incorporates random impurities, and have studied its specific heat and the spin-spin correlation functions of the bulk. This latter study was extremely complicated, when the separation between spins was large, due to the necessity of dealing with Toeplitz determinants of a large dimensionality that is proportional to the separation. These large determinants also arise in the much simpler case of spin-spin correlation functions in the bulk of Onsager's lattice and delayed a full understanding of the asymptotic behavior of these functions for years.[3] However, it has recently been realized[4] that if one considers a half-plane of Ising spins, one may compute the spin-spin correlation functions of spins near the boundary row in terms of determinants of a small dimension that does not increase when the separation between the spins becomes large. In this paper we exploit this fact to study the spin correlation functions on the boundary row of our random Ising model.

We consider a half-plane of Ising spins where the boundary row (called 1) only is allowed to interact with a magnetic field $\mathfrak{H}$ and study the magnetization of this first row

$$\mathfrak{M}_1(\mathfrak{H}) = \langle \sigma_{1,m} \rangle \tag{1.1}$$

and the spin-spin correlation function between two spins in the first row

$$\mathfrak{S}_{1,1}(m,\mathfrak{H}) = \langle \sigma_{1,0}\sigma_{1,m} \rangle . \tag{1.2}$$

The values of $\mathfrak{M}_1$ and $\mathfrak{S}_{1,1}$ are, in general, different for the different lattices in our collection even in the thermodynamic limit. This is in distinct contrast with the free energy and magnetization of the bulk, which are known to approach a value in the thermodynamic limit that is the same, with probability 1, for all lattices of our collection.

A significant aspect of these boundary spin correlation functions is that, even though they are not probability-1 objects themselves, their average values provide lower bounds on certain probability-1 properties of the bulk. In particular, we use a theorem of Griffiths[5] to demonstrate in the Appendix that for all temperatures T

$$\langle \mathfrak{M}_1(\mathfrak{H}) \rangle_{E_2} \le M(H) \tag{1.3a}$$

and

$$\langle \mathfrak{S}_{1,1}(m,\mathfrak{H}) \rangle_{E_2} \le \langle \langle \sigma_{l,0}\sigma_{l,m} \rangle_{\mathrm{bulk}} \rangle_{E_2} = \langle S_m(H) \rangle_{E_2} . \tag{1.3b}$$

Here H is a magnetic field that interacts with the entire lattice and is numerically equal to $\mathfrak{H}$, $M(H)$ is the magnetization of the bulk, and $\langle \cdots \rangle_{E_2}$ denotes an average over the set $\{E_2(j)\}$ that specifies the collection of lattices. These lower bounds may be used to draw conclusions about the critical behavior of the bulk properties if we know that both the bulk and the boundary spontaneous magnetizations vanish at the same temperature as T is increased from zero. We have not been able to show this directly. However, we will

[1] B. M. McCoy and T. T. Wu, Phys. Rev. **176**, 631 (1968). This paper will henceforth be referred to as I.

[2] B. M. McCoy and T. T. Wu, preceding paper, Phys. Rev. **188**, 982 (1969). This paper will be referred to as II.

[3] The limiting value of the correlation function when the separation between spins becomes infinite was announced by L. Onsager, Nuovo Cimento Suppl. **6**, 261 (1949). A derivation of this result was (essentially) first given by C. N. Yang, Phys. Rev. **85**, 808 (1952). See also C. H. Chang, *ibid.* **88**, 1422 (1952). It was also known to L. Onsager [*ibid.* **65**, 117 (1944)] that if $T > T_c$ and the separation between spins tends to infinity, then the approach of the spin correlation function to its limiting value of zero is exponential. However, the complete details of this asymptotic behavior (for $T \le T_c$ as well as $T > T_c$) were obtained only much later, for the case of two spins in the same row by T. T. Wu [*ibid.* **149**, 380 (1966)] and for two spins in different rows by H. Cheng and T. T. Wu [*ibid.* **164**, 719 (1967)].

[4] B. M. McCoy and T. T. Wu, Phys. Rev. **162**, 436 (1967). This paper will henceforth be referred to as IV.

[5] R. Griffiths, J. Math. Phys. **8**, 478 (1967); **8**, 484 (1967).

see in Sec. 2 that the boundary spontaneous magnetization vanishes at the same temperature T_c at which the observable specific heat found in I fails to be analytic. Thus, if we could show that the bulk spontaneous magnetization vanishes at the same temperature at which the specific heat fails to be analytic, we would have

$$\langle \mathfrak{M}_1(0)\rangle_{E_2}=M(0)=0 \quad \text{for } T\geq T_c. \tag{1.3c}$$

Indeed, this identification of the temperature at which the order parameter vanishes with the temperature at which the specific heat is nonanalytic is universally made in the literature on magnetic critical phenomena.[6] No general proof of the validity of this seemingly natural assumption exists, but neither has a counter example been found. Therefore, we will assume it to be the case, and find from (1.3a) and (1.3c) that

$$\langle \partial\mathfrak{M}(\mathfrak{H})/\partial\mathfrak{H}\,|_{\mathfrak{H}=0}\rangle_{E_2}\leq \partial M(H)/\partial H\,|_{H=0}. \tag{1.3d}$$

It has become common to parametrize the behavior of ferromagnets near the critical temperature in terms of a set of "critical exponents."[6] In particular,

$$M(0)\sim \text{const}(T_c-T)^{\beta} \tag{1.4a}$$

as $T\rightarrow T_c-$,

$$\left.\frac{\partial M(H)}{\partial H}\right|_{H=0}\sim \text{const}(T_c-T)^{-\gamma'} \qquad \text{if } T\rightarrow T_c-$$

$$\sim \text{const}(T-T_c)^{-\gamma} \qquad \text{if } T\rightarrow T_c+ \tag{1.4b}$$

$$M(H)\sim \text{sgn}(H)\,\text{const}\,|H|^{1/\delta} \qquad \text{if } T=T_c \tag{1.4c}$$

and if, in addition, m is large,

$$\langle S_m(0)\rangle_{E_2}\sim \text{const}\, m^{2-d-\eta} \qquad \text{if } T=T_c \tag{1.4d}$$

and

$$\langle S_m(0)\rangle_{E_2}\sim M^2(0)+\text{const}\, m^{-a}e^{-m/\xi} \quad \text{if } T\neq T_c. \tag{1.4e}$$

In (1.4d), d is the dimensionality of the lattice, which in our case is 2. In (1.4e) the correlation length ξ depends on T and a may be different for T above or below T_c. If we define a similar set of "critical exponents" for $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ and $\langle\mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2}$, and if we use (1.3), we find

$$\beta_{\text{bulk}}\leq \beta_{\text{boundary}}, \tag{1.5a}$$

$$\gamma_{\text{bulk}}\geq \gamma_{\text{boundary}}, \tag{1.5b}$$

$$\delta_{\text{bulk}}\geq \delta_{\text{boundary}}, \tag{1.5c}$$

$$\eta_{\text{bulk}}\leq \eta_{\text{boundary}}, \tag{1.5d}$$

and, if $T>T_c$,

$$\xi_{\text{bulk}}\leq \xi_{\text{boundary}}. \tag{1.5e}$$

The primary purpose of this paper is to demonstrate that, in general, there is no reason to assume that an impure, and hence realistic, magnetic system near T_c is described by the "critical exponents" defined by (1.4). We demonstrate this by studying $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ and $\langle\mathfrak{S}_{1,1}(\mathfrak{H})\rangle_{E_2}$ for the case of the particular probability distribution considered in I and II:

$$P(E_2)\frac{dE_2}{d\lambda}=\mu(\lambda)=N\lambda_0^{-N}\lambda^{N-1} \quad \text{if } 0\leq\lambda\leq\lambda_0$$

$$=0 \qquad \text{otherwise}, \tag{1.6}$$

where $\lambda=\tanh^2 E_2\beta$ and $\beta=(kT)^{-1}$. In Sec. 2 we study $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ when $T-T_c=O(N^{-2})$ and $\mathfrak{H}=O(N^{-1})$. We find in (2.35) that as $T\rightarrow T_c-$

$$\langle\mathfrak{M}_1(0)\rangle_{E_2}\sim C_M N(T_c-T), \tag{1.7}$$

where

$$C_M=(2\pi)^{1/2}k\beta_c^2(1+z_{2c}^{0-1})z_{1c}^{-1/2}(1+z_{1c}) \times\{E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)\}, \tag{1.8}$$

$$z_1=\tanh E_1\beta, \quad z_2^0=\tanh E_2^0\beta, \tag{1.9}$$

and the subscript c means $T=T_c$. This is in conformity with (1.4a). However, we also find that $\mathfrak{M}_1(\mathfrak{H})$ is not an analytic function of $\mathfrak{H}$ at $\mathfrak{H}=0$ when $T-T_c=O(N^{-2})$. Most striking is the fact [see Eqs. (2.50) and (2.63)] that there is a temperature range about T_c where $\langle\partial\mathfrak{M}_1(\mathfrak{H})/\partial\mathfrak{H}\,|_{\mathfrak{H}=0}\rangle_{E_2}$ does not exist because, when δ, which, as defined by (2.11), is proportional to $N^2(T-T_c)$, is neither zero nor an integer plus $\frac{1}{2}$, one has

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}\sim\langle\mathfrak{M}_1(0)\rangle_{E_2} +C(\delta)\,\text{sgn}(\mathfrak{H})N^{-1}(\beta_c|\mathfrak{H}|N)^{2|\delta|} +O(\mathfrak{H}^{4|\delta|})+O(\mathfrak{H}). \tag{1.10}$$

Here

$$C(\delta)=2^{3(\delta-1/2)}[z_{1c}^{1/2}(1+z_{1c})^{-1}]^{2\delta-1} \times z_{2c}^{0-2\delta}\pi^{1/2}[\Gamma(\delta)]^{-1}\csc\pi(\tfrac{1}{2}-\delta) \tag{1.11a}$$

if $T>T_c$ and

$$C(\delta)=2^{-3(\delta+1/2)}[z_{1c}^{1/2}(1+z_{1c})^{-1}]^{-2\delta-1}z_{2c}^{0\delta}\Gamma(\tfrac{1}{2}-2\delta) \times\Gamma(1+\delta)[\Gamma(-\delta)\Gamma(1-\delta)]^{-1}\csc\pi(\tfrac{1}{2}+\delta) \tag{1.11b}$$

if $T<T_c$. When $T<T_c$, we need the additional restriction that $|\delta|$ is not an integer. For the values of δ at which $C(\delta)$ is zero or infinity the form (1.10) breaks down. These exceptional values are studied also in Sec. 2. In particular, when $T=T_c(\delta=0)$, Eq. (2.71) shows

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}\sim-(\text{sgn}\mathfrak{H})\times 2^{-5/2} \times z_{1c}^{-1/2}(1+z_{1c})\pi^{1/2}N^{-1}[\ln N\beta_c|\mathfrak{H}|]^{-1}. \tag{1.12}$$

Therefore the assumed "critical-exponent" forms (1.4b) and (1.4c) do not at all describe the behavior of

[6] We follow the standard notation as given, for example, by L. P. Kadanoff, W. Götze, D. Hamblen, R. Hecht, E. A. S. Lewis, V. V. Palciauskas, M. Rayl, J. Swift, D. Aspnes, and J. Kane, Rev. Mod. Phys. 39, 395 (1967). This article contains extensive references to earlier work. The standard notation seems to have been applied only to lattices in which the spin-spin correlation function depends only on the relative separation between spins and approaches a limit as the separation becomes infinite. Our definitions (1.4d) and (1.4e) are therefore somewhat more general than have been considered previously.

$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$. Hence (1.3) and (1.5) imply that for our random Ising model γ_{bulk} and δ_{bulk} do not exist.

It is not surprising that forms (1.4b) and (1.4c) are violated in our model because they have been abstracted from calculations on pure, homogeneous lattices in which the only relevant length scale is the coherence length ξ defined by (1.4e). In some sense our random lattice possesses two length scales, the coherence length ξ^0 of the pure Onsager lattice, which is known[3,4] to be proportional to $|T-T_c|^{-1}$, and some sort of length associated with the impurities. Because the specific heat computed in I deviates appreciably from its Onsager value only when $T-T_c=O(N^{-2})$, we expect that this impurity length scale has the order of magnitude N^2. In Sec. 3 we try to make these concepts more precise by studying $\langle\mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2}$ when $m=O(N^2)$, $T-T_c=O(N^{-2})$, and $\mathfrak{H}=O(N^{-1})$. When $m\ll N^2$, this average correlation function approaches its Onsager value. However, when $m\gg N^2$ and $T=T_c$, (3.54) shows that

$$\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2}|_{\delta=0} \sim\tfrac{1}{16}z_{1c}^{-1}(1+z_{1c})^2N^{-2}[\ln N^{-2}m]^{-1}. \quad (1.13)$$

Furthermore, (3.48) shows that if the scaled temperature δ is of order 1 and $m\gg N^2$,

$$\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2}\sim\tfrac{1}{16}z_{1c}^{-1}(1+z_{1c})^2N^{-2} \times\{\max[-\delta,0]+D(\delta)(N^2/m)^{2|\delta|}\}, \quad (1.14)$$

where

$$D(\delta)=4\Gamma(2|\delta|)[\Gamma(|\delta|)]^{-2} \times[\tfrac{1}{4}z_{2c}{}^0z_{1c}^{-1}(1+z_{1c})^2]^{-2|\delta|}. \quad (1.15)$$

Therefore the critical-exponent forms (1.4d) and (1.4e) fail to describe the behavior of $\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2}$ near T_c and (1.5) implies that η_{bulk} and ξ_{bulk} for $T>T_c$ and $T-T_c=O(N^{-2})$ do not exist.

Not only are the average values of the boundary spin correlation functions of interest, but also the probability distribution of these functions gives us additional insight into the microscopic details of our random Ising model. In the previous paper[2] we interpreted $\lim_{m\to\infty}\langle S_m{}^{1/2}\rangle_{E_2}$ as a measure of the local magnetization in a row and used the value of $\lim_{m\to\infty}\langle\ln S_m\rangle_{E_2}$ and the lower bound on $M(0)$ of this paper to speculate about the probability distribution of this local magnetization. In Sec. 4 we study the probability distribution of $\mathfrak{M}_1(\mathfrak{H})$, which is the measure of the local magnetization in the boundary row. We examine $\langle\mathfrak{M}_1{}^n(\mathfrak{H})\rangle_{E_2}$ and show, in particular, from (4.20) that as $T\to T_c-$ and if n is of order 1,

$$\langle\mathfrak{M}_1{}^n(0)\rangle_{E_2}\sim f(n)N^{-n+2}(T_c-T), \quad (1.16)$$

where

$$f(n)=4k\beta_c(1+z_{2c}{}^{0-1})\{E_1(1-z_{1c})+E_2{}^0(1-z_{2c}{}^0)\} \times\Gamma(\tfrac{1}{2}n)[2^{-3/2}z_{1c}^{-1/2}(1+z_{1c})]^n. \quad (1.17)$$

Therefore, as $T\to T_c-$ all moments of $N\mathfrak{M}_1(0)$ have the same order of magnitude.

We discuss this behavior when $N\mathfrak{M}_1$ is of order 1 as $N\to\infty$ by constructing in (4.29) the probability distribution which has the moments (4.20):

$$\mathfrak{P}(\mathfrak{M}_1)=2c_{\mathfrak{m}}Ne^{-(c_{\mathfrak{m}}N\mathfrak{M}_1)^2}(c_{\mathfrak{m}}N\mathfrak{M}_1)^{2|\delta|-1}/\Gamma(|\delta|), \quad (1.18)$$

where

$$c_{\mathfrak{m}}=2^{3/2}z_{1c}{}^{1/2}(1+z_{1c})^{-1}. \quad (1.19)$$

This then serves to make more plausible and precise the discussion of $P(S_\infty{}^{1/2})$ of Sec. 6 of II.

2. AVERAGE BOUNDARY MAGNETIZATION

The general technique for studying spin correlation functions of our random Ising model has been explained in detail in Sec. 2 of II. The modifications needed to study spin correlations near the boundary of a half-place of Ising spins that interact with a magnetic field $\mathfrak{H}$ applied to the boundary row (called 1) have been discussed in Sec. 2 of IV. From these sources we find that for any lattice of our collection

$$\mathfrak{M}_1(\mathfrak{H})=z+(1-z^2)\mathfrak{A}^{-1}(1,0;0,0)_{DU}, \quad (2.1)$$

where

$$\mathfrak{A}^{-1}(1,0;0,0)_{DU}=(2\pi)^{-1}\int_0^{2\pi}d\theta[\mathfrak{B}^{-1}(\theta)]_{1D,0U}. \quad (2.2)$$

The required inverse matrix elements of $B(\theta)$ are computed as

$$[\mathfrak{B}^{-1}]_{jl,j'l'}=\text{cofactor}\,\mathfrak{C}_{j'l',jl}/\det\mathfrak{C}, \quad (2.3)$$

where $l=U,D$, $l'=U,D$ [compare with (7.5) of IV], and

$$\mathfrak{C}(\theta)=\begin{array}{cc|cccccccccc}
 & & 0 & 0 & 1 & 1 & 2 & 2 & & \mathcal{M}-1 & \mathcal{M} & \mathcal{M} \\
 & & D & U & D & U & D & U & \cdots & U & D & U \\ \hline
0 & D & ic & 0 & & & & & & & & \\
0 & U & 0 & -ic & z & & & & & & & \\
1 & D & & -z & ia & b & & & & & & \\
1 & U & & & -b & -ia & z_2(1) & & & & & \\
2 & D & & & & -z_2(1) & ia & b & & & & \\
 & & & & & & & & \ddots & & & \\
2 & U & & & & & -b & -ia & & & & \\
\vdots & & & & & & & \ddots & \ddots & & & \\
\mathcal{M}-1 & U & & & & & & & & ia & z_2(\mathcal{M}-1) & \\
\mathcal{M} & D & & & & & & & & -z_2(\mathcal{M}-1) & ia & b \\
\mathcal{M} & U & & & & & & & & & b & -ia
\end{array} \quad (2.4)$$

[compare with (2.8) of I]. Here

$$c(\theta)=-2\sin\theta|1+e^{i\theta}|^{-2}, \tag{2.5}$$

$$z=\tanh\beta\mathfrak{H}, \tag{2.6}$$

and a and b are defined in (2.7) of I. In terms of $C(1,\mathfrak{M})$, $\bar{D}(1,\mathfrak{M})$, and $D(1,\mathfrak{M})$ defined in Sec. 2 of II we find

$$\det\mathfrak{C}=-c[-cC(1,\mathfrak{M})+z^2\bar{D}(1,\mathfrak{M})]. \tag{2.7}$$

We similarly evaluate the relevant cofactor and find

$$\begin{aligned}[\mathfrak{B}^{-1}]_{1D,0U}&=z\bar{D}(1,\mathfrak{M})[z^2\bar{D}(1,\mathfrak{M})-cC(1,\mathfrak{M})]^{-1}\\&=z[z^2+c\bar{x}(1,\mathfrak{M})]^{-1},\end{aligned} \tag{2.8}$$

where $\bar{x}(1,\mathfrak{M})$ is defined by (2.21) of II. Then we apply the argument of Sec. 2 of II to find

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=z+(1-z^2)z(2\pi)^{-1}\times\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dx\nu(x)[z^2+cx]^{-1}. \tag{2.9}$$

In this paper we will confine our attention to the distribution function $\mu(\lambda)$ given by (1.6). In this case we know from (4.1) of I that T_c is located from

$$\ln[z_{2c}{}^{0-1}(1-z_{1c})/(1+z_{1c})]=-\tfrac{1}{2}N^{-1}. \tag{2.10}$$

Throughout this paper we will make comparisons between quantities computed for the random lattice and for the corresponding Onsager lattice with the same E_1 and T_c. For this Onsager lattice $E_2=\bar{E}_2$, where $\bar{z}_{2c}{}^{-1}(1-z_{1c})/(1+z_{1c})=1$. When N is large, usually the difference between $\bar{E}_2$ and $E_2{}^0$ is important only in locating T_c, so that to leading order in N^{-1} we will often be able to replace $\bar{E}_2$ by $E_2{}^0$. The boundary magnetization $\mathfrak{M}_1{}^O(\mathfrak{H})$ of this Onsager lattice has been studied in detail in Sec. 5 of IV and the boundary spin-spin correlation of that lattice, $\mathfrak{S}_{1,1}{}^O(m,\mathfrak{H})=\langle\sigma_{1,0}\sigma_{1,m}\rangle$, was studied in Sec. 8 of IV.

We confine our attention to the temperature region considered in I and II where

$$\delta=(T-T_c)N^24k\beta_c{}^2(1+z_{2c}{}^{0-1})\times\{E_1(1-z_{1c})+E_2{}^0(1-z_{2c}{}^0)\} \tag{2.11}$$

is of order 1. In addition, it is easily seen from the expression for $\nu(x)$ [(4.4) of I] that unless $z=O(N^{-1})$, (2.9) will not be sensibly different from its value for Onsager's lattice. We therefore define

$$\bar{z}=zN4\lambda_0{}^{-1/2}z_{1c}{}^{1/2}(1+z_{1c})^{-1}, \tag{2.12}$$

and recall the definition of ϕ [(4.16) of I]:

$$\phi=-8\lambda_0{}^{-1/2}z_{1c}(1+z_{1c})^{-2}N^2\theta. \tag{2.13a}$$

Then

$$c=\tfrac{1}{16}\lambda_0{}^{1/2}z_{1c}{}^{-1}(1+z_{1c})^2N^{-2}\phi+O(N^{-3}). \tag{2.13b}$$

When ϕ and δ are of order 1, we find from Sec. 4 of I that

$$x=\lambda_0{}^{1/2}e^{-q}+O(N^{-1}) \tag{2.14}$$

and

$$\nu(x)\frac{dx}{dq}=\hat{U}(q)=[2K_\delta(\phi)]^{-1}e^{-\delta q-(\phi/2)(e^q+e^{-q})}. \tag{2.15}$$

We may now follow the procedure of I and II and break the θ integration in (2.9) up into two regions: one region where θ is of the order N^{-2} and a second where $|\theta|$ is much greater than N^{-2}. The contribution from this second region is (at least to leading order in N) a constant independent of δ and $\bar{z}$. We find

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=\tfrac{1}{2}z_{1c}{}^{-1/2}(1+z_{1c})\pi^{-1}N^{-1}\bar{z}\times\left\{\int_0^{N^2}d\phi\int_{-\infty}^{\infty}dq\hat{U}(q)(\bar{z}^2+\phi e^{-q})^{-1}+\text{const}+O(N^{-1})\right\}. \tag{2.16}$$

We may replace the upper limit of the ϕ integration by ∞ if we use the fact [which is easily seen from (2.15)] that

$$\lim_{\phi\to\infty}\hat{U}(q)=\delta(q), \tag{2.17}$$

to find for large ϕ

$$\int_{-\infty}^{\infty}dq\hat{U}(q)[z^{-2}+\phi e^{-q}]^{-1}\sim\phi^{-1}. \tag{2.18}$$

Therefore

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=\tfrac{1}{2}z_{1c}{}^{-1/2}(1+z_{1c})\pi^{-1}N^{-1}\bar{z}\times\left\{\int_0^{\infty}d\phi\left[\int_{-\infty}^{\infty}dq\hat{U}(q)(\bar{z}^2+\phi e^{-q})^{-1}-(\phi+1)^{-1}\right]+\ln N^2+\text{const}+O(N^{-1})\right\}. \tag{2.19}$$

The constant in (2.19) may be determined by demanding that the $\delta\to\pm\infty$ limits of $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ agree with the $T\sim T_c$ behavior of $\mathfrak{M}_1{}^O(\mathfrak{H})$ that may be obtained from Sec. 5 of IV. For the purpose of this paper we will not need this constant and will not compute it here. However, it is not without interest to study the manner in which (2.19) approaches its Onsager limit. For example, consider the case $T=T_c$ and $\bar{z}\to\infty$. We write

$$\begin{aligned}&\int_0^{\infty}d\phi\left[\int_{-\infty}^{\infty}dq\hat{U}(q)(\bar{z}^2+\phi e^{-q})^{-1}-(\phi+1)^{-1}\right]\\&\sim\int_0^{\infty}d\phi\left\{\int_{-\infty}^{\infty}dq\hat{U}(q)(z^{-2}+\phi)^{-1}\times[1-\phi(e^{-q}-1)(\bar{z}^2+\phi)^{-1}+\phi^2(e^{-q}-1)^2(\bar{z}^2+\phi)^{-2}]-(\phi+1)^{-1}\right\}.\end{aligned} \tag{2.20}$$

We cannot neglect ϕ in comparison with $\bar{z}^2$ in any of these terms. When ϕ is large,

$$\int_{-\infty}^{\infty} dq\hat{U}(q)e^{-q} = K_1(\phi)/K_0(\phi) = -\frac{d}{d\phi}\ln K_0(\phi)$$
$$\sim 1+\tfrac{1}{2}\phi^{-1}-\tfrac{1}{8}\phi^{-2} \quad (2.21)$$

and

$$\int_{-\infty}^{\infty} dq\hat{U}(q)(e^{-q}-1)^2 = [K_0(\phi)-2K_1(\phi)+K_2(\phi)]/K_0(\phi)$$
$$\sim \phi^{-1}[1+(5/4)\phi^{-1}]. \quad (2.22)$$

Using these approximations, we find that (2.20) is approximately given by

$$-\ln\bar{z}^2 - \tfrac{1}{8}\int_0^{\infty} d\phi(\bar{z}^2+\phi)^{-2}(\phi+1)^{-1}+O(\bar{z}^{-4})$$
$$= -\ln\bar{z}^2-\tfrac{1}{8}\bar{z}^{-4}\ln\bar{z}^2+O(\bar{z}^{-4}). \quad (2.23)$$

Consequently for $T=T_c$ as $\bar{z}\rightarrow\infty$

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2} \sim \tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}N^{-1}\bar{z}\{-2\ln(\bar{z}N^{-1})$$
$$+\text{const}-\tfrac{1}{8}\bar{z}^{-4}\ln\bar{z}^2+O(\bar{z}^{-4})\}$$
$$\sim -4\pi^{-1}z_{2c}^{0-1}z\ln z, \quad (2.24)$$

which agrees with (5.31) of IV.

We proceed to analyze (2.19) in several stages. First we will determine the spontaneous magnetization and then will study the behavior when $\bar{z}\sim 0$ for $\delta>0$, $\delta<0$, and $\delta=0$.

A. Spontaneous Magnetization

The average boundary spontaneous magnetization is defined to be

$$\langle\mathfrak{M}_1(0+)\rangle_{E_2} = \lim_{\mathfrak{H}\rightarrow 0+}\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}. \quad (2.25)$$

Clearly the contributions to (2.19) from values of ϕ greater than some small positive number ϵ will vanish as $\bar{z}\rightarrow 0$. Then if we let

$$q=q'-\ln\tfrac{1}{2}\phi \quad (2.26)$$

and

$$\phi=\sqrt{2}|\bar{z}|\alpha, \quad (2.27)$$

we find

$$\lim_{\mathfrak{H}\rightarrow 0+}\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2} = 2^{-1/2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}N^{-1}$$
$$\times\lim_{\bar{z}\rightarrow 0+}\int_0^{\epsilon/\bar{z}\sqrt{2}} d\alpha[2K_\delta(\sqrt{2}\bar{z}\alpha)]^{-1}\int_{-\infty}^{\infty} dq'(2^{-1/2}\bar{z}\alpha)^{\delta}$$
$$\times\exp[-\delta q'-e^{q'}-\tfrac{1}{2}\bar{z}^2\alpha^2e^{-q'}]\,(1+\alpha^2e^{-q'})^{-1}. \quad (2.28)$$

In the $\bar{z}\rightarrow 0$ limit we may omit the term proportional to $\bar{z}^2$ in the exponential. We may also expand for small ϕ

$$2K_\delta(\phi)\sim\Gamma(|\delta|)(\tfrac{1}{2}\phi)^{-|\delta|}, \quad (2.29)$$

and replace the upper limit of α integration by infinity. If $\delta>0$, (2.28) surely is zero. If $\delta<0$,

$$\langle\mathfrak{M}_1(0)\rangle_{E_2} = 2^{-1/2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}N^{-1}$$
$$\times[\Gamma(|\delta|)]^{-1}\int_0^{\infty} d\alpha\int_{-\infty}^{\infty} dq'$$
$$\times\exp[-\delta q'-e^{q'}]\,(1+\alpha^2e^{-q'})^{-1}, \quad (2.30)$$

from which, if we interchange the orders of integration and let

$$\alpha=e^{q'/2}\alpha', \quad (2.31)$$

we obtain

$$\langle\mathfrak{M}_1(0+)\rangle_{E_2} = 2^{-3/2}z_{1c}^{-1/2}(1+z_{1c})N^{-1}$$
$$\times\Gamma(\tfrac{1}{2}+|\delta|)/\Gamma(|\delta|). \quad (2.32)$$

When $\delta\rightarrow-\infty$, (2.32) is approximated as

$$\langle\mathfrak{M}_1(0+)\rangle_{E_2} = 2^{-3/2}z_{1c}^{-1/2}(1+z_{1c})N^{-1}|\delta|^{1/2}$$
$$\times[1-\tfrac{1}{8}|\delta|^{-1}+O(\delta^{-2})]$$
$$\sim z_{1c}^{-1/2}(1+z_{1c})\{(T_c-T)\tfrac{1}{2}\beta_c^2k(1+z_{2c}^0)z_{2c}^{-1}$$
$$\times[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)]\}^{1/2}. \quad (2.33)$$

This last expression is seen to agree with the $T\approx T_c$ behavior of $\mathfrak{M}_1^O(0)$ as given by (5.29) of IV if we note that near T_c

$$\alpha_2\sim 1+(T-T_c)k\beta_c^2[E_1z_{1c}^{-1}(1-z_{1c}^2)+2E_2^0]$$
$$\sim 1+(T-T_c)k\beta_c^2\tfrac{1}{2}z_{1c}^{-1}(1+z_{1c})^2(1+z_{2c})$$
$$\times[E_1(1-z_{1c})+E_2^0(1-z_{2c}^0)]. \quad (2.34)$$

When $\delta\rightarrow 0$,

$$\langle\mathfrak{M}_1(0+)\rangle_{E_2}\sim\pi^{1/2}2^{-3/2}z_{1c}^{-1/2}(1+z_{1c})N^{-1}|\delta|. \quad (2.35)$$

Therefore, the average boundary spontaneous magnetization vanishes linearly as $T\rightarrow T_c-$ as opposed to the square root of the Onsager case. Finally, for the case $E_1=E_2^0$ we compare (2.32) with $\mathfrak{M}_1^O(0+)$ by plotting them in Fig. 1 for the same values of N and δ considered in Fig. 3 of I.

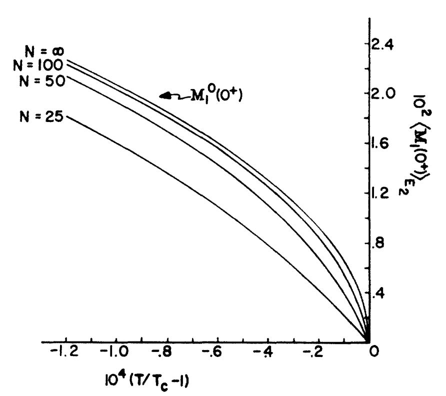

FIG. 1. Comparisons of $\mathfrak{M}_1^O(0+)$ and $\langle\mathfrak{M}_1(0+)\rangle_{E_2}$ for several values of N for the case $E_1=E_2^0$.

B. $\bar{z}$ near Zero, $\delta>0$

Consider first the restriction

$$n-\tfrac{1}{2}<\delta<n+\tfrac{1}{2}, \quad (2.36)$$

where n is a non-negative integer, and write

$$(\bar{z}^2+\phi e^{-q})^{-1}=\phi^{-1}e^q\sum_{k=0}^{n-1}(-\bar{z}^2\phi^{-1}e^q)^k +(-\bar{z}^2\phi^{-1}e^q)^n(\bar{z}^2+\phi e^{-q})^{-1}, \quad (2.37)$$

where the first summation is to be omitted if $n=0$. We may introduce this expression in (2.19) and write

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}\sim\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}N^{-1} \times\Big(\sum_{k=0}^{n-1}I_k+I_n^{(1)}+\text{const}\bar{z}+\bar{z}\ln N^2\Big), \quad (2.38)$$

where

$$I_k=(-1)^k\bar{z}^{2k+1}\int_0^\infty d\phi\Big\{[2K_\delta(\phi)]^{-1}\phi^{-k-1}\int_{-\infty}^\infty dq \times\exp[(k+1-\delta)q-\tfrac{1}{2}\phi(e^q+e^{-q})]-(\phi+1)^{-1}\delta_{k,0}\Big\}$$
$$=(-1)^k\bar{z}^{2k+1}\int_0^\infty d\phi\{\phi^{-k-1}K_{1+k-\delta}(\phi)/K_\delta(\phi) -(\phi+1)^{-1}\delta_{k,0}\}, \quad (k<n) \quad (2.39)$$

and

$$I_n^{(1)}=(-1)^n\bar{z}^{2n+1}\int_0^\infty d\phi[2K_\delta(\phi)]^{-1}\phi^{-n}\int_{-\infty}^\infty dq \times\exp[(n-\delta)q-\tfrac{1}{2}\phi(e^q+e^{-q})](\bar{z}^2+\phi e^{-q})^{-1}. \quad (2.40)$$

Because of the restriction (2.36) the integrals in (2.39) and (2.40) converge. We will see shortly that for δ fixed and greater than $\frac{1}{2}$, $I_n^{(1)}=0(\bar{z})$ as $\bar{z}\to 0$. However, when $\delta\to n-\frac{1}{2}$, I_{n-1} will tend to infinity. More specifically, if (2.36) holds and $\delta\sim n-\frac{1}{2}$, we may study the singular part of I_{n-1} by using approximation (2.29) to write

$$I_{n-1}=(-1)^n\bar{z}^{2n-1}\Gamma(\delta-n)[\Gamma(\delta)]^{-1} \times 2^{-n}\int_0^\epsilon d\phi(\tfrac{1}{2}\phi)^{2(\delta-n)}+O(1)$$
$$=(-1)^{n+1}\bar{z}^{2n-1}\pi^{1/2}[\Gamma(n-\tfrac{1}{2})]^{-1} \times 2^{-n}(\delta-n+\tfrac{1}{2})^{-1}+O(1), \quad (2.41)$$

where $O(1)$ means finite as $\delta\to n-\frac{1}{2}$.

If $\delta>\frac{1}{2}$, we may use (2.38) to write the average boundary zero-field susceptibility as

$$\langle\partial\mathfrak{M}_1(\mathfrak{H})/\partial\mathfrak{H}|_{\mathfrak{H}=0}\rangle_{E_2}=\beta_c2z_{2c}^{0-1}\pi^{-1} \times\Big\{\int_0^\infty d\phi[\phi^{-1}K_{1-\delta}(\phi)/K_\delta(\phi)-(\phi+1)] +\ln N^2+\text{const}+O(N^{-1})\Big\}. \quad (2.42)$$

When $\delta\to\infty$, we may use the relation

$$K_{\delta-1}(\phi)=-\delta\phi^{-1}K_\delta(\phi)-\frac{dK_\delta(\phi)}{d\phi} \quad (2.43)$$

and the asymptotic expansion[7]

$$K_\delta(\phi)\sim(\tfrac{1}{2}\pi)^{1/2}(\delta^2+\phi^2)^{-1/4} \times\exp[-(\delta^2+\phi^2)^{1/2}+\delta\sinh^{-1}(\delta/\phi)] \quad (2.44)$$

to find

$$\langle\partial\mathfrak{M}_1(\mathfrak{H})/\partial\mathfrak{H}|_{\mathfrak{H}=0}\rangle_{E_2}\sim\beta_c2z_{2c}^{0-1}\pi^{-1} \times\{-\ln N^{-2}\delta+\ln 2-1+\text{const}+\delta^{-1}\tfrac{1}{4}\pi\}$$
$$=-\beta_c2z_{2c}^{0-1}\pi^{-1}\ln(T-T_c)+O(1), \quad (2.45)$$

which is seen to agree with $\partial\mathfrak{M}_1^0(\mathfrak{H})/\partial\mathfrak{H}|_{\mathfrak{H}=0}$ given by (5.32) of IV. However, from (2.41) we find that as $\delta\to\frac{1}{2}^+$

$$\langle\partial\mathfrak{M}_1(\mathfrak{H})/\partial\mathfrak{H}|_{\mathfrak{H}=0}\rangle_{E_2} =\pi^{-1}\beta_cz_{2c}^{0-1}(\delta-\tfrac{1}{2})^{-1}+O(1). \quad (2.46)$$

Therefore the average boundary susceptibility at zero field diverges at a temperature *above* T_c. This is completely different from the zero-field susceptibility of either the bulk or the boundary of Onsager's lattice, which are known to be finite for all temperatures $T\neq T_c$.

When $\bar{z}$ is small, we may study $I_n^{(1)}$ exactly as we studied the spontaneous magnetization. The contributions from values of ϕ greater than ϵ are $O(\bar{z}^{2n+1})$. When $\phi\sim 0$, we use (2.26), (2.27), (2.29), and (2.31) and find

$$I_n^{(1)}=\text{sgn}(z)(-1)^n(|\bar{z}|/\sqrt{2})^{2\delta}2^{1/2}[\Gamma(\delta)]^{-1} \times\int_{-\infty}^\infty dq'\exp[\tfrac{1}{2}q'-e^{q'}] \times\int_0^{\epsilon\exp(-q|\bar{z}|^{-1/2})}d\alpha'\alpha'^{2(\delta-n)}(1+\alpha'^2)^{-1} +O(\bar{z}^{2n+1})+O(|\bar{z}|^{4\delta}). \quad (2.47)$$

The α' integral may be approximately evaluated as

$$\int_0^{\epsilon\exp(-q'|\bar{z}|^{-1/2})}d\alpha'\alpha'^{2(\delta-n)}(1+\alpha'^2)^{-1} =\tfrac{1}{2}\{\pi\csc\pi(\tfrac{1}{2}+n-\delta) +\bar{z}^{-2\delta+2n+1}[(\delta-n-\tfrac{1}{2})^{-1}+O(1)]\}, \quad (2.48)$$

where $O(1)$ is a term that does not diverge as $\delta\to n+\frac{1}{2}$. Therefore

$$I_n^{(1)}=(-1)^n\,\text{sgn}(\bar{z})|\bar{z}|^{2\delta}2^{-\delta-1/2}[\Gamma(\delta)]^{-1}\pi^{3/2} \times\csc\pi(\tfrac{1}{2}+n-\delta)+(-1)^n\bar{z}^{2n+1} \times\{\pi^{1/2}[\Gamma(n+\tfrac{1}{2})]^{-1}2^{-n-1}(\delta-n-\tfrac{1}{2})^{-1}+O(1)\} +O(\bar{z}^{4\delta})+O(\bar{z}^{2(\delta+1)}). \quad (2.49)$$

---

[7] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. 2, p. 86.

This contribution to $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ clearly fails to be analytic at $\mathfrak{H}=0$, since the nth derivative does not exist. Indeed, if $0<\delta<\frac{1}{2}$, $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ is not differentiable at all at $\mathfrak{H}=0$, so the zero-field susceptibility will not exist.

It remains to lift the restriction (2.36) by allowing δ to be half an odd integer. We first explicitly exhibit $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ when (2.36) holds as

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}\Big\{\sum_{k=0}^{n-2}I_k+(-1)^{n-1}\bar{z}^{2n-1}[\pi^{1/2}(\Gamma(n-\tfrac{1}{2}))^{-1}2^{-n}(\delta-n+\tfrac{1}{2})^{-1}+O(1)]$$
$$+\operatorname{sgn}(\bar{z})|\bar{z}|^{2\delta}2^{-\delta-1/2}[\Gamma(\delta)]^{-1}\pi^{3/2}\csc\pi(\tfrac{1}{2}-\delta)+(-1)^n\bar{z}^{2n+1}[\pi^{1/2}(\Gamma(n+\tfrac{1}{2}))^{-1}2^{-n-1}(\delta-n-\tfrac{1}{2})^{-1}+O(1)]$$
$$+O(\bar{z}^{4\delta})+O(\bar{z}^{2\delta+2})+\ln\mathcal{N}^2+\text{const}\Big\}. \quad (2.50)$$

In this form we may now let $\delta\to n'-\frac{1}{2}$ from either above or below and find

$$\lim_{\delta\to n'-1/2}\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}\Big\{\sum_{k=0}^{n'-2}I_k+(-1)^{n'}\pi^{1/2}2^{-n'+1}[\Gamma(n'-\tfrac{1}{2})]^{-1}$$
$$\times\operatorname{sgn}(\bar{z})\bar{z}^{2n'-1}\ln|\bar{z}|+O(\bar{z}^{2n'-1})+\ln\mathcal{N}^2+\text{const}\Big\}. \quad (2.51)$$

Therefore, for all positive δ [at least that are $O(1)$ as $\mathcal{N}\to\infty$], $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ is not an analytic function of $\mathfrak{H}$ at $\mathfrak{H}=0$.

C. $\bar{z}$ near Zero, $\delta<0$

The analysis of $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ when $\delta<0$ is only slightly more complicated than the case $\delta>0$ just treated. When $\delta<0$, we may use the results of Sec. 2 A to write

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=\operatorname{sgn}(\bar{z})\langle\mathfrak{M}_1(0+)\rangle_{E_2}+\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}\bar{z}$$
$$\times\left\{\int_0^\infty d\phi\left[\int_0^\infty dq\{[2K_\delta(\phi)]^{-1}e^{-\delta q-(\phi/2)(e^q+e^{-q})}-[\Gamma(|\delta|)]^{-1}(\tfrac{1}{2}\phi)^{|\delta|}e^{-\delta q-(\phi/2)e^q}\}\right.\right.$$
$$\left.\left.\times(\bar{z}^2+\phi e^{-q})^{-1}-(\phi+1)^{-1}\right]+\ln\mathcal{N}^2+\text{const}+O(\mathcal{N}^{-1})\right\}. \quad (2.52)$$

When $\delta<-\frac{1}{2}$, the second term is $O(\bar{z})$ as $\bar{z}\to 0$; therefore

$$\langle\partial\mathfrak{M}_1(\mathfrak{H})/\partial\mathfrak{H}|_{\mathfrak{H}=0}\rangle_{E_2}$$
$$=2\beta_c z_{2c}^{0-1}\pi^{-1}\left\{\int_0^\infty d\phi\left[\phi^{-1}\int_{-\infty}^\infty dq e^q\{[2K_\delta(\phi)]^{-1}e^{-\delta q-(\phi/2)(e^q+e^{-q})}-[\Gamma(|\delta|)]^{-1}(\tfrac{1}{2}\phi)^{|\delta|}e^{-\delta q-(\phi/2)e^q}\}\right.\right.$$
$$\left.\left.-(\phi+1)^{-1}\right]+\ln\mathcal{N}^2+\text{const}+O(\mathcal{N}^{-1})\right\}$$
$$=2\beta_c z_{2c}^{0-1}\pi^{-1}\left\{\int_0^\infty d\phi\{\phi^{-1}[K_{|\delta|+1}(\phi)/K_{|\delta|}(\phi)-2\phi^{-1}|\delta|]-(\phi+1)^{-1}\}+\ln\mathcal{N}^2+\text{const}+O(\mathcal{N}^{-1})\right\}, \quad (2.53)$$

which, if we use the recursion relation[8]

$$K_{|\delta|+1}(\phi)=K_{1-|\delta|}(\phi)+2|\delta|\phi^{-1}K_{|\delta|}(\phi), \quad (2.54)$$

becomes

$$\langle\partial\mathfrak{M}_1(\mathfrak{H})/\partial\mathfrak{H}|_{\mathfrak{H}=0}\rangle_{E_2}=2\beta_c z_{2c}^{0-1}\pi^{-1}$$
$$\times\left\{\int_0^\infty d\phi[\phi^{-1}K_{1-|\delta|}(\phi)/K_{|\delta|}(\phi)-(\phi+1)^{-1}]\right.$$
$$\left.+\ln\mathcal{N}^2+\text{const}+O(\mathcal{N}^{-1})\right\}, \quad (2.42')$$

which is exactly the same as (2.42) except that δ is replaced by $|\delta|$. Thus

$$\langle\partial\mathfrak{M}_1(\mathfrak{H};\delta)/\partial\mathfrak{H}|_{\mathfrak{H}=0}\rangle_{E_2}$$
$$=\langle\partial\mathfrak{M}_1(\mathfrak{H};-\delta)/\partial\mathfrak{H}|_{\mathfrak{H}=0}\rangle_{E_2} \quad (2.55)$$

and the zero-field susceptibility diverges as a simple pole when $\delta\to-\frac{1}{2}^-$ as well as when $\delta\to+\frac{1}{2}^+$.

We may also use the procedures of Sec. 2 B to show that, in addition to a Taylor series in $|\bar{z}|$, $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ contains terms proportional to $|\bar{z}|^{|2|\delta}$. Values of $\phi>\epsilon$ contribute only to the odd terms in this series. Any divergences in these terms as $\delta\to-n'+\frac{1}{2}$ and any other terms come from the region $0<\phi<\epsilon$. Consider first

[8] Reference 7, Vol. 2, p. 79.

the case

$$n-\tfrac{1}{2}<|\delta|<n+\tfrac{1}{2}, \qquad (2.56)$$

$$\delta\neq -n, \qquad (2.57)$$

where n is a non-negative integer. When $\phi\sim 0$, we use the complete expansion

$$2K_{|\delta|}(\phi)=\pi(\sin|\delta|\pi)^{-1} \times\{\sum_{k=0}^{\infty}(\tfrac{1}{2}\phi)^{2k-|\delta|}/[k!\Gamma(k+1-|\delta|)] -\sum_{k=0}^{\infty}(\tfrac{1}{2}\phi)^{2k+|\delta|}/[k!\Gamma(k+1+|\delta|)]\}, \qquad (2.58)$$

to find

$$[2K_{|\delta|}(\phi)]^{-1}=\Gamma(|\delta|)^{-1}(\tfrac{1}{2}\phi)^{|\delta|} \times\{\sum_{k=0}^{\infty}A_k(\tfrac{1}{2}\phi)^{2k}+(\tfrac{1}{2}\phi)^{2|\delta|}\Gamma(1-|\delta|) \times[\Gamma(1+|\delta|)]^{-1}\sum_{k=0}^{\infty}B_k\phi^{2k}+O(\phi^{4|\delta|})\}, \qquad (2.59)$$

where the only properties of A_k and B_k we need are

$$A_0=B_0=1 \qquad (2.60)$$

and, as $\delta\rightarrow -n$,

$$A_k=O(1) \quad \text{for } 0\leq k\leq n-1, \qquad (2.61)$$

$$A_n=-\Gamma(1-|\delta|)/[n!\Gamma(n+1-|\delta|)]. \qquad (2.62)$$

Using these approximations, we find

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=\operatorname{sgn}(\bar{z})\langle\mathfrak{M}_1(0+)\rangle_{E_2}+\operatorname{sgn}(\bar{z})\sum_{k=1}^{2n-2}M^{(k)}|z|^k+(\operatorname{sgn}\bar{z})\times\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}$$
$$\times\{|\bar{z}|^{-2\delta}2^{\delta-1/2}\pi\csc\pi(\tfrac{1}{2}+\delta)\Gamma(\tfrac{1}{2}-2\delta)\Gamma(1+\delta)[\Gamma(1-\delta)\Gamma(-\delta)]^{-1}$$
$$+|\bar{z}|^{2n-1}((-1)^n2^{-n}(\delta+n-\tfrac{1}{2})^{-1}\Gamma(2n-\tfrac{1}{2})\Gamma(\tfrac{3}{2}-n)[\Gamma(n+\tfrac{1}{2})\Gamma(n-\tfrac{1}{2})]^{-1}+O(1))$$
$$+\bar{z}^{2n}(2^{-n-1/2}\pi\Gamma(\tfrac{1}{2}+2n)(n!)^{-1}[(n-1)!]^{-2}(\delta+n)^{-1})$$
$$+|\bar{z}|^{2n+1}((-1)^{n+1}2^{-n-1}(\delta+n+\tfrac{1}{2})^{-1}\Gamma(2n+\tfrac{3}{2})\Gamma(\tfrac{1}{2}-n)[\Gamma(n+\tfrac{3}{2})\Gamma(n+\tfrac{1}{2})]^{-1}+O(1))+O(\bar{z}^{-2\delta+2})+O(\bar{z}^{-4\delta})\}, \qquad (2.63)$$

where the coefficients $M^{(k)}$ with $k\leq 2(n-1)$ are analytic in δ for $|\delta|\leq n+\frac{1}{2}$. Note that the coefficient of $\bar{z}^{2n}$ has a pole at $\delta=-n$ and the coefficient of $\bar{z}^{2n-1}$ has a pole at $\delta=-n+\frac{1}{2}$. We may now lift the restriction (2.57) by letting $\delta\rightarrow -n$ to obtain

$$\lim_{\delta\rightarrow -n}\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=\operatorname{sgn}(\bar{z})\langle\mathfrak{M}_1(0+)\rangle_{E_2} +\operatorname{sgn}(\bar{z})\sum_{k=1}^{2n-1}M^{(k)}|\bar{z}|^k +\operatorname{sgn}(\bar{z})\times\tfrac{1}{2}z_{1c}^{-1/2}(1-z_{1c})\pi^{-1}\mathcal{N}^{-1}|\bar{z}|^{2n} \times\{2^{-n-1/2}\pi\Gamma(\tfrac{1}{2}+2n)(n!)^{-1}[(n-1)!]^{-2} \times\ln\bar{z}^2+O(1)\}+O(\bar{z}^{2n+1}), \qquad (2.64)$$

and we may lift restriction (2.56) by letting $\delta\rightarrow -n'+\frac{1}{2}$ to obtain

$$\lim_{\delta\rightarrow -n'+1/2}\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}=\operatorname{sgn}(\bar{z})\langle\mathfrak{M}_1(0+)\rangle_{E_2} +\operatorname{sgn}(\bar{z})\sum_{k=1}^{2n'-2}M^{(k)}|\bar{z}|^k +(\operatorname{sgn}\bar{z})\times\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}|\bar{z}|^{2n'-1} \times\{(-1)^{n'}2^{-n'}\Gamma(2n'-\tfrac{1}{2})\Gamma(\tfrac{3}{2}-n') \times[\Gamma(n'+\tfrac{1}{2})\Gamma(n'-\tfrac{1}{2})]^{-1}\ln\bar{z}^2+O(1)\}. \qquad (2.65)$$

Therefore, not only does $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ fail to be analytic at $\mathfrak{H}=0$ because of the presence of terms proportional to $\operatorname{sgn}(\bar{z})|\bar{z}|^{2k}$, but also

$$\lim_{\mathfrak{H}\rightarrow 0+}\frac{\partial^n}{\partial\mathfrak{H}^n}\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$$

does not exist if $n\geq|\delta|+\frac{1}{2}$. In particular, the zero-field susceptibility does not exist if $-\frac{1}{2}\leq\delta\leq 0$.

D. $\bar{z}$ near Zero, $\delta=0$

It remains to study the case $\delta=0$. The previous approximations fail in this case because neglected terms $O(\bar{z}^{4|\delta|})$ become important. From (2.19) we have

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}|_{\delta=0}=\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}\bar{z} \times\left\{\int_0^{\infty}d\phi\left[\int_{-\infty}^{\infty}dq[2K_0(\phi)]^{-1} \times\exp[-\tfrac{1}{2}\phi(e^q+e^{-q})] \times(\bar{z}^2+\phi e^{-q})^{-1}-(\phi+1)^{-1}\right] +\ln\mathcal{N}^2+\text{const}+O(\mathcal{N}^{-1})\right\}. \qquad (2.66)$$

As before, values of $\phi > \epsilon$ contribute only to a Taylor series in odd powers of $\bar{z}$. When $\phi \ll 1$, we approximate

$$K_0(\phi) \sim -[\ln\tfrac{1}{2}\phi + \gamma] + o(1) = -[\ln(\phi A 2^{-1/2}) + o(1)], \quad (2.67)$$

where $\gamma \approx 0.577216$ is Euler's constant, and

$$A = 2^{-1/2} e^{\gamma}. \quad (2.68)$$

Then using (2.26) and (2.27), we may approximate

$$\langle \mathfrak{M}_1(\mathfrak{H}) \rangle_{E_2} |_{\delta=0} \sim -(\text{sgn}\bar{z}) \times 2^{-3/2} z_{1c}^{-1/2} (1+z_{1c}) \pi^{-1} N^{-1} \times \int_0^{\epsilon/|\bar{z}|} d\alpha [\ln|\bar{z}| \alpha A]^{-1} \times \int_{-\infty}^{\infty} dq' e^{-\epsilon q'} (1+\alpha^2 e^{-q'})^{-1}. \quad (2.69)$$

Because $\epsilon \ll 1$, we may expand

$$[\ln|\bar{z}| \alpha A]^{-1} \sim [\ln|\bar{z}| A]^{-1} \times \{1 - \ln\alpha / \ln|\bar{z}| A + \cdots\}. \quad (2.70)$$

Then we may obtain the leading terms in the approximation by interchanging orders of integration and replacing the upper limit of the α integration by ∞ to obtain

$$\langle \mathfrak{M}_1(\mathfrak{H}) \rangle_{E_2} |_{\delta=0} \sim -(\text{sgn}\bar{z}) \times 2^{-5/2} \times z_{1c}^{-1/2} (1+z_{1c}) \pi^{1/2} N^{-1} [\ln|\bar{z}| A]^{-1} \times \{1 - \tfrac{1}{2}[\ln|\bar{z}| A]^{-1} \psi(\tfrac{1}{2}) + \cdots\}, \quad (2.71)$$

where

$$\psi(\tfrac{1}{2}) = \Gamma'(\tfrac{1}{2}) / \Gamma(\tfrac{1}{2}) = -\gamma - 2\ln 2. \quad (2.72)$$

Clearly (2.71) vanishes more slowly than any fractional power of $\mathfrak{H}$ as $\mathfrak{H} \to 0$ and is therefore not of the form (1.4c), which is parametrized by the "critical exponent" δ.

3. AVERAGE SPIN-SPIN CORRELATION FUNCTIONS

There are, of course, many distinct spin-spin correlation functions for two spins near the boundary of our half-plane of Ising spins, and we will confine our interest to the special case when both spins are in the boundary row 1. We then use the formalism of Sec. 8 of IV and find that for any lattice in our collection

$$\mathfrak{S}_{1,1}(m,\mathfrak{H}) = (1-z^2)\{[\mathfrak{A}^{-1}(1,0;0,0)_{DU} + (z^{-1}-z)^{-1}]^2 - [\mathfrak{A}^{-1}(1,m;0,0)_{DU}]^2 - \mathfrak{A}^{-1}(1,0;1,m)_{DD} \mathfrak{A}^{-1}(0,0;0,m)_{UU}\}, \quad (3.1)$$

where, from Sec. 2,

$$\mathfrak{A}^{-1}(1,m;0,0)_{DU} = (2\pi)^{-1} \int_0^{2\pi} d\theta e^{im\theta} [\mathfrak{B}^{-1}(\theta)]_{1D,0U} = (2\pi)^{-1} z \int_0^{2\pi} d\theta e^{im\theta} [z^2 + c\bar{x}(1,\mathfrak{N};\theta)]^{-1}, \quad (3.2a)$$

and, using an argument similar to Sec. 2,

$$\mathfrak{A}^{-1}(1,0;1,m)_{DD} = (2\pi i)^{-1} \int_0^{2\pi} d\theta e^{-im\theta} [\bar{x}(1,\mathfrak{N};\theta) + z^2 c^{-1}]^{-1}, \quad (3.2b)$$

$$\mathfrak{A}^{-1}(0,0;0,m)_{UU} = -(2\pi i)^{-1} \int_0^{2\pi} d\theta e^{-im\theta} [c + z^2 \bar{x}(1,\mathfrak{N};\theta)^{-1}]^{-1}. \quad (3.2c)$$

The first term in (3.1) is recognized as $\mathfrak{M}_1(\mathfrak{H})^2$. We may apply the discussion of Sec. 2 of II to average (3.1) over all $\{E_2\}$ by use of the two-variable function $\nu(x_1,x_2)$ and obtain

$$\langle \mathfrak{S}_{1,1}(m,\mathfrak{H}) \rangle_{E_2} = \langle \mathfrak{M}_1(\mathfrak{H})^2 \rangle_{E_2} - (1-z^2)(2\pi)^{-2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \times \nu(x_1,x_2) \{z^2 e^{im\theta_1} e^{im\theta_2} [z^2 + c_1 x_1]^{-1} [z^2 + c_2 x_2]^{-1} + e^{im\theta_1} e^{im\theta_2} [x_1 + z^2 c_1^{-1}]^{-1} [c_2 + z^2 x_2]^{-1}\}, \quad (3.3)$$

where

$$\langle \mathfrak{M}_1(\mathfrak{H})^2 \rangle_{E_2} = z^2 \left\{ 1 + (1-z^2)\pi^{-1} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dx \nu(x) [z^2 + cx]^{-1} + (1-z^2)^2 (2\pi)^{-2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \times \nu(x_1,x_2) [z^2 + c_1 x_1]^{-1} [z^2 + c_2 x_2]^{-1} \right\} \quad (3.4)$$

and

$$c_j = c(\theta_j). \quad (3.5)$$

To study the leading term of (3.3) when $\delta = O(1)$ we make the substitutions (2.11)–(2.14). We also define

$$\bar{m} = \tfrac{1}{8} \lambda_0^{1/2} z_{1c}^{-1} (1+z_{1c})^2 N^{-2} m. \quad (3.6)$$

As in the previous sections, we then consider the contribution to the θ_j integrals from $0 < \theta_j \sim N^{-2}$ and $\theta_j \gg N^{-2}$ separately. For $\bar{m} \neq 0$, the contributions to $\mathfrak{A}^{-1}(1,m;0,0)_{DU}$ and $\mathfrak{A}^{-1}(0,0;0,m)_{UU}$ from these large θ_j may be neglected. The contribution to $\mathfrak{A}^{-1}(1,0;1,m)_{DD}$

from $\theta \gg N^{-2}$ is to lowest order in N independent of δ and $\bar{z}$, and depends only on m. Therefore, we have

$$\langle \mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2} = \langle \mathfrak{M}_1^2(\mathfrak{H})\rangle_{E_2} + [\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}]N^{-2}\int_0^{N^2} d\phi_1 \int_0^{N^2} d\phi_2 \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2$$

$$\times \hat{U}(q_1,q_2)\left[\frac{\sin\bar{m}\phi_1 \sin\bar{m}\phi_2}{(e^{-q_1}+\phi_1^{-1}\bar{z}^2)(\phi_2+e^{q_2}\bar{z}^2)} - \frac{\bar{z}^2 \cos\bar{m}\phi_1 \cos\bar{m}\phi_2}{(\phi_1 e^{-q_1}+\bar{z}^2)(\phi_2 e^{-q_2}+\bar{z}^2)}\right] + f(m)\int_0^{\infty} d\phi \int_{-\infty}^{\infty} dq \hat{U}(q)\frac{\sin\bar{m}\phi}{\phi+e^{q}\bar{z}^2} + o(N^{-2}), \quad (3.7)$$

where $f(m)$ is the contribution to $2\pi^{-1}\mathfrak{A}^{-1}(1,0;1,m)_{DD}$ from $|\theta| \gg N^{-2}$,

$$\langle \mathfrak{M}_1^2(\mathfrak{H})\rangle_{E_2} = [\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}]^2 N^{-2}\bar{z}^2 \Bigg\{ \int_0^{\infty} d\phi_1 \int_0^{\infty} d\phi_2 \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2$$

$$\times \hat{U}(q_1,q_2)[(\bar{z}^2+\phi_1 e^{-q_1})^{-1} - (\phi_1+1)^{-1}][(\bar{z}^2+\phi_2 e^{-q_2})^{-1} - (\phi_2+1)^{-1}]$$

$$+2\ln N^2 \int_0^{\infty} d\phi \int_{-\infty}^{\infty} dq \hat{U}(q)[(\bar{z}^2+e^{-q})^{-1} - (\phi+1)^{-1}] + (\ln N^2)^2$$

$$+\text{const}\left[\int_0^{\infty} d\phi \int_{-\infty}^{\infty} dq \hat{U}(q)[(\bar{z}^2+\phi e^{-q})^{-1} - (\phi+1)^{-1}] + \ln N^2\right] + \text{const}' + O(N^{-1})\Bigg\}, \quad (3.8)$$

and

$$\hat{U}(q_1,q_2)dq_1 dq_2 = \nu(x_1,x_2)dx_1 dx_2. \quad (3.9)$$

In (3.8) the upper limits of ϕ integration have been replaced by ∞ and the N^2 dependence made explicit by recalling from Sec. 4 of II that as $\phi_1 \to \infty$

$$\hat{U}(q_1,q_2) \to \delta(q_1)\hat{U}(q_2) \quad (3.10)$$

and following the procedure of Sec. 2. The constants in (3.8) may be explicitly computed but they are not needed for the purposes of this paper. The upper limits of ϕ integration in (3.7) may be replaced by infinity if we use (3.10) to show that for ϕ_2 large

$$\int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 \hat{U}(q_1,q_2)\left[\frac{\sin\bar{m}\phi_1 \sin\bar{m}\phi_2}{(e^{-q_1}+\phi_1^{-1}\bar{z}^2)(\phi_2+e^{q_2}\bar{z}^2)} - \frac{\bar{z}^2\cos\bar{m}\phi_1\cos\bar{m}\phi_2}{(\phi_1 e^{-q_1}+\bar{z}^2)(\phi_2 e^{-q_2}+\bar{z}^2)}\right]$$

$$\sim \phi_2^{-1}\sin\bar{m}\phi_2 \int_{-\infty}^{\infty} dq_1 \hat{U}(q_1)\sin\bar{m}\phi_1 (e^{-q_1}+\phi_1\bar{z}^2)^{-1} - \bar{z}^2\phi_2^{-1}\cos\bar{m}\phi_2 \int_{-\infty}^{\infty} dq_1 \hat{U}(q_1)\cos\bar{m}\phi_1 (\phi_1 e^{-q_1}+\bar{z}\ )^{-1}, \quad (3.11)$$

which is integrable for large ϕ_2. If ϕ_1 is large, the left-hand side of (3.11) tends to

$$\sin\bar{m}\phi_1 \int_{-\infty}^{\infty} dq_2 \hat{U}(q_2)\sin\bar{m}\phi_2 (q_2+e^{q_2}\bar{z}^2)^{-1} - \bar{z}^2\phi_1^{-1}\cos\bar{m}\phi_1 \int_{-\infty}^{\infty} dq_1 \hat{U}(q_1)\cos\bar{m}\phi_2 (\phi_2 e^{-q_2}+\bar{z}^2)^{-1}, \quad (3.12)$$

which is not integrable for large ϕ_1. Therefore,

$$\langle \mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2} = \langle \mathfrak{M}_1^2(\mathfrak{H})\rangle_{E_2} + [\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}]^2 N^{-2}\int_0^{\infty} d\phi_1 \int_0^{\infty} d\phi_2 \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2$$

$$\times \hat{U}(q_1,q_2)\left[(\sin\bar{m}\phi_1)\left(\frac{1}{e^{-q_1}+\phi_1^{-1}\bar{z}^2} - 1\right)\frac{\sin\bar{m}\phi_2}{\phi_2+e^{q_2}\bar{z}^2} - \frac{\bar{z}^2\cos\bar{m}\phi_1\cos\bar{m}\phi_2}{(\phi_1 e^{-q_1}+\bar{z}^2)(\phi_2 e^{-q_2}+\bar{z}^2)}\right]$$

$$+f(m)\int_0^{\infty} d\phi \int_{-\infty}^{\infty} dq \hat{U}(q)\frac{\sin\bar{m}\phi}{\phi+e^{q}\bar{z}^2} + O(N^{-2}), \quad (3.13)$$

where we have redefined $f(m)$ of (3.7) by absorbing a multiple of $m^{-1}\cos m$.

One way (though by no means the only way) of determining $f(m)$ is to study the $\delta=0$, $\bar{z}\to\infty$ limit of (3.13). In this limit the terms in $\langle \mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2}$ that are independent of N must agree with the $T=T_c$,

$\sim m^{-2}$ approximation to $\mathfrak{S}_{1,1}{}^{O}(m,\mathfrak{H})$ given by (8.96) of IV. We study (3.13) in this limit by writing

$$\phi_j=\alpha_j\bar{z}^2 \tag{3.14}$$

and using approximations (3.10) and (2.17) to obtain

$$\begin{aligned}\mathfrak{S}_{1,1}\,(m,\mathfrak{H})\rangle_{E_2}|_{\delta=0}\rightarrow&\langle\mathfrak{M}_1{}^2(\mathfrak{H})\rangle_{E_2}\\&-[\tfrac{1}{2}z_{1c}{}^{-1/2}(1+z_{1c})\pi^{-1}]^2N^{-2}\bar{z}^2\\&\times\left\{\left[\int_0^\infty d\alpha(\alpha+1)^{-1}\sin\bar{z}^2\bar{m}\alpha\right]^2\right.\\&\left.+\left[\int_0^\infty d\alpha(\alpha+1)^{-1}\cos\bar{z}^2\bar{m}\alpha\right]^2\right\}\\&+f(m)\int_0^\infty d\alpha(\alpha+1)^{-1}\sin\bar{z}^2\bar{m}\alpha. \end{aligned}\tag{3.15}$$

Now

$$\begin{aligned}\int_0^\infty &d\alpha(\alpha+1)^{-1}\sin\bar{z}^2\bar{m}\alpha\\&=(2i)^{-1}\left\{\int_0^\infty d\alpha(\alpha+1)^{-1}e^{i\bar{z}^2\bar{m}\alpha}-\int_0^\infty d\alpha(\alpha+1)^{-1}e^{-i\bar{z}^2\bar{m}\alpha}\right\}\\&=\int_0^\infty d\xi e^{-\bar{z}^2\bar{m}\xi}(1+\xi^2)^{-1}\end{aligned}\tag{3.16}$$

and

$$\int_0^\infty d\alpha(\alpha+1)^{-1}\cos\bar{z}^2\bar{m}\alpha=\int_0^\infty d\xi e^{-\bar{z}^2\bar{m}\xi}\xi(1+\xi^2)^{-1}, \tag{3.17}$$

so that

$$\begin{aligned}\langle\mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2}|_{\delta=0}\rightarrow&\langle\mathfrak{M}_1{}^2(\mathfrak{H})\rangle_{E_2}-4\pi^{-2}z_{2c}{}^{0-2}\bar{z}^2\\&\times\left\{\left[\int_0^\infty d\xi e^{-\bar{z}^2\bar{m}\xi}(1+\xi^2)^{-1}\right]^2\right.\\&\left.+\left[\int_0^\infty d\xi e^{-\bar{z}^2\bar{m}\xi}\xi(1+\xi^2)^{-1}\right]^2\right\}\\&+f(m)\int_0^\infty d\xi e^{-\bar{z}^2\bar{m}\xi}(1+\xi^2)^{-1}.\end{aligned}\tag{3.18}$$

Since

$$\bar{z}^2\bar{m}=2z_{2c}{}^{0-1}z^2m, \tag{3.19}$$

(3.18) will agree with (8.96) of IV if

$$f(m)=2\pi^{-2}z_{2c}{}^{0-1}m^{-1}. \tag{3.20}$$

Therefore $\langle\mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2}$ is completely expressed as

$$\begin{aligned}\langle\mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2}=&\langle\mathfrak{M}_1{}^2(\mathfrak{H})\rangle_{E_2}+\tfrac{1}{2}[z_{1c}{}^{-1/2}(1+z_{1c})\pi^{-1}]^2N^{-2}\left\{\int_0^\infty d\phi_1\int_0^\infty d\phi_2\int_{-\infty}^\infty dq_1\int_{-\infty}^\infty dq_2\right.\\&\times\hat{U}(q_1,q_2)\left[\sin\bar{m}\phi_1\left(\frac{1}{e^{-q_1}+\phi_1{}^{-1}\bar{z}^2}-1\right)\frac{\sin\bar{m}\phi_2}{\phi_2+e^{q_2}\bar{z}^2}-\frac{\bar{z}^2\cos\bar{m}\phi_1\cos\bar{m}\phi_2}{(\phi_1e^{-q_1}+\bar{z}^2)(\phi_2e^{-q_2}+\bar{z}^2)}\right]\\&\left.+\bar{m}^{-1}\int_0^\infty d\phi\int_{-\infty}^\infty dq\hat{U}(q)\frac{\sin\bar{m}\phi}{\phi+e^q\bar{z}^2}\right\}+o(N^{-2}).\end{aligned}\tag{3.21}$$

In general, a study of (3.21) requires detailed knowledge of the two-variable function $\hat{U}(q_1,q_2)$. Fortunately, in the important special case $\mathfrak{H}=0$, (3.21) simplifies to an expression involving only the one-variable function $\hat{U}(q)$. To see this, consider first the term in the integral over ϕ_1 and ϕ_2 proportional to $\bar{z}^2$. Values of ϕ_1 and ϕ_2 greater than ϵ do not contribute in the $\bar{z}\rightarrow 0$ limit. Therefore, in this term ϕ_j is small, $\cos\bar{m}\phi\sim 1$, and this term is easily seen to cancel the similar integral that occurs in $\langle\mathfrak{M}_1{}^2(\mathfrak{H})\rangle_{E_2}$ of (3.8). In the remaining term in the integral over ϕ_1 and ϕ_2 of (3.21), $\bar{z}$ may be set equal to zero and the q_2 integral explicitly evaluated, since [(4.30) of II]

$$\int_{-\infty}^\infty dq_2\hat{U}(q_1,q_2)=\hat{U}(q_1). \tag{3.22}$$

Therefore,

$$\begin{aligned}\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2}=&\tfrac{1}{8}z_{1c}{}^{-1}(1+z_{1c})^2\pi^{-1}N^{-2}\\&\times\left\{\int_0^\infty d\phi\int_{-\infty}^\infty dq\hat{U}(q)(e^q-1)\sin\bar{m}\phi+\bar{m}^{-1}\right\}\\&+o(N^{-2}).\end{aligned}\tag{3.23}$$

We will, in the remainder of this paper, confine ourselves to this special case.

To analyze (3.23), it is useful to write

$$\begin{aligned}\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2}&=\tfrac{1}{8}z_{1c}{}^{-1}(1+z_{1c})^2\pi^{-1}N^{-2}\lim_{\epsilon\rightarrow 0}\int_0^\infty d\phi e^{-\epsilon\phi}[2K_\delta(\phi)]^{-1}\\&\quad\times(\sin\bar{m}\phi)\int_{-\infty}^\infty dq e^{(1-\delta)q-(\phi/2)(e^q+e^{-q})}+o(N^{-2})\\&=\tfrac{1}{8}z_{1c}{}^{-1}(1+z_{1c})^2\pi^{-1}N^{-2}\lim_{\epsilon\rightarrow 0}\int_0^\infty d\phi e^{-\epsilon\phi}\sin\bar{m}\phi\\&\quad\times K_{1-\delta}(\phi)/K_\delta(\phi)+o(N^{-2}),\end{aligned}\tag{3.24}$$

which, using (2.43), may be reexpressed as

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}=\tfrac{1}{8}z_{1c}^{-1}(1-z_{1c})^2\pi^{-1}N^{-2} \times\left\{-\tfrac{1}{2}\pi\delta-\lim_{\epsilon\to 0}\int_0^\infty d\phi e^{-\epsilon\phi}(\sin\bar{m}\phi) \times\frac{d}{d\phi}\ln K_\delta(\phi)\right\}. \quad (3.25)$$

Furthermore,

$$\lim_{\epsilon\to 0}\int_0^\infty d\phi e^{-\epsilon\phi}(\sin\bar{m}\phi)\frac{d}{d\phi}\ln K_{|\delta|}(\phi) =-\tfrac{1}{2}\pi|\delta|+(2i)^{-1}\lim_{\epsilon\to 0} \times\left\{\int_0^\infty d\phi e^{(i\bar{m}-\epsilon)\phi}\frac{d}{d\phi}\ln[\phi^{|\delta|}K_{|\delta|}(\phi)] -\int_0^\infty d\phi e^{-(i\bar{m}+\epsilon)\phi}\frac{d}{d\phi}\ln[\phi^{|\delta|}K_{|\delta|}(\phi)]\right\}. \quad (3.26)$$

In the first (second) integral, we may deform the contour of integration to the positive (negative) imaginary ϕ axis. Then, using

$$(e^{\pm\pi i/2}\xi)^{|\delta|}K_{|\delta|}(e^{\pm\pi i/2}\xi) =\tfrac{1}{2}\xi^{|\delta|}\pi[Y_{|\delta|}(\xi)\mp iJ_{|\delta|}(\xi)], \quad (3.27)$$

where $J_{|\delta|}(\xi)$ and $Y_{|\delta|}(\xi)$ are the standard Bessel functions of the first and second kind,[9] we find the desired result

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}=\tfrac{1}{8}z_{1c}^{-1}(1+z_{1c})^2\pi^{-1}N^{-2} \times\left\{\tfrac{1}{2}\pi[-\delta+|\delta|]+2\pi^{-1}\int_0^\infty d\xi e^{-\bar{m}\xi}\xi^{-1} \times[Y_{|\delta|}^2(\xi)+J_{|\delta|}^2(\xi)]^{-1}\right\}+o(N^{-2}). \quad (3.28)$$

In this form, it is clear that

$$\begin{aligned}\lim_{m\to\infty}\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}&=\tfrac{1}{8}z_{1c}^{-1}(1+z_{1c})^2N^{-2}|\delta| \quad &\text{if } T<T_c\\ &=0 &\text{if } T>T_c\\ &=[\mathfrak{M}_1{}^O(0+)]^2, & (3.29)\end{aligned}$$

where the last equation may be obtained from (2.33). This is exactly the value that is obtained in the Onsager lattice. This contrasts strongly with the value of $\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}$ obtained in Sec. 2 and is a vivid demonstration of the fact that $\mathfrak{S}_{1,1}(m,\mathfrak{H})$ is not a probability-1 object. It is expected that (3.29) should be equal to $\lim_{\mathfrak{H}\to 0+}\langle\mathfrak{M}_1{}^2(\mathfrak{H})\rangle_{E_2}$ where this limit is calculated directly from expression (3.8). This is indeed the case, as we will demonstrate in detail in Sec. 4.

[9] Reference 7, Vol. 2, p. 6.

In general, $\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2}$ cannot be expressed in terms of tabulated functions. When $|\delta|=\frac{1}{2}$ or $\frac{3}{2}$, however, more simplification is possible.

i. $|\delta|=\frac{1}{2}$

In this case

$$J_{1/2}{}^2(\xi)+Y_{1/2}{}^2(\xi)=2/\pi\xi, \quad (3.30)$$

and we find

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}=\tfrac{1}{8}z_{1c}^{-1}(1+z_{1c})^2 \times\{\pi^{-1}N^{-2}\tfrac{1}{2}\pi[-\delta+\tfrac{1}{2}]+\bar{m}^{-1}\}+o(N^{-2}). \quad (3.31)$$

To leading order in N the difference between $\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2}$ and $\langle\mathfrak{S}_{1,1}(\infty,0)\rangle_{E_2}$ is precisely equal to the leading term of the $T=T_c$, $\mathfrak{H}=0$ expansion of $\mathfrak{S}_{1,1}{}^O(m,0)$ for large m given by (8.89) of IV. Furthermore, (3.31) approaches its $\bar{m}\to\infty$ limit as $\bar{m}^{-1}$. But since $T\neq T_c$, this slow algebraic approach to the $\bar{m}\to\infty$ limit contrasts dramatically with the exponential approach to the $\bar{m}\to\infty$ limit exhibited by $\mathfrak{S}_{1,1}{}^O(m,0)$ and assumed by the critical-exponent description of correlation functions discussed in the Introduction.

ii. $|\delta|=\frac{3}{2}$

In this case

$$J_{3/2}{}^2(\xi)+Y_{3/2}{}^2(\xi)=(2/\pi\xi)[1+\xi^{-2}], \quad (3.32)$$

and, using (3.16), we find

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}=\tfrac{1}{8}z_{1c}^{-1}(1+z_{1c})^2\pi^{-1}N^{-2} \times\left\{\tfrac{1}{2}\pi[-\delta+\tfrac{3}{2}]+\bar{m}^{-1}-\int_0^\infty d\alpha(\alpha+1)^{-1}\sin\bar{m}\alpha\right\} +o(N^{-2}). \quad (3.33)$$

Letting $\alpha'\bar{m}^{-1}=\alpha+1$ we obtain

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2} =\tfrac{1}{8}z_{1c}^{-1}(1+z_{1c})^2\pi^{-1}N^{-2}\{\tfrac{1}{2}\pi[-\delta+\tfrac{3}{2}]+\bar{m}^{-1} -\sin\bar{m}\ \mathrm{Ci}\bar{m}+\cos\bar{m}\ \mathrm{si}\bar{m}\}+o(N^{-2}), \quad (3.34)$$

where $\mathrm{Ci}\bar{m}$ and $\mathrm{si}\bar{m}$ are defined as the sine and cosine integrals.[10] When $\bar{m}$ is small, this may usefully be written as

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}=\tfrac{1}{8}z_{1c}^{-1}(1+z_{1c})^2\pi^{-1}N^{-2} \times\left\{\tfrac{1}{2}\pi[-\delta+\tfrac{3}{2}]+\bar{m}^{-1}-\sin\bar{m} \times\left[\gamma+\ln\bar{m}+\sum_{n=1}^\infty\frac{(-1)^n\bar{m}^{2n}}{(2n)!2n}\right]-\cos\bar{m} \times\left[\tfrac{1}{2}\pi-\sum_{n=1}^\infty\frac{(-1)^n\bar{m}^{2n+1}}{(2n+1)!(2n+1)}\right]\right\}+o(N^{-2}). \quad (3.35)$$

To leading order in N $\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2}-\langle\mathfrak{S}_{1,1}(\infty,0)\rangle_{E_2}$ is clearly not *equal* to the leading term of the $T=T_c$, $\mathfrak{H}=0$ expansion of $\mathfrak{S}_{1,1}{}^O(m,0)$ given in IV but *approaches* it

[10] Reference 7, Vol. 2, p. 145.

as $\bar{m}\to 0$. When $\bar{m}$ is large, we have the asymptotic expansion

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2} \sim \tfrac{1}{8} z_{1c}^{-1}(1+z_{1c})^2 \pi^{-1} N^{-2} \times \{\tfrac{1}{2}\pi[-\delta+\tfrac{3}{2}] - \sum_{n=1}^{\infty}(-1)^n (2n)!\bar{m}^{-2n-1}\}. \quad (3.36)$$

Again we see that the approach to the $\bar{m}\to\infty$ limit is algebraic rather than exponential.

With the orientation provided by these two special cases, we turn to the general case and study the most interesting limiting cases of (3.28): (a) $\bar{m}$ fixed, $|\delta|\to\infty$; (b) δ fixed, $\bar{m}\to 0$; (c) $\delta\neq 0$ fixed, $\bar{m}\to\infty$; and (d) $\delta=0$, $\bar{m}\to\infty$.

(a) $\bar{m}$ fixed, $|\delta|\to\infty$. In this limiting case, when $|\delta|\sim N^2$, we expect the term which is independent of N to agree with the leading term of either the $T\to T_c$ limit of expansions (8.41) or (8.52) of IV of $\mathfrak{S}_{1,1}{}^O(m,0)$, which are valid for $m|T-T_c|\gg 1$, or with the leading term of the $t\to\infty$ ($t'\to\infty$) behavior of $\mathfrak{S}_{1,1}{}^O(m,0)$ given by expansion (8.87) or (8.88) of IV, which are valid for $m|T-T_c|=O(1)$. We do not expect to be able to reproduce more than the first term that depends on m of these expansions because our approximations to $\langle\mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2}$ neglects all terms of order $o(N^{-2})$.

To obtain the desired expansion, it is convenient to return to (3.24) and use the asymptotic expansion (2.44) to obtain

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2} \xrightarrow[|\delta|\to\infty]{} \tfrac{1}{8} z_{1c}^{-1}(1+z_{1c})^2 \pi^{-1} N^{-2} \times \left\{ -\tfrac{1}{2}\pi\delta + \lim_{\epsilon\to 0}\int_0^\infty d\phi\, e^{-\epsilon\phi}\sin\bar{m}\phi \times [\phi^{-1}(\phi^2+\delta^2)^{1/2} + \tfrac{1}{2}\phi(\phi^2+\delta^2)^{-1}] \right\}. \quad (3.37)$$

Since

$$\lim_{\epsilon\to 0}\int_0^\infty d\phi\, e^{-\epsilon\phi}\sin\bar{m}\phi\,\phi^{-1}(\phi^2+\delta^2)^{1/2} = \tfrac{1}{2}\pi|\delta| + |\delta|\int_1^\infty d\xi\,\xi^{-1}e^{-\bar{m}|\delta|\xi}(\xi^2-1)^{1/2} = \tfrac{1}{2}\pi|\delta| + |\delta|\int_{\bar{m}|\delta|}^\infty d\xi\,\xi^{-1}K_1(\xi) \quad (3.38)$$

and

$$\int_0^\infty d\phi\,\phi(\phi^2+\delta^2)^{-1}\sin\bar{m}\phi = \tfrac{1}{2}\pi e^{-\bar{m}|\delta|}, \quad (3.39)$$

we explicitly find

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2} \xrightarrow[|\delta|\to\infty]{} \tfrac{1}{8} z_{1c}^{-1}(1+z_{1c})^2 \pi^{-1} N^{-2} \times \left\{ -\tfrac{1}{2}\pi[\delta-|\delta|] + |\delta|\int_{\bar{m}|\delta|}^\infty d\xi\,\xi^{-1}K_1(\xi) + \tfrac{1}{4}\pi e^{-\bar{m}|\delta|} \right\}. \quad (3.40)$$

The first two terms in this expansion do not depend on N when $|\delta|\sim N^2$ and, as expected, agree with the leading term of (8.87) of IV if $T>T_c$ and the first two terms of (8.88) of IV if $T<T_c$.

(b) δ fixed, $\bar{m}\to 0$. In this case, also, we expect to make contact with IV. In particular, we expect that when $\bar{m}\sim N^{-2}$, the term in $\langle\mathfrak{S}_{1,1}(m,0)\rangle_{E_2} - \langle\mathfrak{S}_{1,1}(\infty,0)\rangle_{E_2}$ which is independent of N should agree with the $T=T_c$, $m\to\infty$ behavior of $\mathfrak{S}_{1,1}{}^O(m,0)$ given in (8.89) of IV. We have previously seen in the special case $|\delta|=\tfrac{1}{2}$ and $|\delta|=\tfrac{3}{2}$ that this is the case but that in the case of $|\delta|=\tfrac{3}{2}$ the approach to this limit is somewhat complicated by the presence of terms involving $\ln\bar{m}$.

To study this limit in the general case, we note[11] that for large ξ

$$J_{|\delta|}{}^2(\xi) + Y_{|\delta|}{}^2(\xi) \sim \frac{2}{\pi\xi}\sum_{k=0}^{\infty}\{1\cdot 3\cdots(2k-1)\}\Gamma(\tfrac{1}{2}+|\delta|+k) \times [\Gamma(\tfrac{1}{2}+|\delta|-k)]^{-1}2^{-k}\xi^{-2k} \sim (2/\pi\xi)\{1+\tfrac{1}{8}(4\delta^2-1)\xi^{-2}+\cdots\}, \quad (3.41)$$

where for our limited purpose we retain only the first two terms. We then write

$$\int_0^\infty d\xi\, e^{-\bar{m}\xi}(2/\pi)\xi^{-1}[Y_{|\delta|}{}^2(\xi)+J_{|\delta|}{}^2(\xi)]^{-1} = \int_0^\infty d\xi\, e^{-\bar{m}\xi}\{(2/\pi)\xi^{-1}[Y_{|\delta|}{}^2(\xi)+J_{|\delta|}{}^2(\xi)]^{-1} - 1 + \tfrac{1}{8}(4\delta^2-1)(1+\xi^2)^{-1}\} + \int_0^\infty d\xi\, e^{-\bar{m}\xi}\{1-\tfrac{1}{8}(4\delta^2-1)(1+\xi^2)^{-1}\}. \quad (3.42)$$

In the first integral, we write

$$e^{-\bar{m}\xi} = 1 - \bar{m}\xi + \tfrac{1}{2}\bar{m}^2\xi^2 + (e^{-\bar{m}\xi} - 1 + \bar{m}\xi - \tfrac{1}{2}\bar{m}^2\xi^2). \quad (3.43)$$

The last integral is of the same form as was studied in the special case $|\delta|=\tfrac{3}{2}$. Thus

$$\int_0^\infty d\xi\, e^{-\bar{m}\xi}(2/\pi)\xi^{-1}[Y_{|\delta|}{}^2(\xi)+J_{|\delta|}{}^2(\xi)]^{-1} = A_0(\delta) + \bar{m}A_1(\delta) + \tfrac{1}{2}\bar{m}^2 A_2(\delta) + o(\bar{m}^2) + \bar{m}^{-1} - \tfrac{1}{8}(4\delta^2-1)[\sin\bar{m}\,\mathrm{Ci}\bar{m} - \cos\bar{m}\,\mathrm{si}\bar{m}], \quad (3.44)$$

where

$$A_j(\delta) = \int_0^\infty d\xi\,\xi^j\{(2/\pi)\xi^{-1}[Y_{|\delta|}{}^2(\xi)+J_{|\delta|}{}^2(\xi)]^{-1} - 1 + \tfrac{1}{8}(4\delta^2-1)(1+\xi^2)^{-1}\}, \quad j=0,1,2. \quad (3.45)$$

[11] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1945), p. 449.

We may use the expansions of Ci and si and obtain

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2} \underset{\bar{m}\to 0}{\longrightarrow} \tfrac{1}{8} z_{1c}^{-1}(1+z_{1c})^2\pi^{-1}\mathcal{N}^{-2}$$
$$\times\{\tfrac{1}{2}\pi[-\delta+|\delta|]+\bar{m}^{-1}+A_0(\delta)$$
$$-\tfrac{1}{16}\pi(4\delta^2-1)-\tfrac{1}{8}(4\delta^2-1)\bar{m}\ln\bar{m}$$
$$+\bar{m}[A(\delta)-\tfrac{1}{8}(4\delta^2-1)(\gamma+1/18)]$$
$$+\bar{m}^2\tfrac{1}{2}A_2(\delta)+O(\bar{m}^2)\}. \quad (3.46)$$

As expected, when $\bar{m}\sim\mathcal{N}^{-2}$, the term independent of $\mathcal{N}$ agrees with (8.89) of IV.

(c) $\bar{m}\to\infty$, $\delta\neq 0$. The most unusual feature of (3.28) is its asymptotic behavior as $\bar{m}\to\infty$. We have already seen in the special cases $|\delta|=\frac{1}{2}$ and $|\delta|=\frac{3}{2}$ that this behavior is algebraic rather than exponential and is therefore not of the form assumed by the critical-exponent parametrization. To see that this behavior holds for general values of δ we approximate for small ξ and $|\delta|\neq 1$

$$[Y_{|\delta|}^2(\xi)+J_{|\delta|}^2(\xi)]^{-1}$$
$$=\pi^2[\Gamma(|\delta|)]^{-2}(\tfrac{1}{2}\xi)^{2|\delta|}\{1+2(\tfrac{1}{2}\xi)^2(1-|\delta|)^{-1}$$
$$+2(\tfrac{1}{2}\xi)^{2|\delta|}\cos(|\delta|\pi)\Gamma(-|\delta|+1)[\Gamma(|\delta|+1)]^{-1}$$
$$+O(\xi^{4|\delta|})+O(\xi^{2|\delta|+2})\}, \quad (3.47)$$

and thus find that (3.28) asymptotically becomes

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2} \underset{\bar{m}\to \infty}{\longrightarrow} \tfrac{1}{8} z_{1c}^{-1}(1+z_{1c})^2\mathcal{N}^{-2}$$
$$\times\{\tfrac{1}{2}[-\delta+|\delta|]+2[\Gamma(|\delta|)]^{-2}$$
$$\times[(2\bar{m})^{-2|\delta|}\Gamma(2|\delta|)+2(2\bar{m})^{-2|\delta|-2}$$
$$\times\Gamma(2|\delta|+2)(1-|\delta|)^{-1}+2(2\bar{m})^{-4|\delta|}$$
$$\times\cos(|\delta|\pi)\Gamma(1-|\delta|)\Gamma(4|\delta|)(\Gamma(1+|\delta|))^{-1}$$
$$+O(\bar{m}^{-6|\delta|})+O(\bar{m}^{-4|\delta|-2})]\}. \quad (3.48)$$

When $|\delta|=\frac{1}{2}$ or $\frac{3}{2}$, the terms in $\bar{m}^{-1}$ that are explicitly given here agree with the preceding results. We may study the case $|\delta|=1$ by letting $|\delta|\to 1$ in (3.48) and find

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}|_{|\delta|=1} \underset{\bar{m}\to \infty}{\longrightarrow} \tfrac{1}{8} z_{1c}^{-1}(1+z_{1c})^2\mathcal{N}^{-2}$$
$$\times\{\tfrac{1}{2}[-\delta+|\delta|]+\tfrac{1}{2}\bar{m}^{-2}$$
$$+\tfrac{1}{2}\bar{m}^{-4}[-3\ln\bar{m}+4]+O(\bar{m}^{-4})\}. \quad (3.49)$$

When $0<|\delta|\leq\frac{1}{2}$, while the leading term in (3.48) is still correct, the neglected terms of order $O(\bar{m}^{-6|\delta|})$ are now larger than the retained terms of order $O(\bar{m}^{-2|\delta|-2})$ and thus these higher terms are no longer meaningful. Indeed, when $|\delta|\to 0$, (3.48) loses its validity altogether.

(d) $\bar{m}\to\infty$, $\delta=0$. The final limiting case of $\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}$ to be considered is $\bar{m}\to\infty$ and $T=T_c$. This is the one temperature at which the critical-exponent description allows a spin-spin correlation function to approach its limiting value in a power-law fashion as parametrized by (1.4d). In the bulk of the two-dimensional Ising model, $\eta=\frac{1}{4}$, while on the boundary, $\eta=1$. However, we have just seen that, if $|\delta|$ is made sufficiently small, then in our random model, even if $T\neq T_c$, $\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}$ may be made to approach its $\bar{m}\to\infty$ value in a power-law fashion with a power as close to zero as we please. Therefore, it is expected that at T_c, $\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}$ will not be of the form (1.4d). To see that this is indeed so, we approximate for small ξ

$$[Y_0^2(\xi)+J_0^2(\xi)]^{-1}\sim[4\pi^{-2}(\ln\xi/A')^2+1]^{-1}$$
$$\sim\tfrac{1}{4}\pi^2(\ln\xi/A')^{-2}$$
$$\times\{1-\tfrac{1}{4}\pi^2(\ln\xi/A')^{-2}+\cdots\}, \quad (3.50)$$

where

$$A'=2e^{-\gamma+1}. \quad (3.51)$$

Therefore,

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}|_{\delta=0}$$
$$\sim\tfrac{1}{16}z_{1c}^{-1}(1+z_{1c})^2\mathcal{N}^{-2}\int_0^\infty d\xi e^{-\bar{m}\xi}\xi^{-1}(\ln\xi/A')^{-2}$$
$$\times\{1-\tfrac{1}{4}\pi^2(\ln\xi/A')^{-2}\}$$
$$=-\bar{m}\tfrac{1}{16}z_{1c}^{-1}(1+z_{1c})^2\mathcal{N}^{-2}\int_0^\infty d\xi e^{-\bar{m}\xi}$$
$$\times[(\ln\xi/A')^{-1}-\tfrac{1}{12}\pi^2(\ln\xi/A')^{-3}], \quad (3.52)$$

where to obtain the last expression we have integrated by parts. We now may let $x=\bar{m}\xi$ and expand the logarithms as

$$[\ln x/\bar{m}A']^{-1}\sim(-\ln\bar{m}A')^{-1}[1+(\ln x)(\ln\bar{m}A')^{-1}$$
$$+(\ln x)^2(\ln\bar{m}A')^{-2}+\cdots] \quad (3.53)$$

to obtain the desired result

$$\langle \mathfrak{S}_{1,1}(m,0)\rangle_{E_2}|_{\delta=0}\sim\tfrac{1}{16}z_{1c}^{-1}(1+z_{1c})^2\mathcal{N}^{-2}(\ln A'\bar{m})^{-1}$$
$$\times\{1-\gamma(\ln A'\bar{m})^{-1}+(\gamma^2-\tfrac{1}{12}\pi^2)(\ln A'\bar{m})^{-2}\}. \quad (3.54)$$

This vanishes more slowly than any power of $\bar{m}$ and is obviously not of the form (1.4d).

4. PROBABILITY DISTRIBUTION OF $\mathfrak{M}_1$

We now turn to the question of the probability distribution that describes the random variable $\mathfrak{M}_1(\mathfrak{H})$ in the thermodynamic limit $\mathfrak{M}\to\infty$, $\mathfrak{N}\to\infty$. A convenient way to discuss this distribution is by studying its moments $\langle\mathfrak{M}_1^n(\mathfrak{H})\rangle_{E_2}$. The first moment ($n=1$) was discussed in detail in Sec. 2. Only in the $\mathfrak{H}\to 0$ limit were the results particularly simple. Similarly, in Sec. 3 we discussed $\lim_{m\to\infty}\langle\mathfrak{S}_{1,1}(m,\mathfrak{H})\rangle_{E_2}$. An explicit answer was obtained for the case $\mathfrak{H}=0$, but a general discussion requires a detailed discussion of the two-variable function $\nu(x_1,x_2)$. However, our principal interest lies in this $\mathfrak{H}=0$ case because of the analogy $\mathfrak{M}_1(0+)$ has with $\lim_{m\to\infty}\langle S_m^{1/2}\rangle$, which was interpreted in II as a measure of the local magnetization in a row. For these reasons we will, in this paper, confine ourselves to the special case $\mathfrak{H}\to 0+$.

As we remarked in Sec. 3, we expect

$$\lim_{m\to\infty}\langle \mathfrak{S}_{1,1}(m,0+)\rangle_{E_2}=\lim_{\mathfrak{H}\to 0+}\langle \mathfrak{M}_1^2(\mathfrak{H})\rangle_{E_2}, \tag{4.1}$$

where the left-hand side is given by (3.29). We will directly verify this by computing the right-hand side starting from (3.8). In so doing we will need a few properties of the two-variable function $\nu(x_1,x_2)$ which have been discussed in II in a different context. We will here recast some of that analysis in a form which is immediately generalizable to a ν function of an arbitrary number of variables.

We begin by remarking that if in (3.8) we use the explicit form of $\hat{U}(q)$, Eq. (2.15), as was done in Sec. 2, we see that no term containing $\hat{U}(q)$ can contribute to the $\mathfrak{H}\to 0+$ limit. Therefore

$$\begin{aligned}\lim_{\mathfrak{H}\to 0+}\langle \mathfrak{M}_1^2(\mathfrak{H})\rangle_{E_2}&=[\tfrac{1}{2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}N^{-1}]^2\\&\times\lim_{\bar{z}\to 0+}\bar{z}^2\int_0^\infty d\phi_1\int_0^\infty d\phi_2\int_{-\infty}^\infty dq_1\int_{-\infty}^\infty dq_2\\&\times\hat{U}(q_1,q_2;\phi_1,\phi_2)[(\bar{z}^2+\phi_1e^{-q_1})^{-1}-(\phi_1+1)^{-1}]\\&\times[(\bar{z}^2+\phi_2e^{-q_2})^{-1}-(\phi_2+1)^{-1}]+o(N^{-2}).\end{aligned}\tag{4.2}$$

As in Sec. 2, the contributions to the ϕ_1 and ϕ_2 integrals from the region $\phi_1>\epsilon$ and $\phi_2>\epsilon$ give no contribution to the $\mathfrak{H}\to 0+$ limit. We therefore follow a reduction analogous to Sec. 2 by writing

$$q_j=q_j'-\ln\tfrac{1}{2}\phi_j \tag{4.3}$$

and

$$\phi_j=\sqrt{2}\bar{z}\alpha_j \tag{4.4}$$

to find

$$\begin{aligned}\lim_{\mathfrak{H}\to 0+}\langle \mathfrak{M}_1^2(\mathfrak{H})\rangle_{E_2}&=[2^{-1/2}z_{1c}^{-1/2}(1+z_{1c})\pi^{-1}N^{-1}]^2\\&\times\lim_{\bar{z}\to 0+}\int_0^{\epsilon/\bar{z}} d\alpha_1\int_0^{\epsilon/\bar{z}} d\alpha_2\int_{-\infty}^\infty dq_1'\int_{-\infty}^\infty dq_2'\\&\times\hat{U}(q_1'-\ln\bar{z}\alpha_1 2^{-1/2},\, q_2'-\ln\bar{z}\alpha_2 2^{-1/2};\, \sqrt{2}\bar{z}\alpha_1,\, \sqrt{2}\bar{z}\alpha_2)\\&\times(1+\alpha_1^2e^{-q_1'})^{-1}(1+\alpha_2^2e^{-q_2'})^{-1}+o(N^{-2}).\end{aligned}\tag{4.5}$$

From Sec. 4 of II we find that $\hat{U}(q_1,q_2)$ satisfies the partial differential equation [which is accurate to order $o(1)$ as $N\to\infty$]

$$\begin{aligned}\left[\frac{\partial}{\partial q_1}+\frac{\partial}{\partial q_2}\right]^2\hat{U}(q_1,q_2)&+\frac{\partial}{\partial q_1}\{\delta-\tfrac{1}{2}\phi_1[e^{-q_1}-e^{q_1}]\}\hat{U}(q_1,q_2)\\&+\frac{\partial}{\partial q_2}\{\delta-\tfrac{1}{2}\phi_2[e^{-q_2}-e^{q_2}]\}\hat{U}(q_1,q_2)=0.\end{aligned}\tag{4.6}$$

To compute the $\bar{z}\to 0+$ limit in (4.5) it is important to note that we only need to know $\hat{U}(q_1,q_2)$ in the region where q_1 and q_2 are large. In this region $\hat{U}(q_1,q_2)$ may be approximated by $\tilde{U}(q_1,q_2)$, which satisfies the simpler equation

$$\begin{aligned}\left[\frac{\partial}{\partial q_1}+\frac{\partial}{\partial q_2}\right]^2\tilde{U}(q_1,q_2)&+\frac{\partial}{\partial q_1}\{\delta+\tfrac{1}{2}\phi_1e^{q_1}\}\tilde{U}(q_1,q_2)\\&+\frac{\partial}{\partial q_2}\{\delta+\tfrac{1}{2}\phi_2e^{q_2}\}\tilde{U}(q_1,q_2)=0.\end{aligned}\tag{4.7}$$

If we note that

$$\begin{aligned}\phi_1e^{q_1}\frac{\partial}{\partial q_1}\delta(q_1+\ln\phi_1-q_2-\ln\phi_2)&\\+\phi_2e^{q_2}\frac{\partial}{\partial q_2}\delta(q_1+\ln\phi_1-q_2-\ln\phi_2)&=0,\end{aligned}\tag{4.8}$$

we find that a solution of (4.7) is

$$\begin{aligned}\tilde{U}(q_1,q_2)&=\text{const}\,\delta(q_1+\ln\phi_1-q_2-\ln\phi_2)\\&\times\exp[-\tfrac{1}{2}\delta(q_1+q_2)-\tfrac{1}{2}(\phi_1\phi_2)^{1/2}e^{(q_1+q_2)/2}].\end{aligned}\tag{4.9}$$

Now the exact $\hat{U}(q_1,q_2)$ must satisfy the subsidiary condition (3.22). When ϕ_1 and ϕ_2 are small and q_1 and q_2 are large, this gives a subsidiary condition on $\tilde{U}$ of

$$\int_{-\infty}^\infty dq_2\tilde{U}(q_1,q_2)=(\tfrac{1}{2}\phi_1)^{|\delta|}[\Gamma(|\delta|)]^{-1}e^{-\delta q_1-(\phi_1/2)e^{q_1}} \tag{4.10a}$$

and

$$\int_{-\infty}^\infty dq_1\tilde{U}(q_1,q_2)=(\tfrac{1}{2}\phi_2)^{|\delta|}[\Gamma(|\delta|)]^{-1}e^{-\delta q_2-(\phi_2/2)e^{q_2}}, \tag{4.10b}$$

where we have used the approximation to $K_\delta(\phi)$, Eq. (2.29). It is now easily seen that (4.9) will satisfy (4.10) if we choose the constant so that

$$\begin{aligned}\tilde{U}(q_1,q_2)&=[\Gamma(|\delta|)]^{-1}[\tfrac{1}{4}\phi_1\phi_2]^{|\delta|/2}\delta(q_1+\ln\phi_1-q_2-\ln\phi_2)\\&\times\exp[-\tfrac{1}{2}\delta(q_1+q_2)-\tfrac{1}{2}(\phi_1\phi_2)^{1/2}e^{(q_1+q_2)/2}].\end{aligned}\tag{4.11}$$

This is recognized as (4.56) of II written in a form that is suitable for the $\delta=O(1)$ region and that exhibits the symmetry in the q_1 and q_2 variables. The difficult task of II was to demonstrate that the particular solution (4.9) to the partial differential equation (4.7) is indeed the correct solution and to investigate the sense in which $\tilde{U}$ is an approximation to $\hat{U}$. The conclusion of that analysis is that for suitably limited purposes such as envisaged in (4.5) we may indeed replace $\hat{U}(q_1,q_2)$

by $\tilde{U}(q_1,q_2)$ of (4.11). Making this replacement, we find

$$\lim_{\mathfrak{H}\to 0+}\langle\mathfrak{M}_1{}^2(\mathfrak{H})\rangle_{E_2}=[2^{-1/2}z_{1c}{}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}]^2[\Gamma(|\delta|)]^{-1}\lim_{\bar{z}\to 0+}\bar{z}^{[|\delta|+\delta]}\int_0^{\epsilon/\bar{z}}d\alpha_1\int_0^{\epsilon/\bar{z}}d\alpha_2\int_{-\infty}^{\infty}dq_1'\int_{-\infty}^{\infty}dq_2'$$

$$\times\delta[q_1'-q_2'][\tfrac{1}{2}\alpha_1\alpha_2]^{[|\delta|+\delta]/2}\exp[-\tfrac{1}{2}\delta(q_1'+q_2')-e^{(q_1'+q_2')/2}](1+\alpha_1{}^2e^{-q_1'})^{-1}(1+\alpha_2{}^2e^{-q_2'})^{-1}+o(\mathcal{N}^{-2}). \quad (4.12)$$

If $\delta>0$,

$$\lim_{\mathfrak{H}\to 0+}\langle\mathfrak{M}_1{}^2(\mathfrak{H})\rangle_{E_2}=0. \quad (4.13)$$

If $\delta<0$, we replace $\epsilon/\bar{z}$ by ∞ and find

$$\begin{aligned}\lim_{\mathfrak{H}\to 0+}\langle\mathfrak{M}_1{}^2(\mathfrak{H})\rangle_{E_2}&=[2^{-1/2}z_{1c}{}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}]^2[\Gamma(|\delta|)]^{-1}\int_0^{\infty}dq\,e^{-\delta q-e^q}\left(\int_0^{\infty}d\alpha(1+\alpha^2e^{-q})\right)^2+o(\mathcal{N}^{-2})\\&=[2^{-3/2}z_{1c}{}^{-1/2}(1+z_{1c})\mathcal{N}^{-1}]^2\Gamma(1-\delta)/\Gamma(|\delta|)+o(\mathcal{N}^2)\\&=\lim_{m\to\infty}\langle\mathfrak{S}_{1,1}(m,0+)\rangle_{E_2},\end{aligned} \quad (4.14)$$

as expected.

It is now straightforward to generalize this procedure to study $\lim_{\mathfrak{H}\to 0+}\langle\mathfrak{M}_1{}^n(\mathfrak{H})\rangle_{E_2}$, at least if n is not too large. We may write out a complete expression for $\langle\mathfrak{M}_1{}^n(\mathfrak{H})\rangle_{E_2}$ when $\delta=O(1)$ and $\bar{z}=O(1)$ as was done in (3.8) in terms of $\hat{U}(q_1,\cdots,q_n)$ and $\hat{U}$ functions of a smaller number of variables. However, only that term involving $\hat{U}(q_1,\cdots,q_n)$ contributes to the $\bar{z}\to 0+$ limit and we have

$$\lim_{\mathfrak{H}\to 0+}\langle\mathfrak{M}_1{}^n(\mathfrak{H})\rangle_{E_2}=[\tfrac{1}{2}z_{1c}{}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}]^n\lim_{\bar{z}\to 0+}\bar{z}^n\int_0^{\infty}d\phi_1\cdots\int_0^{\infty}d\phi_n\int_{-\infty}^{\infty}dq_1\cdots\int_{-\infty}^{\infty}dq_n$$

$$\times\hat{U}(q_1,\cdots,q_n)\prod_{j=1}^{n}[(\bar{z}^2+\phi_je^{-q_j})^{-1}-(\phi_j+1)^{-1}]+o(\mathcal{N}^{-n}), \quad (4.15)$$

which, if we note that the region $\phi_j>\epsilon$ gives no contribution in the $\bar{z}\to 0$ limit, may be written using (4.3) and (4.4) as

$$\lim_{\mathfrak{H}\to 0+}\langle\mathfrak{M}_1{}^n(\mathfrak{H})\rangle_{E_2}=[2^{-1/2}z_{1c}{}^{-1/2}(1+z_{1c})\pi^{-1}\mathcal{N}^{-1}]^n\lim_{\bar{z}\to 0+}\int_0^{\epsilon/\bar{z}}d\alpha_1\cdots\int_0^{\epsilon/\bar{z}}d\alpha_n\int_{-\infty}^{\infty}dq_1'\cdots\int_{-\infty}^{\infty}dq_n'$$

$$\times\hat{U}(q_1'-\ln\bar{z}\alpha_12^{-1/2},\cdots,q_n'-\ln\bar{z}\alpha_n2^{-1/2};\sqrt{2}\bar{z}\alpha_1,\cdots,\sqrt{2}\bar{z}\alpha_n)\prod_{j=1}^{n}[(1+\alpha_j{}^2e^{-q_j'})^{-1}]+o(\mathcal{N}^{-n}). \quad (4.16)$$

We obtain a partial differential equation for $\hat{U}(q_1,\cdots,q_n)$ from the integral equation (2.47) of II for $\nu(x_1,\cdots,x_n)$ in a manner identical to that of Sec. 4 of II. We find

$$\left[\sum_{j=1}^{n}\frac{\partial}{\partial q_j}\right]^2\hat{U}(q_1,\cdots,q_n)+\sum_{j=1}^{n}\frac{\partial}{\partial q_j}\{\delta-\tfrac{1}{2}\phi_j[e^{-q_j}-e^{q_j}]\}\hat{U}(q_1,\cdots,q_n)=0, \quad (4.17)$$

where, in addition, $\hat{U}(q_1,\cdots,q_n)$ satisfies the subsidiary condition

$$\int_{-\infty}^{\infty}dq_j\hat{U}(q_1,\cdots,q_j,\cdots,q_n)=\hat{U}(q_1,\cdots,q_{j-1},q_{j+1},\cdots,q_n). \quad (4.18)$$

When all the q_j are large of the order $-\ln\phi_j$, we approximate $\hat{U}(q_1,\cdots,q_n)$ by $\tilde{U}(q_1,\cdots,q_n)$, which satisfies (4.17) with the terms in e^{-q_j} omitted. We then find that a solution to that equation which satisfies the subsidiary condition (4.18) for $\tilde{U}$ is

$$\tilde{U}(q_1,\cdots,q_n)=[\Gamma(|\delta|)]^{-1}[\prod_{j=1}^{n}\tfrac{1}{2}\phi_j]^{|\delta|/n}\times\prod_{j=1}^{n-1}\delta(q_j+\ln\phi_j-q_{j+1}-\ln\phi_{j+1})\times\exp\{-n^{-1}\delta\sum_{j=1}^{n}q_j-\tfrac{1}{2}[\prod_{j=1}^{n}\phi_j]^{1/n}\times\exp(n^{-1}\sum_{j=1}^{n}q_j)\}. \quad (4.19)$$

The major difference separating the special case $n=2$ of (4.11) from the general case $n>2$ is that in the general case we do not possess the detailed analysis of II that makes precise the sense in which $\tilde{U}$ is an approximation to $\hat{U}$. If, however, we are willing to accept (4.19) without an elaborate justification such as in II, it is a simple matter to use (4.19) in (4.16) to find

$$\lim_{\mathfrak{H}\to 0+} \langle \mathfrak{M}_1{}^n(\mathfrak{H})\rangle_{E_2}$$
$$=[2^{-3/2}z_{1c}{}^{-1/2}(1+z_{1c})N^{-1}]^n$$
$$\times\Gamma(\tfrac{1}{2}n-\delta)/\Gamma(|\delta|)+o(N^{-n}) \quad \text{if } T<T_c$$
$$=0 \qquad\qquad \text{if } T>T_c. \quad (4.20)$$

If $\delta\to-\infty$,

$$\langle\mathfrak{M}_1{}^n(0+)\rangle_{E_2}\to[\mathfrak{M}_1{}^O(0+)]^n+O(\delta^{-1}), \quad (4.21)$$

while if $\delta\to 0$,

$$\langle\mathfrak{M}_1{}^n(0+)\rangle_{E_2}$$
$$\sim[2^{-3/2}z_{1c}{}^{-1/2}(1+z_{1c})N^{-1}]^n|\delta|\Gamma(\tfrac{1}{2}n)+O(\delta^2). \quad (4.22)$$

We also note that the approximations leading to (4.20) are surely invalid for sufficiently large n. In particular, $\mathfrak{M}_1(0+)$ must, for any set $\{E_2\}$, be less than the value it would have if we replaced all $E_2(j)$ by $E_2{}^0$. We know from I that such a replacement raises T_c by $O(N^{-1})$. Since from IV we know that $\mathfrak{M}_1{}^O(0+)\sim\text{const}(T_c-T)^{1/2}$ as $T\to T_c{}^-$, we conclude that

$$\mathfrak{M}_1(0+)<\text{const}N^{-1/2}. \quad (4.23)$$

Therefore the approximations in (4.20) must fail for

$$n=O(N). \quad (4.24)$$

We now may use (4.20) to investigate $\mathfrak{P}(\mathfrak{M}_1)$, the probability that at $\mathfrak{H}=0$, $\mathfrak{M}_1(0)$ assumes a value $\mathfrak{M}_1$ in the interval $d\mathfrak{M}_1$. Clearly if $T>T_c$, $\mathfrak{P}(\mathfrak{M}_1)=\delta(\mathfrak{M}_1)$, so we restrict ourselves to $T<T_c$. Because of the limitation (4.24) we cannot hope to compute $\mathfrak{P}(\mathfrak{M}_1)$ exactly but it is not difficult to find an approximation to $\mathfrak{P}(\mathfrak{M}_1)$ in the sense that it will yield the moments (4.20). Define a scaled magnetization

$$\mathfrak{m}=2^{3/2}z_{1c}{}^{1/2}(1+z_{1c})^{-1}N\mathfrak{M}_1(0+) \quad (4.25)$$

and let

$$\mathfrak{p}(\mathfrak{m})d\mathfrak{m}=\mathfrak{P}(\mathfrak{M}_1)d\mathfrak{M}_1. \quad (4.26)$$

Then from (4.20)

$$\int_0^\infty d\mathfrak{m}\,\mathfrak{m}^n\mathfrak{p}(\mathfrak{m})=\Gamma(\tfrac{1}{2}n+|\delta|)/\Gamma(|\delta|). \quad (4.27)$$

From its definition and from (4.23) it is clear that

$$\mathfrak{p}(\mathfrak{m})=0 \quad \text{if} \quad \mathfrak{m}>\text{const}N^{1/2}. \quad (4.28)$$

If we ignore this restriction, then it is easily verified that

$$\mathfrak{p}(\mathfrak{m})=2e^{-\mathfrak{m}^2}\mathfrak{m}^{2|\delta|-1}/\Gamma(|\delta|) \quad (4.29)$$

will satisfy (4.27) for all n. This function is surely incorrect when $\mathfrak{m}=O(N^{1/2})$ because it fails to satisfy (4.28). However, if $n=O(1)$, the region of $\mathfrak{m}=O(N^{1/2})$ makes a negligible contribution to the integrals (4.26)

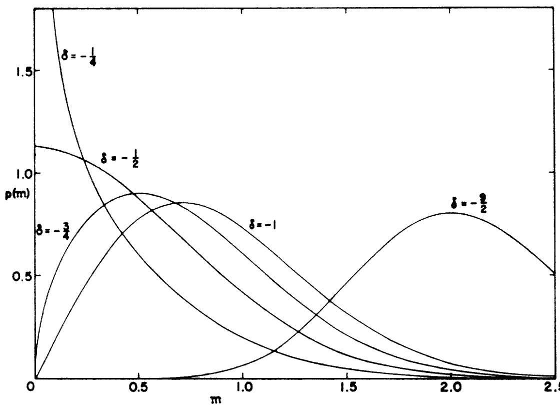

FIG. 2. Plot of the probability density function $\mathfrak{p}(\mathfrak{m})$ versus $\mathfrak{m}$ for several values of δ.

and hence we conclude that (4.29) is an accurate approximation to $\mathfrak{p}(\mathfrak{m})$ if $\mathfrak{m}=O(1)$.

Finally, we may use (4.29) to make contact with the discussion of Sec. 6 of II. We plot $\mathfrak{p}(\mathfrak{m})$ of (4.29) in Fig. 2 for several values of $|\delta|$ and note that while it diverges at $\mathfrak{m}=0$ if $|\delta|<\frac{1}{2}$, there is always a tail to the distribution that extends out to values of $\mathfrak{m}$ of order 1. It is this tail that is giving the important contributions to $\langle\mathfrak{M}_1{}^n\rangle$ for $n\geq 1$ even though the major contribution to the normalization integral comes from the integrable divergence at $\mathfrak{m}=0$. This peaking at $\mathfrak{m}=0$ and long tail were inferred indirectly in II by making use of the smallness of $\exp[\langle\ln M(0+)\rangle_{E_2}]$. In the present case we may compute $\langle\ln\mathfrak{M}_1(0+)\rangle_{E_2}$ as

$$\langle\ln\mathfrak{M}_1(0+)\rangle_{E_2}=\ln[2^{-3/2}z_{1c}{}^{-1/2}(1+z_{1c})N^{-1}]$$
$$+\int_0^\infty d\mathfrak{m}\,\mathfrak{p}(\mathfrak{m})\ln\mathfrak{m}. \quad (4.30)$$

Since $\ln\mathfrak{m}=o(\mathfrak{m})$ as $\mathfrak{m}\to\infty$, approximation (4.29) is accurate enough to give the terms in the integral of (4.30) that are $O(1)$ as $N\to\infty$. We therefore find

$$\langle\ln\mathfrak{M}_1(0+)\rangle_{E_2}=\ln[2^{-3/2}z_{1c}{}^{-1/2}(1+z_{1c})N^{-1}]$$
$$+\tfrac{1}{2}\Psi(|\delta|)+o(1). \quad (4.31)$$

This is valid for δ negative and of order 1. We now may let $\delta\to 0-$ and find[12]

$$\langle\ln\mathfrak{M}_1(0+)\rangle_{E_2}\sim-\ln N-\tfrac{1}{2}|\delta|^{-1}-\tfrac{1}{2}\gamma$$
$$+\ln[2^{-3/2}z_{1c}{}^{-1/2}(1+z_{1c})]+O(|\delta|). \quad (4.32)$$

Therefore we infer that the geometric mean of $\mathfrak{S}_{1,1}(\infty;0)$ as $\delta\to 0-$ is

$$N^{-2}e^{1/\delta}\tfrac{1}{8}z_{1c}{}^{-1}(1+z_{1c})^2e^{-\gamma}. \quad (4.33)$$

This is to be compared with the corresponding geometric mean of S_∞ in the bulk as $\delta\to 0-$, which is given by (6.36) of II. Both geometric means vanish exponentially rapidly as $\delta\to 0-$ and the geometric mean of $\mathfrak{S}_{1,1}(\infty,0)$

[12] Reference 7, Vol. 1, p. 47.

vanishes more rapidly than the geometric mean of S_∞ of the bulk.

5. CONCLUSION

When taken together, the results of this paper, as summarized in the Introduction, demonstrate dramatically that the magnetic properties of impure ferromagnets may be drastically different from those of pure ferromagnets. Indeed, it would be most desirable to compare the qualitative features of our results with precise experimental data as we did previously for the specific heat computed in I.[13] However, to our knowledge, sufficiently detailed measurements of magnetizations, magnetic susceptibilities, and asymptotic behavior of spin-spin correlation functions have not yet been made in the temperature regime near T_c where the specific heat of the sample rounds off. Only such experiments can decide the question of whether the description of magnetic phase transitions in terms of critical exponents, besides failing to provide a complete description of this random Ising model, also fails to provide a complete description of real, impure samples upon which all experiments are ultimately conducted.

ACKNOWLEDGMENT

I am greatly indebted to Professor T. T. Wu for most helpful discussions.

APPENDIX

Griffiths[5] has proven that in any Ising model whose interaction energies are never negative, if we increase (decrease) the strength of any bond, we cannot decrease (increase) the value of any spin-spin correlation function. Furthermore, the interaction of a magnetic field with some sites of the lattice is equivalent to having an extra spin which can only take on the value $+1$ and interacts with those same spins with a strength equal to that of the magnetic field. If we call this extra spin σ_0, then, if H interacts with all spins, the magnetization may be written as a sum of spin correlation functions

$$M(H)=\lim_{\mathfrak{M}\to\infty,\mathfrak{N}\to\infty}(4\mathfrak{M}\mathfrak{N})^{-1}\langle\sum_{j,k}\sigma_{j,k}\sigma_0\rangle\,, \qquad \text{(A1)}$$

where

$$-\mathfrak{M}+1<j<\mathfrak{M}\,, \qquad \text{(A2)}$$

$$-\mathfrak{N}+1<k<\mathfrak{N}\,, \qquad \text{(A3)}$$

and we impose cyclic boundary conditions in the horizontal direction. Now $M(H)$, being a property of the lattice as a whole, is a probability-1 object, so

$$M(H)=\langle\langle\sigma_{1,0}\sigma_0\rangle\rangle_{E_2}\,. \qquad \text{(A4)}$$

Consider any lattice out of the collection of lattices specified by a set of energies $\{E_2(j)\}$ where j satisfies (A2). The magnetic field $\mathfrak{H}$ interacts with the row $j=1$ only. Therefore, Griffiths's theorem says that for any $\{E_2(j)\}$, if H is numerically equal to $\mathfrak{H}$,

$$M(\mathfrak{H})\leq M(H)\,. \qquad \text{(A5)}$$

We are interested in the relation between $M(H)$ and

$$\mathfrak{M}_1(\mathfrak{H})=\langle\sigma_{1,0}\sigma_0\rangle_{HP}\,, \qquad \text{(A6)}$$

where $\langle\cdots\rangle_{HP}$ means a thermal average in an Ising lattice where the rows j satisfy

$$1\leq j\leq\mathfrak{M} \qquad \text{(A7)}$$

instead of (A2). If we replace all vertical bonds between the row $j=0$ and $j=1$ in the original lattice specified by (A2) by zero, we may apply Griffiths's theorem again to find

$$\mathfrak{M}_1(\mathfrak{H})\leq M(H)\,. \qquad \text{(A8)}$$

But this inequality holds for every collection of bonds $\{E_2\}$, so it holds for the average as well; so

$$\langle\mathfrak{M}_1(\mathfrak{H})\rangle_{E_2}\leq M(H)\,, \qquad \text{(A9)}$$

which proves (1.3a). A similar argument applied to $S_m(H)=\langle\sigma_{0,0}\sigma_{0,m}\rangle$ establishes (1.3b).

[13] B. M. McCoy and T. T. Wu, Phys. Rev. Letters **21**, 549 (1968).