PHYSICAL REVIEW VOLUME 187, NUMBER 3 15 NOVEMBER 1969

Size Effects in the Resonances of Nonlocal Helicon Waves

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(Received 20 May 1969)

The theory of nonlocal helicon waves, valid for metal samples considered as bulk material, is extended in order to include the effect of the finite thickness of such samples. The main problem is to solve an integro-differential equation for the electric field of the waves. This is done by an iteration procedure which yields the solution for the field as an infinite series, the first term of which is the field of the nonlocal bulk theory. With these solutions the response of the sample to an external ac magnetic field is calculated. The main results are that in a first approximation the resonance frequencies are nearly exactly the same as given by the nonlocal bulk theory, whereas the Q values of the resonances are reduced; also the phase of the response is changed. The reduction of the Q values is interpreted as an ac size effect in the resistivity. An experimental verification of this ac size effect is proposed in connection with the dc size effect.

I. INTRODUCTION

THE interesting phenomena connected with the propagation of electromagnetic waves in pure metals placed in a strong dc magnetic field have been the subject of many investigations in the last few years. The phenomena are observed in very pure samples of an uncompensated metal at low temperature, so that the metal has a Hall angle sufficiently near to $\frac{1}{2}\pi$. For instance, it is well established that low-frequency electromagnetic (EM) waves, called helicon waves because of their polarization properties,[1] can propagate with very low phase velocity and relatively little attenuation in such metals. The existence of the waves also shows up in helicon resonance experiments, in which the metal sample acts as a resonance cavity for helicon waves.

In a simple resonance experiment, a thin metal plate is placed simultaneously in a strong dc magnetic field H_0 along a z axis perpendicular to the plate and a weak homogeneous ac magnetic field $(h_{0x}\,e^{i\omega t},0,0)$ perpendicular to $\mathbf{H}_0$. The response of the sample to the exciting ac field, for instance, measured by the pickup in a coil with its axis in the xy plane, shows resonances at various frequencies which depend on the thickness of the plate. An alternative way to carry out the experiment is to study the oscillations in the decay of the eddy currents after a constant field h_x is suddenly switched off.

The mathematical description of the resonance experiment on an infinitely large plate of thickness $2d$ is simplified by the fact that all ac fields and currents are perpendicular to the z axis and depend only on z. According to Maxwell's equations, we then have, if we neglect the dielectric displacement current and put the magnetic permeability equal to 1,

$$\frac{\partial^2\mathbf{E}(z)}{\partial z^2}=\frac{4\pi i\omega}{c^2}\mathbf{j}(z)\,, \tag{1}$$

together with the boundary conditions

$$\begin{aligned}\frac{e}{i\omega}\left(\frac{\partial E_y(z)}{\partial z}\right)_{z=\pm d}&=(H_x(z))_{z=\pm d}=h_{0x}\,,\\ -\frac{e}{i\omega}\left(\frac{\partial E_x(z)}{\partial z}\right)_{z=\pm d}&=(H_y(z))_{z=\pm d}=0\,.\end{aligned} \tag{2}$$

The further steps are extremely simple if a local relationship between $\mathbf{j}$ and $\mathbf{E}$ is assumed to hold. We then have for isotropic materials

$$\mathbf{E}(z)=\rho\mathbf{j}(z)+R_H(\mathbf{j}(z)\times\mathbf{H}_0)\,, \tag{3}$$

in which R_HH_0 is assumed to be larger than ρ. In this paper R_H is taken to be positive, i.e., we assume electron conduction. The field $\mathbf{H}(z)$ that satisfies (1)–(3) is composed of a right circularly polarized field

$$H^+(z)=H_x(z)+iH_y(z)=h_{0x}\cos k^+z/\cos k^+d \tag{4a}$$

and a left circularly polarized field

$$H^-(z)=H_x(z)-iH_y(z)=h_{0x}\cos k^-z/\cos k^-d\,, \tag{4b}$$

with $k^\pm$ given by the dispersion relation

$$\omega=\frac{(k^\pm)^2c^2}{4\pi}(\pm R_HH_0+i\rho)\,, \tag{5}$$

where ω is real and $k^\pm$ complex.

The resonance character of the solution stems from $\cos k^+d$ being nearly zero for

$$\omega\cong[(2n+1)\pi/2d]^2R_HH_0c^2/4\pi\,.$$

Note that the left circularly polarized component does not show resonances; its field has a cutoff character rather than a standing-wave character. All this has been treated in great detail by Chambers and Jones.[2] The question to be discussed in this paper is what happens if the local relation between $\mathbf{j}$ and $\mathbf{E}$ no longer holds. The question is particularly relevant since the mean

[1] *Proceedings of the Symposium on Plasma Effects in Solids, Paris, 1964* (Academic Press Inc., New York, 1965).

[2] R. G. Chambers and B. K. Jones, Proc. Roy. Soc. (London) **A270**, 417 (1962).

free path l of the charge carriers in the pure metal concerned might easily be greater than the helicon wavelength $4d/(2n+1)$ at the nth resonance, while the local approximation is only justified if l is much smaller than the wavelength, i.e., $kl<1$. Experimentally, clear resonances are observed if $kl>1$, including the case $l/d>1$.[3]

Without explicitly solving the problem, Sheard[4] has theoretically discussed this point. Sheard suggests that the only difference between the local and the nonlocal theory is that in the latter theory the dispersion relation (5) must be replaced by the nonlocal dispersion relation derived for wave propagation in an infinitely large medium.[5] For a simple Sommerfeld free-electron model, the nonlocal dispersion relation reads, if $kl/\omega_c\tau<1$, $\omega_c\tau>1$, and $\omega_c\gg\omega$,

$$\omega=\frac{(k^{\pm})^2c^2}{4\pi}\frac{\pm\omega_c\tau+i}{\sigma_0}\left(1-\frac{(k^{\pm})^2l^2}{5\omega_c\tau^2}\right), \tag{6}$$

where $\omega_c=eH_0/mc$ is the cyclotron frequency, τ the relaxation time, and $\sigma_0=Ne^2\tau/m$, N being the electron concentration. The effect of the nonlocal correction factor $1-(k^{\pm})^2l^2/5\omega_c^2\tau^2$ has been experimentally demonstrated.[3,6–8]

Sheard's considerations do not take into account the specific influence of the boundaries on the electron velocity distribution. If there is diffuse surface scattering, there will be a thin layer with a thickness of the order $l/\omega_c\tau$ near the boundaries in which the average current will have a sizable component parallel to the electric field rather than flowing perpendicular to it as in the bulk. This will give rise to an additional damping and also to a modification of the EM field near the boundaries, so that, for instance (4) will no longer be correct. We shall deal with these effects in this paper. We wish to observed that the additional power dissipation near the surface is of the same type as that occurring in the dc size effect of a metal placed in a strong dc magnetic field. We shall have the opportunity to return to this point in the discussion.

It is well known that formal nonlocal ac transport theory in the presence of boundaries leads to an integrodifferential equation for the (electric) field. If there is only one boundary, the Laplace-transform technique allows one to obtain an explicit solution (cf. the theory of the anomalous skin effect[9,10]). If there are two boundaries, the situation is considerably more complex.[10–12] Therefore, in this paper we shall resort to an expansion of the solution in powers of a small parameter, for which we shall take $\omega\sigma_0 l^2/c^2\omega_c^3\tau^3$. In the local limit this parameter corresponds to $k^2l^2/\omega_c^2\tau^2$. In Sec. III we shall obtain the series expansion by solving the integrodifferential equation formulated in Sec. II, by means of an iteration procedure discussed in the Appendix. The interpretation of the results obtained in Sec. IV is given in Sec. V.

II. TRANSPORT THEORY

We shall consider a thin metallic sample of thickness $2d$ in a perpendicular magnetic field H_0 along the z axis. The surfaces of the sample are at $z=\pm d$. The electrons in the metal are described by a distribution function f which will be written in the form

$$f=f_0+f_1. \tag{7}$$

Here f_0 is the undisturbed Fermi distribution, which will be assumed to be spherical; f_1 is the deviation from equilibrium due to an external disturbance. In the presence of an ac electric field perpendicular to the z axis, varying in time as

$$\mathbf{E}(z,t)=\mathbf{E}(z)e^{i\omega t}, \tag{8}$$

and of the dc magnetic field H_0, the linearized Boltzmann equation takes the form

$$(1+i\omega\tau)f_1+\tau v_z\frac{\partial f_1}{\partial z}-\frac{eH_0\tau}{mc}\left(v_y\frac{\partial f_1}{\partial v_x}-v_x\frac{\partial f_1}{\partial v_y}\right)=e\tau\mathbf{E}\cdot\mathbf{v}\frac{\partial f_0}{\partial\epsilon}, \tag{9}$$

where ϵ is the energy and $-e$ the charge of the electrons. To solve Eq. (9), we write f_1 in the form

$$f_1=(v_xg_x+v_yg_y)\frac{\partial f_0}{\partial\epsilon}, \tag{10}$$

where g_x and g_y do not depend explicitly on v_x and v_y. Substitution of Eq. (10) into Eq. (9) leads to two coupled equations for g_x and g_y; if we introduce the complex quantities $g^{\pm}$ defined by

$$g^{\pm}=g_x\pm ig_y \tag{11}$$

and circularly polarized components

$$E^{\pm}=E_x\pm iE_y, \tag{12}$$

these equations for g_x and g_y can be written as independent equations for $g^{\pm}$:

$$\gamma^{\pm}g^{\pm}+\tau v_z\frac{\partial g^{\pm}}{\partial z}=e\tau E^{\pm},\qquad \gamma^{\pm}=1+i\omega\tau\mp i\omega_c\tau. \tag{13}$$

[3] M. T. Taylor, J. R. Merril, and R. Bowers, Phys. Letters **6**, 159 (1963).
[4] F. W. Sheard, Phys. Rev. **129**, 2563 (1963).
[5] J. J. Quinn and S. Rodriguez, Phys. Rev. **133A**, 1489 (1964).
[6] S. W. Hui, Phys. Letters **24A**, 265 (1967).
[7] M. T. Taylor, Phys. Rev. **137A**, 1145 (1965).
[8] J. L. Stanford and S. A. Stern, Phys. Rev. **144**, 534 (1966).
[9] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).
[10] E. H. Sondheimer, Advan. Phys. **1**, 1 (1952).
[11] P. Cotti, Physik Kondensierten Materie **3**, 40 (1964).
[12] G. A. Baraff, J. Math. Phys. **9**, 372 (1968); Phys. Rev. **167**, 625 (1968).

Introducing the boundary conditions for diffuse scattering at the surfaces $z=\pm d$,

$$g^{\pm}(d,v_z,v)_{v_z<0}=0\,,\quad g^{\pm}(-d,v_z,v)_{v_z>0}=0\,, \tag{14}$$

we find

$$g^{\pm}(z,v_z,v)_{v_z<0} = -\frac{e}{v_z}\int_z^d dz'\,E^{\pm}(z')\exp[-\gamma^{\pm}(z-z')/\tau v_z]\,, \tag{15a}$$

$$g^{\pm}(z,v_z,v)_{v_z>0} = \frac{e}{v_z}\int_{-d}^z dz'\,E^{\pm}(z')\exp[-\gamma^{\pm}(z-z')/\tau v_z]\,. \tag{15b}$$

The current density is

$$\mathbf{j}(z)=-2e\left(\frac{m}{h}\right)^3\int d\mathbf{v}\,\mathbf{v}f(z,\mathbf{v})\,. \tag{16}$$

Introducing polar coordinates v, φ, and ϑ with $v_z=v\cos\vartheta=vu$, we have for a degenerate electron gas

$$j^{\pm}(z)=\frac{3Ne^2}{m}\int_{-1}^{+1}du(1-u^2)g^{\pm}(z,v_Fu,v_F) \tag{17}$$

and find after some trivial manipulations

$$j^{\pm}(z)=\frac{3\sigma_0}{4l}\int_{-d}^{+d}dz'\,E^{\pm}(z')\int_1^\infty ds\left(\frac{1}{s}-\frac{1}{s^3}\right)\times\exp(-\gamma^{\pm}|z-z'|sl^{-1}) =\frac{3\sigma_0}{4l}\int_{-d}^{+d}dz'\,E^{\pm}(z')K^{\pm}(z-z')\,. \tag{18}$$

We now substitute the above equation for the current in the combined Maxwell equations given by Eq. (1); the result is the following integrodifferential equation for the electric field:

$$\frac{\partial^2E^{\pm}(z)}{\partial z^2}=\frac{4\pi i\omega\sigma_0}{c^2\gamma^{\pm}}\frac{3\gamma^{\pm}}{4l}\int_{-d}^{+d}dz'\,E^{\pm}(z')K^{\pm}(z-z')\,, \tag{19}$$

which will be solved in Sec. III. We want, however, to remark that the general solution for the field $E^{\pm}$ as a function of z is more than we need. In a resonance experiment one is interested in the average magnetic induction in the sample induced by an external ac magnetic field h. For an ac magnetic field of the form $(h_x(\omega),0,0)$ applied to the surfaces of the sample the average magnetic induction can be expressed in the following form (the magnetic permeability is assumed to be equal to 1):

$$B^{\pm}(\omega)=\frac{1}{2d}\int_{-d}^{+d}dz\,H^{\pm}(z,\omega)=\mu^{\pm}(\omega)h_x(\omega)\,, \tag{20}$$

where $\mu^{\pm}(\omega)$ is given by

$$\mu^{\pm}(\omega)=E^{\pm}(d)/d\cdot(\partial E^{\pm}(z)/\partial z)_{z=d}\,. \tag{21}$$

Here we have made use of the symmetry of the problem, i.e.,

$$H^{\pm}(z)=H^{\pm}(-z)\,,\quad E^{\pm}(z)=-E^{\pm}(-z)\,. \tag{22}$$

III. GENERAL METHOD

We must find a solution of the integrodifferential equation (19), satisfying the symmetry relation (22). In view of Eq. (21), it suffices to know the solution apart from a multiplication factor. It is shown in the Appendix that Eq. (19) can be solved by iteration:

$$E(z)=\sum_{n=0}^{\infty}E_n(z)\,, \tag{23a}$$

$$E_0(z)=\sin kz\,, \tag{23b}$$

$$E_n(z)=-\frac{k_0^2l^2}{\gamma^2}\frac{3\gamma}{4l}\int_{-d}^{+d}dz'\,E_{n-1}(z')\int_1^\infty ds\left(\frac{1}{s^3}-\frac{1}{s^5}\right)\times\frac{\exp(-\gamma|z-z'|sl^{-1})}{1+k^2l^2/\gamma^2s^2}\,, \tag{23c}$$

where $k_0^2=-4\pi i\omega\sigma_0/c^2\gamma$. Thus, k_0 is the helicon wavenumber corresponding to the frequency ω according to local theory, while k is determined by the nonlocal dispersion relation given in the Appendix [Eq. (A9) and in approximation by Eq. (A10)]. For convenience we have omitted the reference $\pm$ to the two polarizations in k, k_1, γ, and E.

It is shown in the Appendix that the iteration procedure converges for sufficiently small values of $|kl/\gamma|$, i.e., for values smaller than a number η of the order of but smaller than 1. In the same sense we may say that the procedure can only be used for $|k_0l/\gamma|<\eta_0$ or $\omega<\eta_0c^2|\gamma|^3/\sigma_0l^3$. The expressions for $E_1(z)$ and its derivative $E_1'(z)$ are easily calculated:

$$E_1(z)=\frac{3k_0^2l^2}{2\gamma^2}\int_1^\infty ds\left(\frac{1}{s^4}-\frac{1}{s^6}\right)\left(\frac{1}{1+k_0^2l^2/\gamma^2s^2}\right)^2\times[\sinh(\gamma szl^{-1})(E_0(d)+lE_0'(d)/\gamma s)\times\exp(-\gamma sdl^{-1})-E_0(z)]\,, \tag{24}$$

$$E_1'(z)=\frac{3k_0l}{2\gamma}\int_1^\infty ds\left(\frac{1}{s^3}-\frac{1}{s^5}\right)\left(\frac{1}{1+k_0^2l^2/\gamma^2s^2}\right)^2\times[\cosh(\gamma szl^{-1})(k_0E_0(d)+k_0lE_0'(d)/\gamma s)\times\exp(-\gamma sdl^{-1})-k_0lE_0'(z)/\gamma s]\,. \tag{25}$$

As already remarked, we are mainly interested in the values of $E(z)$ and $E'(z)$ for $z=d$.

It can be shown from Eqs. (23)–(25) that the general form of $E(d)$ and $E'(d)$ is given by

$$E(d)=E_0(d)P+[lE_0'(d)/\gamma]Q, \quad (26)$$

$$E'(d)=E_0'(d)R+k_0E_0(d)(k_0l/\gamma)S, \quad (27)$$

where

$$P=1+O(k_0^2l^2/\gamma^2), \quad (28)$$

$$Q=O(k_0^2l^2/\gamma^2), \quad (29)$$

$$R=1+O(k_0^2l^2/\gamma^2), \quad (30)$$

$$S=\frac{3}{16}+\frac{3}{4}\int_1^\infty ds\left(\frac{1}{s^3}-\frac{1}{s^5}\right)\times\exp(-2\gamma sdl^{-1})+O(k_0^2l^2/\gamma^2). \quad (31)$$

IV. RESONANCES OF THIN METALLIC SAMPLE

We are now able to calculate the response to an external excitation of a thin metallic sample, placed in a strong dc magnetic field; the configuration is that given in the previous sections. According to Eqs. (20) and (21), the average magnetic induction in the sample induced by an ac magnetic field can be expressed in terms of a response function $\mu(\omega)$ given by

$$\mu(\omega)=E(d)/d\times E'(d). \quad (32)$$

After substitution of Eqs. (26) and (27) into Eq. (32), we can write

$$\mu(\omega)=\frac{A\tan(kd-\phi)}{k_0d}+B, \quad (33)$$

where A, B, and ϕ are given by

$$A=1+O(k_0^2l^2/\gamma^2), \quad (34)$$

$$B=(l/d\gamma)[1+O(k_0^2l^2/\gamma^2)], \quad (35)$$

$$\phi=\tan^{-1}\left(\frac{k_0l}{\gamma}\frac{S}{R}\right). \quad (36)$$

With the aid of

$$\tan z=\sum_{n=1,3,5,\cdots}^{\infty}\frac{2z}{z^2-(n\pi/2)^2}, \quad (37)$$

Equation (33) can be written as

$$\mu(\omega)=\frac{2A(kd-\phi)}{k_0d}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{(kd-\phi)^2-(n\pi/2)^2}+B. \quad (38)$$

In the local limit $k_0l/\gamma\rightarrow 0$ this expression reduces to the well-known expression[2]

$$\mu(\omega)_{\mathrm{loc}}=\sum_{n=1,3,\cdots}^{\infty}\frac{2}{k_0^2d^2-(n\pi/2)^2}. \quad (39)$$

The right-hand side of Eq. (38) is a function of the (real) variable ω, since A, B, ϕ, and k depend on k_0, and $k_0^2=-4\pi i\omega\sigma_0/c^2\gamma$. Since in practice $\omega\ll\omega_c$, we may neglect the ω dependence of γ. For the right circularly polarized mode we then have $\gamma=1-i\omega_c\tau$, and $\mu^+(\omega)$ shows resonant character near the frequencies

$$\omega_n=(n\pi/2d)^2c^2\omega_c\tau/\sigma_0$$

provided $\omega_c\tau>1$. Similarly, $\mu^-(\omega)$ shows resonances near $\omega=-\omega_n$. We must remember, however, that Eq. (38) is valid only for sufficiently low frequencies:

$$\omega<c^2|\gamma|^3/\sigma_0l^2.$$

We now want to evaluate the position and the width of the resonances occurring in this low-frequency region. For that purpose we retain in the denominators the first relevant terms by which $kd-\phi$ differs from k_0d. In the frequency range considered we use

$$k^2=k_0^2\cdot(1-k_0^2l^2/5\gamma^2)$$

and $\phi=k_0lS/\gamma$. Since near the resonances, $l/2d\gamma\cong kl/n\pi\gamma$, we may assume $l/2d\gamma<1$ and hence approximate S by $\frac{3}{16}$. Indeed, from Eq. (31) and $l/2d\gamma<1$ one easily concludes that

$$S=3/16+O(l^2/2^2d^2\gamma^2)+O(k_0^2l^2/\gamma^2). \quad (40)$$

Thus, in this approximation we have

$$(kd-\phi)^2=k_0^2d^2\left[1-\frac{k_0^2l^2}{5\gamma^2}-\frac{3l}{8d\gamma}-\frac{9l^2}{256d^2\gamma^2}+\cdots\right]. \quad (41)$$

We now insert Eq. (41) into Eq. (38) for $\mu(\omega)$ and consider the resulting expression as a function of a complex variable $\hat{\omega}$. The function has an infinite number of poles at $\hat{\omega}=\hat{\omega}_n$, $n=1, 3, 5, \ldots, \infty$. We shall separate out the contributions of those poles, $n=1, 3, \ldots, M$, within a unit circle of radius $c^2|\gamma|^3/\sigma_0l^2$, since only these poles can, generally speaking, determine any resonances found in the relevant range of frequecies ω. The remaining part we shall collect in a term $T(\omega)$ that has a smooth background character in the relevant frequency range. The position of the poles, $n=1, 3, \ldots, M$, can be most easily found by observing that in the local limit $k_0^2=(n\pi/2d)^2$ at such a pole. With the aid of Eq. (41), and making use of the definition of $k_0^2=-4\pi i\omega\sigma_0/c^2\gamma$, we find that

$$\frac{4\pi i\hat{\omega}_n\sigma_0}{c^2\gamma}\left(1-\frac{n\pi}{2}\frac{l^2}{5d^2\gamma^2}-\frac{3l}{8d\gamma}+\frac{9l^2}{256d^2\gamma^2}+\cdots\right)+\left(\frac{n\pi}{2d}\right)^2=0, \quad (42)$$

so that Eq. (38) can be written as

$$\mu(\omega)=\sum_{n=1,3,\cdots}^{M}\frac{2\hat{A}(\hat{k}d-\hat{\phi})}{\hat{k}_0d}\left(\frac{2}{n\pi}\right)^2\frac{\hat{\omega}_n}{\omega-\hat{\omega}_n}\times[1+O(\hat{k}_0^2l^2/\gamma^2)]+T(\omega). \quad (43)$$

Here $\hat{A}$, $\hat{k}$, $\hat{k}_0$, and $\hat{\phi}$ are the values of A, k, k_0, and ϕ at the complex eigenfrequencies $\hat{\omega}_n$. Note that while $\hat{\omega}_n$ is correctly given by Eq. (42) to the order in which this equation is written out, the terms in (43) are correct apart from contributions of the order $(kl/\gamma)^2$. In this approximation it is correct to put $A=1$, $k_0d=\frac{1}{2}n\pi$, and $kd-\phi=(\frac{1}{2}n\pi)\times(1-3l/16d\gamma)$. We thus have

$$\mu^{\pm}(\omega)=2\left(1-\frac{3l}{16d\gamma^{\pm}}\right)\times\sum_{n=1,3,\cdots}^{M}\left(\frac{2}{n\pi}\right)^2\frac{\hat{\omega}_n{}^{\pm}}{\omega-\hat{\omega}_n{}^{\pm}}+T^{\pm}(\omega), \quad (44)$$

where we have explicitly written out as a reminder the $\pm$ sign referring to the right and left circularly polarized modes. Quite generally, we have

$$[\mu^+(\omega)]^*=\mu^-(-\omega). \quad (45)$$

We shall now be interested in the component of the magnetic induction in a direction that makes and angle ϑ with the direction of excitation. We have

$$\bar{B}_\vartheta=\bar{B}_x\cos\vartheta+\bar{B}_y\sin\vartheta =\tfrac{1}{2}[\mu^+\exp(-i\vartheta)+\mu^-\exp(i\vartheta)]h_x=\mu_\vartheta h_x. \quad (46)$$

Writing

$$1-3l/16d\gamma^{\pm}=r\exp(\mp i\varphi), \quad (47)$$

$$\hat{\omega}_n=\pm\omega_n\exp(\pm i\chi), \quad (48)$$

we find, using Eq. (44), that

$$\mu_\vartheta(\omega)=r\cos(\varphi+\vartheta)\times\sum_{n=1,3,\cdots}^{M}\left(\frac{2}{n\pi}\right)^2\frac{-1+2iQ\omega_n/\omega}{1+iQ(\omega/\omega_n-\omega_n/\omega)}+r\sin(\varphi+\vartheta)\times\sum_{n=1,3,\cdots}^{M}\left(\frac{2}{n\pi}\right)^2\frac{\cot\chi}{1+iQ(\omega/\omega_n-\omega_n/\omega)}+T_\varphi(\omega), \quad (49)$$

where

$$Q=1/2\sin\chi=\tfrac{1}{2}(1+\cot^2\chi)^{1/2}. \quad (50)$$

It is easily shown that in the approximation used, r, φ, ω_n, and Q are given by

$$r=\left(1-\frac{2\beta}{1+\omega_c{}^2\tau^2}+\frac{\beta^2}{1+\omega_c{}^2\tau^2}\right)^{1/2}, \quad (51)$$

$$\varphi=\tan^{-1}\left(\frac{\beta\omega_c\tau}{1+\omega_c{}^2\tau^2-\beta}\right), \quad (52)$$

$$\omega_n=\left(\frac{n\pi}{2d}\right)^2\frac{c^2}{4\pi\sigma_0}(1+\omega_c{}^2\tau^2)^{1/2}\times\left[1+\frac{1}{5}\left(\frac{n\pi l}{2d\omega_c\tau}\right)^2-\frac{2\beta}{1+\omega_c{}^2\tau^2}+\frac{\beta^2}{1+\omega_c{}^2\tau^2}\right]^{-1}, \quad (53)$$

$$Q=(1+\omega_c{}^2\tau^2)^{1/2}/2(1+2\beta), \quad (54)$$

where we have introduced the parameter β:

$$\beta=\tfrac{3}{8}l/2d. \quad (55)$$

By taking the limit $\beta\rightarrow0$, one obtains the formula of the nonlocal bulk theory, whereas the results of the local theory are obtained if one takes also the limit $\hat{k}l/\omega_c\tau=n\pi l/2d\omega_c\tau\rightarrow0$.

V. DISCUSSION

The results of this paper differ from those of the local theory in two aspects.

(a) There are corrections due to the nonlocal bulk effect. These corrections depend on $\hat{k}^2l^2/\gamma^2$ and are directly and only a consequence of the nonlocal relationship between k and ω [see Eq. (A9)]. In our approximation they show up only in the expression for the resonance frequencies and are the same as already given by Sheard.

(b) There are corrections which depend on $\beta\propto l/d$ and which therefore may be called size-effect corrections. They are caused by the modification of the phase and amplitude of the field and currents at the boundaries of the sample. These modifications can be seen, for instance, in the expression for the electric field $E_1(z)$. Besides a part varying as $E_0=\sin kz$, $E_0(z)$ contains additional terms, which at distances $\zeta>v_F/\omega_c$ from the surface fall off as $(v_F/\zeta\omega_c)^2\exp(-\zeta/l)$ and oscillate with wave vector ω_c/v_F.

Gantmakher and Kaner[13] have shown that surface excitations of this kind can be observed directly if one chooses the frequency or magnetic field such that $|kl/\gamma|>1$; the helicon wave is then damped out in a skin layer of thickness $\delta<v_F/\omega_c<d,l$ and only the surface excitations propagate over distances of the order l.

In principle, the size-effect corrections show up in the amplitude, phase, resonance frequencies, and Q values of the resonances; in practice, they are only important for the phase and Q values.

The corrections in the amplitude are small, of the order of $\beta/\omega_c{}^2\tau^2$, $\beta^2/\omega_c{}^2\tau^2$, where $\beta/\omega_c\tau<|kl/\gamma|<1$ and $\omega_c\tau>1$, and not uniquely determined because of the nonresonant part of the response function $T_\vartheta(\omega)$.

The same argument of smallness applies to the size-effect corrections in ω_n, where $\hat{k}^2l^2/5\omega_c{}^2\tau^2>\beta^2/\omega_c{}^2\tau^2$, $\beta/\omega_c{}^2\tau^2$. This is in accordance with the experimental results that the resonance frequencies are very well described by using the nonlocal dispersion relation and for $kl/\omega_c\tau$ up to 0.8 also with the numerical results we have obtained; see the Appendix.

The phase shift φ, however, which in our approximation is equal to $\beta/\omega_c\tau$ if $\beta<\omega_c{}^2\tau^2$, should be a measurable

[13] V. F. Gantmakher and E. A. Kaner, Zh. Eksperim. i Teor. Fiz. **48**, 1572, 1965 [English transl.: Soviet Phys.—JETP **21**, 1053 (1965)].

quantity; it can be determined by measuring the angle ϑ at which the (transverse) response is exactly in phase with the exciting field, $\varphi=90°-\vartheta$. It should be noted that this phase shift is a new phenomenon, which is entirely due to the size effect. The greatest effect, however, is the reduction of the Q's of the resonances by the factor $(1+2\beta)^{-1}$. This should be easily measurable if $\omega_c\tau>1$ and provided $\omega_c\tau$ is known from other experimental data. If, on the other hand, one wishes to determine $\omega_c\tau$ from the Q values of nonlocal helicon resonances, size-effect corrections have to be taken into account. The size-effect correction in Q is of such a nature that it is as if the bulk resistivity were increased by a factor $1+2\beta$. This may be compared with the size effect in the dc resistivity. In that case the size effect makes the sample behave as if its resistivity were increased by a factor $1+\beta$, as is well known from Sondheimer's work[10] and also follows immediately from Eq. (18) with E independent of z. To understand this difference we have to resort to the picture already sketched in the Introduction. An electron scattered diffusely at the surface starts drifting in phase with the electric field rather than flowing perpendicular to it, increasing the resistivity and giving rise to additional absorption in a surface layer of thickness v_F/ω_c. This applies to both the dc and ac situations, where for the same value of the electric field at $z=d$ the additional absorption is the same. However, apart from the disturbances at the surfaces, $j_{\rm dc}(E_{\rm dc})$ is uniform over the thickness of the sample, whereas $j_{\rm ac}(E_{\rm ac})=j_{\rm dc}(E_{\rm dc})\times\sin kz/\sin kd$. Since now in resonance $\langle j_{\rm ac}E_{\rm ac}\rangle_{\rm av}=\frac{1}{2}j_{\rm dc}E_{\rm dc}$, the over-all bulk absorption in the ac situation is half that in the dc situation. The relative importance of the additional absorption in the surface layers is therefore twice as large, giving a size effect twice as large.

We suggest that the existence of the ac size effect can be shown by measuring the Q of the fundamental resonance of a single metal plate and that of the fundamental resonance of a flat box consisting of two of these plates, described in Ref. 14.

According to our theory the Q of the single plate should in the limit $l/2d>1$, $l/2d\omega_c\tau<1$ be given by $4\omega_c\tau\times 2d/3l$, whereas Q the of the flat box, which for the fundamental is determined by the dc size effect, is in this limit $8\omega_c\tau\times 2d/3l$, being twice as large. One could of course also compare the Q of the fundamental resonance of the flat box with the Q of the first harmonic, since the latter is determined by the ac size effect. Thus it seems that the only advantage of the use of helicon waves for the determination of the size effect is that it allows a contactless measurement of the ac and dc size effects.

[14] C. A. A. J. Greebe, W. F. Druyvesteyn, and W. J. A. Goossens, Phys. Letters **24A**, 727 (1967).

APPENDIX

The integrodifferential equation to be solved is

$$\frac{\partial^2 E(z)}{\partial z^2}=-k_0^2\frac{3\gamma}{4l}\int_{-d}^{+d}dz'\,E(z')K(z-z'),\quad -d\le z\le +d \tag{A1}$$

with $k_0^2=-4\pi i\omega\sigma_0/c^2\gamma$ and $K(z)$ given by

$$K(z)=\int_1^\infty ds\left(\frac{1}{s}-\frac{1}{s^3}\right)\exp(-\gamma|z|sl^{-1}). \tag{A2}$$

Equation (A1) can be written as

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)E(z)=\int_{-d}^{+d}dz'\left(-k_0^2\frac{3\gamma}{4l}K(z-z')+k^2\delta(z-z')\right)E(z'). \tag{A3}$$

The above equation can be solved by iteration:

$$E(z)=\sum_{n=0,1,\ldots}^{\infty}E_n(z), \tag{A4}$$

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)E_0(z)=0, \tag{A5}$$

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)E_n(z)=\int_{-d}^{+d}dz'\,E_{n-1}(z')\times\left(-k_0^2\frac{3\gamma}{4l}K(z-z')+k^2\delta(z-z')\right),\quad n=1,2,\ldots,\infty. \tag{A6}$$

We have

$$E_0(z)=\sin kz, \tag{A7}$$

while $E_n(z)$ can be defined in terms of the function $E_{n-1}(z)$ by

$$E_n(z)=-\frac{k_0^2l^2}{\gamma^2}\frac{3\gamma}{4l}\int_{-d}^{+d}dz'\,E_{n-1}(z')\int_1^\infty dz\left(\frac{1}{s^3}-\frac{1}{s^5}\right)\times\frac{\exp(-\gamma s|z-z'|l^{-1})}{1+k^2l^2/\gamma^2s^2}, \tag{A8}$$

provided we choose

$$k^2=k_0^2\times\frac{3}{2}\int_1^\infty ds\left(\frac{1}{s^2}-\frac{1}{s^4}\right)\frac{1}{1+k^2l^2/\gamma^2s^2}, \tag{A9}$$

which is the nonlocal dispersion relation. For $|k^2l^2/\gamma^2|<1$ this relation can be approximated by

$$k^2=k_0^2\left(1-\frac{k_0^2l^2}{5\gamma^2}\right), \tag{A10}$$

from which it is seen that in this limit $k^2 \cong k_0^2$. To prove the convergence of the iteration procedure we have to consider Eq. (A8). From that equation it follows that

$$|E_n(z)| \le \left|\frac{k_0^2 l^2}{\gamma^2}\right| \left|\frac{3\gamma}{4l}\right| |E_{n-1}(z')|_{\max} \times \int_{-d}^{+d} dz' \left| \int_1^{\infty} ds \frac{s^2-1}{s^5} \frac{\exp(-\gamma|z-z'|sl^{-1})}{1+k^2l^2/\gamma^2 s^2} \right|. \quad \text{(A11)}$$

We replace the integration over s between 1 and ∞ by an integration between 0 and ∞, with respect to a new real variable t defined by $s=1+\gamma^{-1}|\gamma|t$. Along the new path of integration we have

$$|(s+1)s^{-1}| \le 2, \quad \text{(A12)}$$

$$|s^4| \ge (1+t^2)^2, \quad \text{(A13)}$$

$$|\gamma l^{-1}| \int_{-d}^{+d} dz' \exp(-\gamma|z-z'|sl^{-1}) < 2/t, \quad \text{(A14)}$$

$$|1+k^2l^2/\gamma^2 s^2| > 1-|k^2l^2/\gamma^2|/(1+t^2), \quad \text{(A15)}$$

so that

$$|E_n(z)| \le |E_{n-1}(z')|_{\max} \times \left|3\frac{k_0^2 l^2}{\gamma^2}\right| \int_0^{\infty} dt \frac{1}{1+t^2} \times 1/(1-|k_0^2 l^2/\gamma^2|/(1+t^2)). \quad \text{(A16)}$$

From this expression and (A9) it is obvious that the iteration procedure converges for sufficiently small values of $|k_0^2 l^2/\gamma^2|$. For instance, a rough estimate shows that for $|kl/\gamma| < 0.5$ the convergence is guaranteed. Explicit formulas suggest that the first-order approximation carried through in this paper is still reliable for this value.

In order to see up to which value of $|kl/\gamma|$ the theory makes sense and gives reliable results, we have calculated on the machine in the limit $\omega_c\tau \to \infty$ the exact value of the real and imaginary parts of $\mu^{\pm}$ for different values of the dimensionless parameters $K=k^+v_F/\omega_c$ and $D=d\omega_c/v_F$. The limit $\omega_c\tau \to \infty$ was chosen for mathematical convenience; physically it means that there is only damping due to the surfaces. For practical reasons we have only calculated the fundamental resonance, i.e., $K \cong \pi/2D$. The exact values of $\mu^{\pm}$ have been obtained by solving numerically the two sets of coupled equations for the real and imaginary parts of $E^{\pm}(z)$. These coupled equations are obtained from Eq. (A1) with $(l/\gamma^{\pm})_{\omega_c\tau\to\infty} = \pm iv_F/\omega_c$; note that in this problem v_F/ω_c appears as the characteristic length instead of $l=v_F\tau$. We have also calculated two types of approximate values for $\mu^{\pm}$. In the first type of approximation $\mu^{\pm}$ was calculated from $E(d)$ and $E'(d)$ given by the nonlocal bulk theory, i.e.,

$$\mu^{\pm} = \frac{\sin k^{\pm}d}{k^{\pm}d \cos k^{\pm}d}, \quad \text{(A17)}$$

where $k^{\pm}$ is exactly calculated from Eq. (A9). Therefore, the effect of the nonlocal dispersion relation is fully accounted for.

In the second type of approximation the values of $\mu^{\pm}$ were calculated from

$$\mu^{\pm} = \frac{\sin k^{\pm}d}{k^{\pm}d \cos k^{\pm}d + k_0^{\pm}d(3k_0^{\pm}l/16\gamma^{\pm}) \sin k^{\pm}d}, \quad \text{(A18)}$$

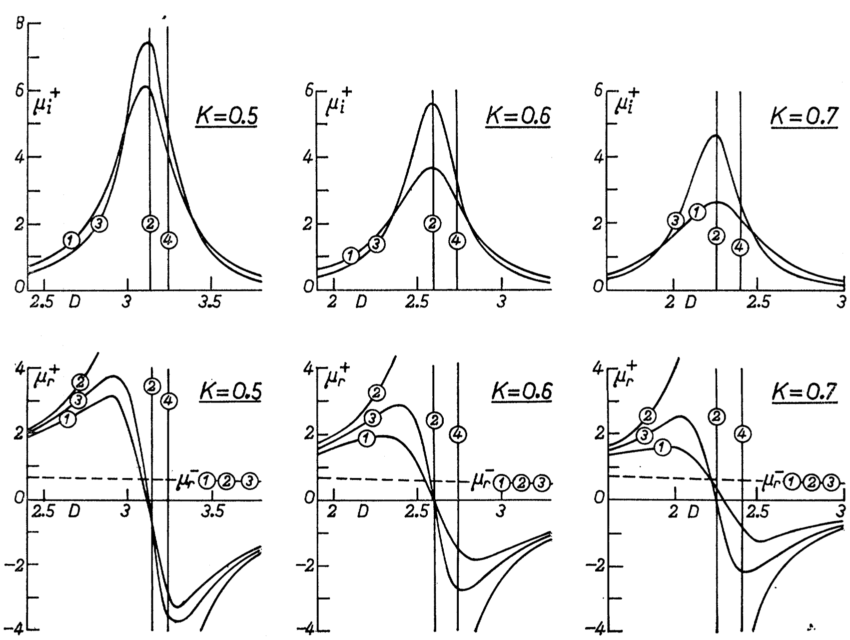

FIG. 1. Real and imaginary parts of the circular response functions $\mu^{\pm}$ in the limit $\omega_c\tau \to \infty$, calculated numerically as a function of $D=d\omega_c/v_F$ for different values of the parameter $K=k^+v_F/\omega_c$; $K=0.5$, 0.6, 0.7. The curves indicated by ① are calculated from equation (A1), those indicated by ② and ③ from Eqs. (A18) and (A19), respectively; the vertical lines indicated by ④ give the position of the resonances according to the local theory.

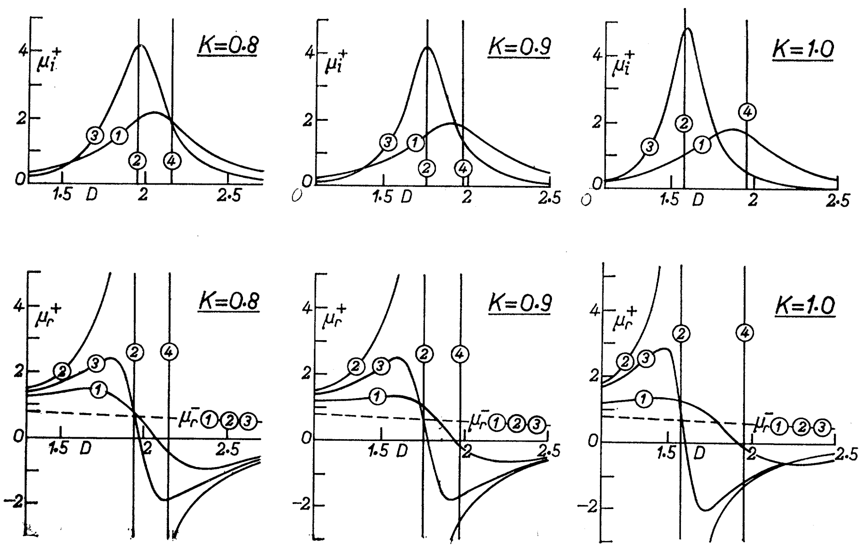

FIG. 2. Real and imaginary part of the circular response functions $\mu^{\pm}$ in the limit $\omega_c\tau\to\infty$, calculated numerically as a function of $D=d\omega_c/v_F$ for different values of the parameter $K=k^+v_F/\omega_c$; $K=0.8$, 0.9, 1.0. The curves indicated by ① are calculated from Eq. (A1), those indicated by ② and ③ from Eqs. (A18) and (A19), respectively; the vertical lines indicated by ④ give the position of the resonances according to the local theory.

which corresponds to the lowest-order approximation of Eqs. (26) and (27). For completeness we have also calculated the position of the resonances in the local theory. In Figs. 1 and 2 the calculated values of the real and imaginary parts of $\mu^{\pm}$ are given as a function of D for different values of K. The imaginary part of μ^- is not drawn because it nearly coincides with the abscissa; note that the imaginary part of μ^+ for the local and the nonlocal bulk theory is zero everywhere except in resonance, where it is infinite.

Concerning these results, we wish to make the following remarks.

(1) The position of the resonances in the two approximations is nearly the same and for values of K up to 0.8 in good agreement with the exact position. For $K\rangle 0.8$, however, the use of the nonlocal theory seems to give incorrect results; this can be said with certainly only for the fundamental resonance. There is, however, no reason to believe that things are worse for the higher resonances. On the contrary, if in a given sample a higher harmonic is studied at such a high K value, a higher value of $\omega_c\tau$ will be required. This means that the size-effect correction on the position of the resonance, which, at least in a first approximation, is proportional to $(l/d\omega_c\tau)^2$, can be expected to be smaller than for the first resonance.

The difference in the position of the maximum $\mu_i^+-\mu_i^-$ and of the zero $\mu_r^+-\mu_r^-$ shows that there is a phase shift in the response. This shift, which does not not occur in the nonlocal bulk theory, does agree quite well up to $K=0.8$.

(2) Whereas in the limit $\omega_c\tau\to\infty$ the nonlocal bulk theory gives infinitely sharp resonances, the resonances according the second approximation have a finite width. Compared to the width of the exactly calculated resonances, we can only say that they are of the right order of magnitude, the agreement getting worse the higher the value of K. These features correspond to the fact that in the limit $|\omega_c\tau\to\infty$ the nonlocal bulk theory gives Q values that are infinite, whereas the second approximation gives Q values that are finite, due to the damping at the surfaces. Since this damping is in a first approximation proportional to $v_F/d\omega_c$, it will for the same reason as already mentioned in (1) be smaller for a higher harmonic than for the first resonance at the same K value. We may therefore expect that the agreement between the approximate and exact results for the higher harmonics will be better.